A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6

APPLIED COMPLEX VARIABLES

TRINITY TERM 2017
FRIDAY, 2 JUNE 2017, 2.30pm to 4.15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

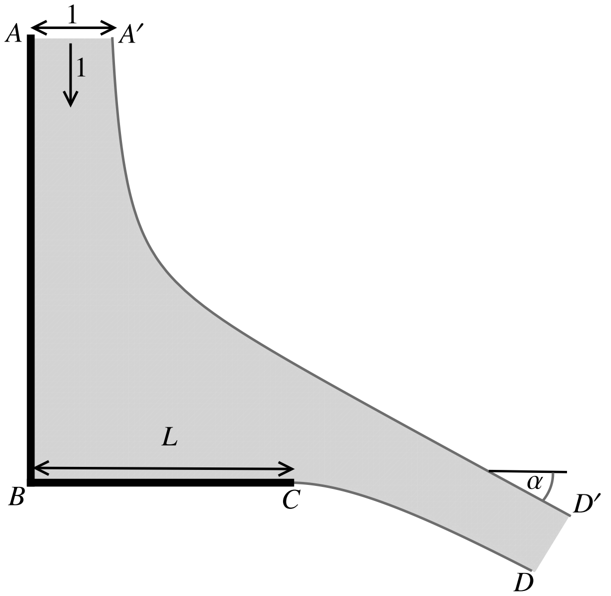
You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. Consider the steady two-dimensional inviscid flow illustrated below. A semi-infinite straight impenetrable wall AB is connected to a finite straight impermeable wall BC of length L through a corner at B with interior angle $\pi/2$. A jet of unit thickness flows in at unit speed from AA' and flows out at DD' in a direction making a clockwise angle $\alpha \in (0, \pi/2)$ with the wall BC as shown. The free surface CD separates tangentially at the point C.



You may assume that the complex potential $w(z)$ satisfies the boundary conditions $\operatorname{Im} w(z) = 0$ on $ABCD$, $\operatorname{Im} w(z) = 1$ on $A'D'$, $|w'(z)| = 1$ on the free surfaces CD and $A'D'$. Without loss of generality, the value of w at B is set to zero.

(a) [6 marks] Sketch the images of the fluid domain in the potential and hodograph planes. Indicate the images of the points A, B, C, D, D', A', and give brief explanations for the location of each image point.

(b) [12 marks] By conformally mapping the fluid domain onto the upper half-plane, show that w satisfies the equation

$$e^{\pi w} = \frac{\left(1 + (w')^2\right)^2}{1 - 2(w')^2 \cos(2\alpha) + (w')^4}.$$

Deduce that the value of w at C is given by $(2/\pi) \log(\operatorname{cosec} \alpha)$.

(c) [7 marks] Show that the length L of BC and the deflection angle α satisfy the relation

$$L = \frac{8 \cos^2 \alpha}{\pi} \int_0^1 \frac{\left(1 - u^2\right)\, du}{(1 + u^2)\,(1 - 2u^2 \cos(2\alpha) + u^4)}.$$

[*Hint: note that* $w' = u \in \mathbb{R}$ *on* BC.]

2. In this question, let $\Gamma \subset \mathbb{C}$ denote the open line segment $\Gamma = \{x + \mathrm{i}y : y = 0, -c < x < c\}$ and let $\overline{\Gamma}$ denote the closure $\overline{\Gamma} = \{x + \mathrm{i}y : y = 0, -c \leqslant x \leqslant c\}$.

   (a) [4 marks] State the *Plemelj formulae*, clearly defining any terms used.

   (b) [9 marks] Clearly define the branch of the function $\left(z^2 - c^2\right)^{1/2}$ that is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$ and takes positive real values when $z = x \in \mathbb{R}$ and $x > c$.

   Show that
$$\frac{1}{\pi}\int_{-c}^{c}\left(a + \frac{b}{1+x^2}\right)\frac{\mathrm{d}x}{\sqrt{c^2 - x^2}} = a + \frac{b}{\sqrt{1+c^2}}.$$

   [*Hint: consider a suitable integral around a large circle.*]

   (c) [12 marks] It is required to find a function $w(z)$ that is holomorphic on $\mathbb{C}\setminus\overline{\Gamma}$, has at worst inverse square root singularities as $z \to \pm c$, and satisfies the conditions
$$\operatorname{Im} w_+(x) = \operatorname{Im} w_-(x) = g(x) \quad \text{on } \Gamma, \qquad\qquad w(z) \to 0 \quad \text{as } z \to \infty,$$
   where $g : \overline{\Gamma} \mapsto \mathbb{R}$ is a given continuous function and $w_\pm(x) = \lim_{y\to 0\pm} w(x + \mathrm{i}y)$.

   (i) Show that, if w is written in the form
$$w(z) = \frac{1}{2\pi\mathrm{i}}\int_{-c}^{c}\frac{f(\xi)\,\mathrm{d}\xi}{\xi - z},$$
   then f must satisfy the singular integral equation
$$\frac{1}{\pi}⨍_{-c}^{c}\frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} = -2g(x).$$

   (ii) Show that the solution for w takes the form
$$w(z) = \frac{\mathrm{i}}{\left(z^2 - c^2\right)^{1/2}}\left(\frac{1}{\pi}\int_{-c}^{c}\frac{g(\xi)\sqrt{c^2 - \xi^2}}{\xi - z}\,\mathrm{d}\xi + H(z)\right),$$
   where H is holomorphic on $\mathbb{C} \setminus \{-c, c\}$ and real on Γ. Explain clearly why H must be a constant.

   (iii) For the case where
$$g(x) = 1 - \frac{\beta}{1+x^2},$$
   show that a *bounded* solution for w exists only if $\beta = \sqrt{1+c^2}$.

3. (a) [10 marks] By writing $w(z)$ as a generalised Fourier integral of the form

$$w(z) = \int_\Gamma g(\zeta)\, \mathrm{e}^{z\zeta}\, \mathrm{d}\zeta,$$

where both the kernel $g(\zeta)$ and the integration contour Γ are to be determined, find two linearly independent solutions for $\operatorname{Re} z > 0$ of the differential equation

$$zw''(z) + 3w'(z) + (2\alpha + 1 - z)w(z) = 0,$$

where $\alpha \in (0, 1)$.

[*You may express the solutions as unevaluated integrals.*]

(b) [15 marks] The function $u(x, y)$ satisfies the modified Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \epsilon^2 u = 0$$

in $y > 0$, subject to the mixed boundary conditions

$$u(x, 0) = 0 \quad \text{for } x < 0, \qquad\qquad \frac{\partial u}{\partial y}(x, 0) = \cos(x) \quad \text{for } x > 0,$$

and $u(x, y) \to 0$ as $y \to \infty$, where ϵ is a positive constant.

(i) Let

$$u_+(x) = \begin{cases} 0 & x < 0, \\ u(x, 0) & x > 0, \end{cases} \qquad\qquad h_-(x) = \begin{cases} \partial u/\partial y(x, 0) & x < 0, \\ 0 & x > 0. \end{cases}$$

Assuming that $u_+(x)$ is bounded as $x \to +\infty$ and that $h_-(x) = O\left(\mathrm{e}^{\epsilon x}\right)$ as $x \to -\infty$, show that the Fourier transforms of u_+ and h_- satisfy the equation

$$\left(k^2 + \epsilon^2\right)^{1/2} \overline{u}_+(k) + \overline{h}_-(k) = -\frac{\mathrm{i}}{2}\left(\frac{1}{k+1} + \frac{1}{k-1}\right). \qquad (\star)$$

Clearly define an appropriate branch of the multifunction $\left(k^2 + \epsilon^2\right)^{1/2}$, and state where in the complex k-plane equation $(\star)$ holds.

(ii) Hence determine $\overline{u}_+(k)$.

[*You may assume without proof that* $\overline{u}_+(k) = O\left(k^{-3/2}\right)$ *and* $\overline{h}_-(k) = O\left(k^{-1}\right)$ *as* $k \to \infty$.]

(iii) Deduce that, in the limit $\epsilon \to 0^+$, $\overline{u}_+(k)$ is given by

$$\overline{u}_+(k) = \frac{1}{2k^{1/2}}\left(\frac{1}{k-1} - \frac{\mathrm{i}}{k+1}\right),$$

where you should clearly define the appropriate branch of $k^{1/2}$.

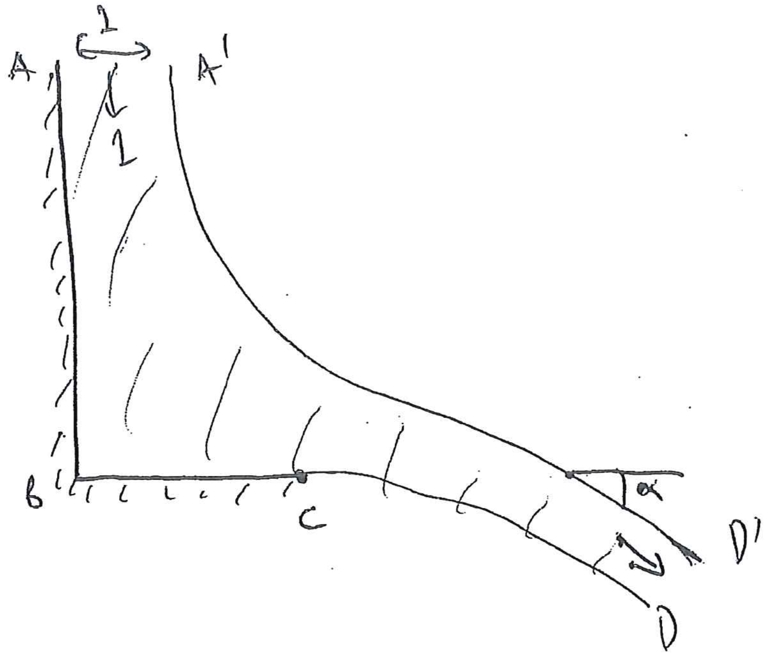
Describe a suitable inversion contour to compute $u_+(x)$.

(iv) Deduce that

$$u_+(x) = -\cos x + \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{(1+s)\,\mathrm{e}^{-sx}}{(1+s^2)\sqrt{s}}\, \mathrm{d}s.$$

C5.6

1(a)

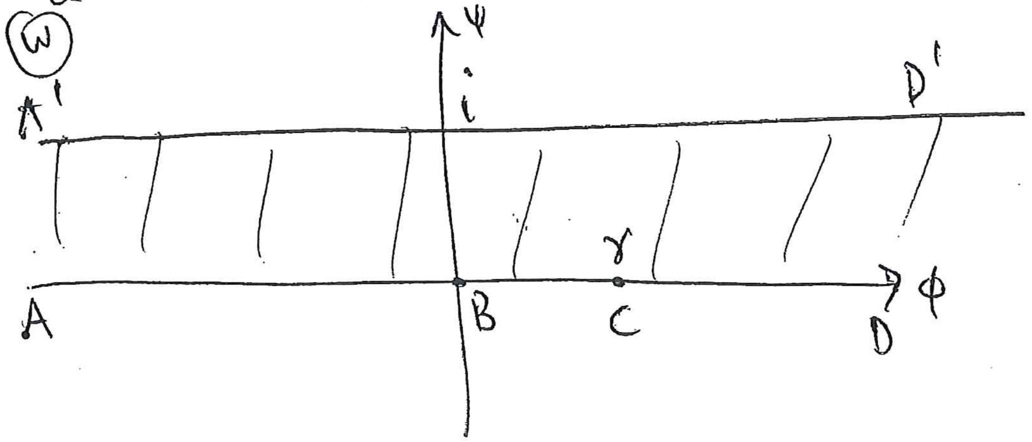


We have $\psi = \operatorname{Im} w = \begin{cases} 0 & \text{on } ABCD \\ 1 & \text{on } A'D' \end{cases}$

$w \sim +iz$ at AA' ⇒ $\phi \sim -y \to -\infty$ at AA'

$w \sim e^{i\alpha} z$ at DD' ⇒ $\phi \sim x\cos\alpha - y\sin\alpha \to +\infty$ at DD'

& we are given $w=0$ at B:



Value γ of ϕ at C is to be determined.

③

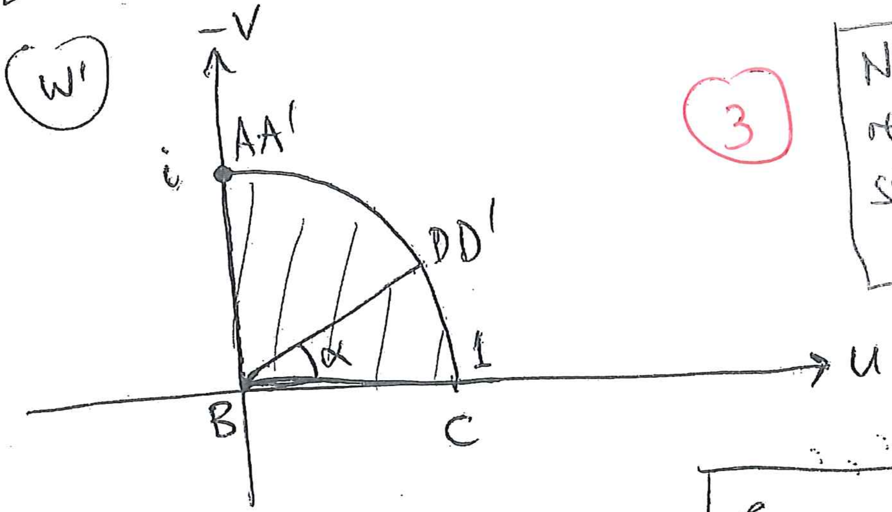
For hodograph plane, $w'(z) = u - iv \to i$ at AA'

On AB, $u=0$ and v decreases from 1 to 0 (since angle at B is $< \pi$)

On BC, $v=0$ and u increases from 0 to 1.

On CD, $|w'|=1$ and $w' = e^{i\theta}$, where θ increases from 0 at C to α at DD'

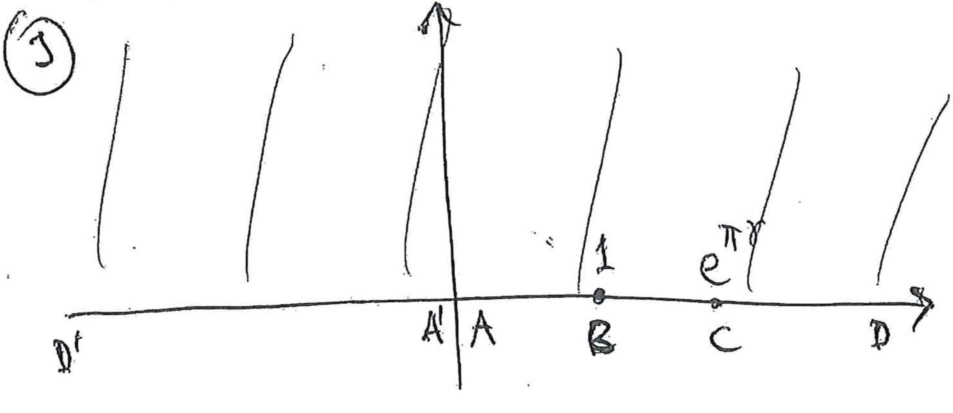
On A'D', $w' = e^{i\theta}$ where θ decreases from $\frac{\pi}{2}$ at A' to α at D'.



③

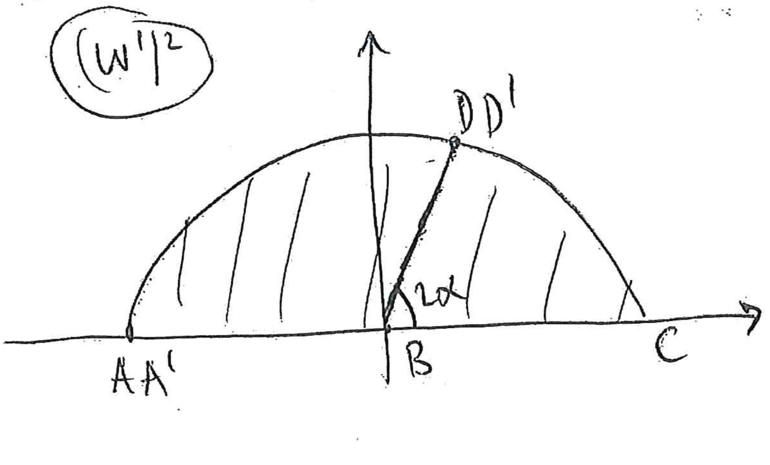
New but reasonably standard example

Map potential plane onto UHP: $\boxed{\zeta = e^{\pi w}}$



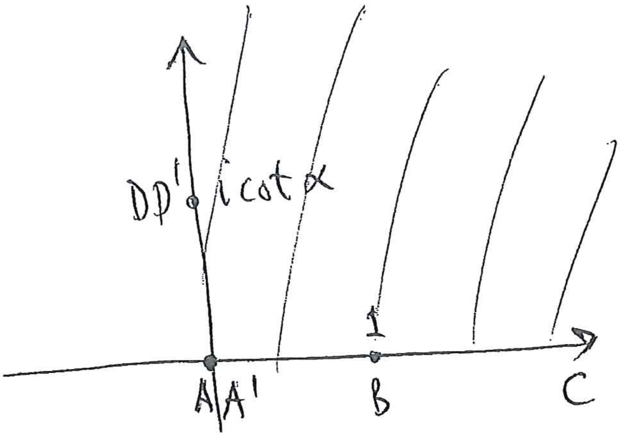
②

Do same for hodograph plane. First square:



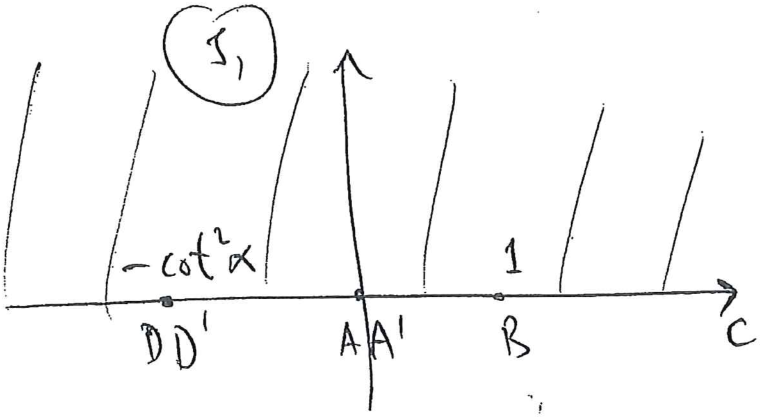
Next Möbius:

$$\boxed{\frac{1+(w')^2}{1-(w')^2}}$$



Square again:

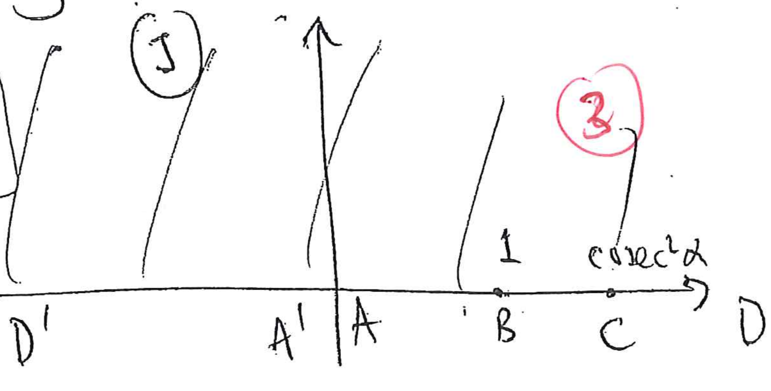
$$\zeta_1 = \left(\frac{1+(w')^2}{1-(w')^2}\right)^2$$



③

Finally shuffle the poles using Möbius:

$$\boxed{\zeta = \frac{\zeta_1}{\zeta_1\sin^2\alpha+\cos^2\alpha}}$$



③

Since positions of A, B, D are the same, by Riemann mapping theorem, these two maps must be identical, i.e.

$$e^{\pi w} = \frac{\zeta_1}{\zeta_1\sin^2\alpha+\cos^2\alpha} = \frac{\left(1+(w')^2\right)^2}{\left(1+(w')^2\right)^2\sin^2\alpha+\left(1-(w')^2\right)^2\cos^2\alpha}$$

$$\boxed{e^{\pi w} = \frac{\left(1+(w')^2\right)^2}{1+(w')^4-2(w')^2\cos(2\alpha)}}$$

②

and moreover, location of C must be the same:

$$e^{\pi \gamma} = \operatorname{cosec}^2 \alpha$$

$$\Rightarrow \boxed{\gamma = \frac{2}{\pi} \log \operatorname{cosec} \alpha}$$

New example

?

On BC, $w' = u \in [0,1]$

$$\therefore \quad e^{\pi w} = \frac{(1+u^2)^2}{1-2u^2\cos 2\alpha + u^4}$$

$$\therefore \quad \pi e^{\pi w} w' \frac{dz}{du} = \frac{\pi (1+u^2)^2}{1-2u^2\cos 2\alpha + u^4} \cdot u \cdot \frac{dz}{du}$$ ③

$$= \frac{4u(1+u^2)}{1-2u^2\cos 2\alpha + u^4} - \frac{(1+u^2)^2(4u^3 - 4u\cos 2\alpha)}{(1-2u^2\cos 2\alpha + u^4)^2}$$

$$= \frac{4u(1+u^2)}{(1-2u^2\cos 2\alpha + u^4)^2}\left\{1-2u^2\cos 2\alpha + u^4 - (1+u^2)(u^2-\cos 2\alpha)\right\}$$

where $\{\ \} = 1 + \cos 2\alpha - u^2(1+\cos 2\alpha) = 2\cos^2\alpha\,(1-u^2)$

$$\therefore \quad \frac{dz}{du} = \frac{8\cos^2\alpha\,(1-u^2)}{\pi(1+u^2)(1-2u^2\cos 2\alpha + u^4)}$$ ③

Integrate from $u=0$ at B to $u=1$ at C:

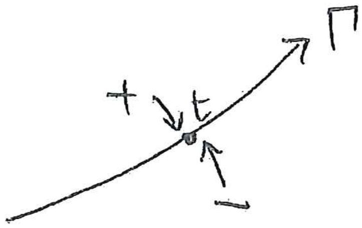
$$\boxed{L = \frac{8\cos^2\alpha}{\pi}\int_0^1 \frac{(1-u^2)\,du}{(1+u^2)(1-2u^2\cos 2\alpha + u^4)}}$$ ①

New example

2(a) Plemelj formulae

Let $w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$ where f is continuous on Γ and integrable on $\bar{\Gamma}$

Let $W_\pm(t) = \lim_{z\to t} w(z)$ where $t \in \Gamma$ and limit is taken from the left (+) or right (−) side relative to the orientation of Γ:



Then

$$\boxed{W_\pm(t) = \frac{1}{2\pi i}\fint_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - t} \pm \frac{1}{2} f(t)}$$

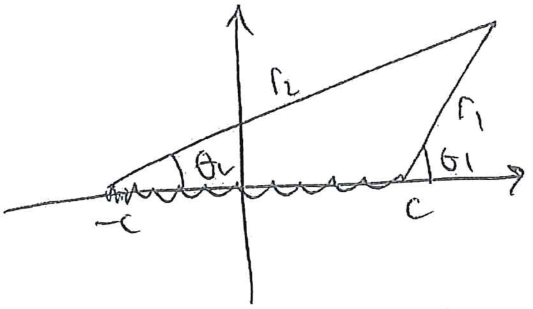
where $\fint$ denotes the principal value integral

$$\fint_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - t} = \lim_{\varepsilon\to 0}\int_{\Gamma\setminus\gamma_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - t}$$

and $\gamma_\varepsilon = \Gamma \cap D(t;\varepsilon)$

④ Bookwork

2(b) Define $(z^2-c^2)^{\frac{1}{2}} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$



with $r_1 = |z-c|$

$r_2 = |z+c|$

$\theta_1 = \arg(z-c) \in (-\pi, \pi)$

$\theta_2 = \arg(z+c) \in (-\pi, \pi)$

$\boxed{\text{Standard approach}}$ ②

Now consider
$$\oint_{C_R} \left(a + \frac{b}{1+z^2}\right) \frac{dz}{(z^2-c^2)^{\frac{1}{2}}} = I$$

where $C_R = \{Re^{i\theta} : 0 \leq \theta < 2\pi\}$

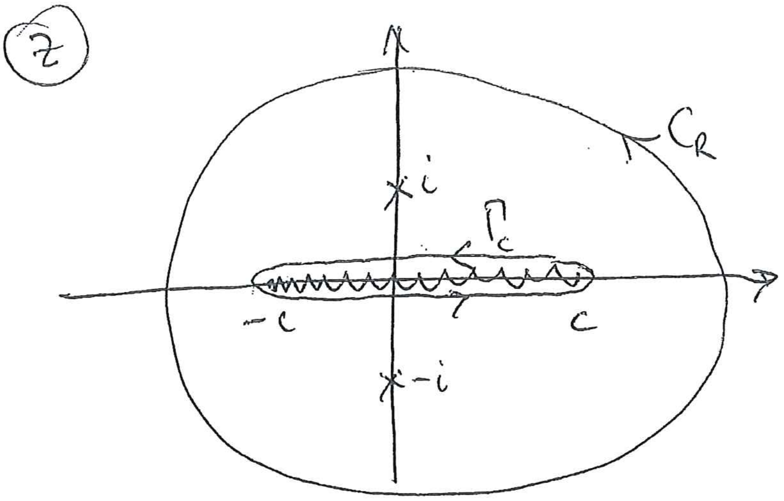
Then as $R \to \infty$

$$I = \oint_{C_R} \left[a + O\left(\frac{1}{z^2}\right)\right]\left(1 - \frac{c^2}{z^2}\right)^{-\frac{1}{2}} \frac{dz}{z}$$

$$= 2\pi i a + O\left(\frac{1}{R^2}\right)$$

$\therefore \boxed{I = 2\pi i a}$ (since I is independent of R for R sufficiently large)

②



$\Gamma_c \Rightarrow$ enclosing branch cut

By contour deformation,

$$I = \int_{\Gamma_c}\left(a+\frac{b}{1+z^2}\right)\frac{dz}{(z^2-c^2)^{\frac{1}{2}}} + 2\pi i\,\mathrm{Res}\Big|_{z=i} + 2\pi i\,\mathrm{Res}\Big|_{z=-i}$$ ①

Now $(z^2-c^2)^{\frac{1}{2}} = \pm i\sqrt{c^2-x^2}$ when $z = x \pm 0i$

So $$\int_{\Gamma_c} = 2\int_{-c}^{c}\left(a+\frac{b}{1+x^2}\right)\frac{dx}{-i\sqrt{c^2-x^2}} = 2i\int_{-c}^{c}\left(a+\frac{b}{1+x^2}\right)\frac{dx}{\sqrt{c^2-x^2}}$$ ②

and $(z^2-c^2)^{\frac{1}{2}} = \pm i\sqrt{1+c^2}$ at $z = \pm i$

So $$\left.\begin{aligned}\mathrm{Res}\Big|_{z=-i} &= \frac{b}{(-2i)(-i\sqrt{1+c^2})} = -\frac{b}{2\sqrt{1+c^2}} \\ \mathrm{Res}\Big|_{z=+i} &= \frac{b}{(2i)(i\sqrt{1+c^2})} = -\frac{b}{2\sqrt{1+c^2}}\end{aligned}\right\}$$ ②

$$\therefore\ 2\pi i a = 2i\int_{-c}^{c}\left(a+\frac{b}{1+x^2}\right)\frac{dx}{\sqrt{c^2-x^2}} - 2\pi i\frac{b}{\sqrt{1+c^2}}$$

Rearrange:

$$\boxed{\frac{1}{\pi}\int_{-c}^{c}\left(a+\frac{b}{1+x^2}\right)\frac{dx}{\sqrt{c^2-x^2}} = a + \frac{b}{\sqrt{1+c^2}}}$$

new example

②

2(c) Let $W_\pm(x) = u_\pm(x) + i g(x)$ on Γ

where $u_\pm, g \in \mathbb{R}$.

Apply Plemelj:

$$W_+(x) + W_-(x) = u_+(x) + u_-(x) + 2ig(x) = \frac{1}{\pi i}\int_{-c}^{c}\frac{f(\xi)\,d\xi}{\xi - x}$$

$$W_+(x) - W_-(x) = u_+(x) - u_-(x) = f(x)$$

So $f(x) \in \mathbb{R}$ and $u_+(x) + u_-(x) = 0$

$$\text{So} \quad i\left(W_+(x) + W_-(x)\right) = \boxed{\frac{1}{\pi}\int_{-c}^{c}\frac{f(\xi)\,d\xi}{\xi - x} = -2g(x)}$$

$\boxed{\text{Standard result}}$ ②

Let $W(z) = (z^2 - c^2)^{\frac{1}{2}} w(z)$

and write $W(z) = \frac{1}{2\pi i}\int_\Gamma \frac{F(\xi)\,d\xi}{\xi - z}$

$$\text{Then} \quad F(x) = W_+(x) - W_-(x) = i\sqrt{c^2 - x^2}\left(w_+(x) + w_-(x)\right)$$
$$= -2\sqrt{c^2 - x^2}\, g(x)$$

which gives a solution

$$W(z) = \frac{i}{\pi}\int_\Gamma \frac{\sqrt{c^2 - \xi^2}\, g(\xi)\,d\xi}{\xi - z}$$ ②

For general solution, consider the homogeneous problem with $g = 0$, i.e. with $W(z)$ continuous across Γ and Re $W = 0$ on Γ (and so Im $W = 0$ on Γ). This is satisfied by $W(z) = iH(z)$ where $H(z)$ is holomorphic on $\mathbb{C}\setminus\{\pm c\}$ and real on Γ.

So general solution is

$$W(z) = \frac{i}{\pi}\int_{\Gamma} \frac{\sqrt{c^2-\zeta^2}\; g(\zeta)\,d\zeta}{\zeta - z} + iH(z)$$

& hence

$$\boxed{w(z) = \frac{i}{(z^2-c^2)^{\frac{1}{2}}}\left(\frac{1}{\pi}\int_{-c}^{c}\frac{\sqrt{c^2-\zeta^2}\; g(\zeta)\,d\zeta}{\zeta - z} + H(z)\right)}$$

standard result ②

$H(z)$ holomorphic on $\mathbb{C}\setminus\{\pm c\}$ ⇒ any singularities at $z = \pm c$ must be isolated.

Requirement w has at worst inverse square root singularities ⇒ $H = O(1)$ as $z \to \pm c$ and hence $H(z)$ is entire.

Requirement $w(z) \to 0$ as $z \to \infty$ ⇒ $H(z) = O(1)$ as $z \to \infty$

and hence by Liouville $H(z) \equiv$ constant ☒

standard approach ②

For $w(z)$ to be bounded, the term in brackets must $\to 0$ as $z \to \pm c$, i.e.

$$-\pi H = \text{const.} = \int_{-c}^{c} \frac{\sqrt{c^2-\xi^2}\, g(\xi)}{\xi - c}\, d\xi = \int_{-c}^{c} \frac{\sqrt{c^2-\xi^2}\, g(\xi)\, d\xi}{\xi + c}$$

$$\Rightarrow \boxed{\int_{-c}^{c} \frac{g(\xi)\, d\xi}{\sqrt{c^2-\xi^2}} = 0}$$ ②

Now put in $g(x) = 1 - \frac{\beta}{1+x^2}$ and use part (b)

$$\frac{1}{\pi}\int_{-c}^{c} \frac{g(x)}{\sqrt{c^2-x^2}}\, dx = 1 - \frac{\beta}{\sqrt{1+c^2}}$$

So necessary condition is $\boxed{\beta = \sqrt{1+c^2}}$ ∎

$\boxed{\text{New example}}$ ②

3(a) With $w(z) = \int_\Gamma g(\zeta)\, e^{z\zeta}\, d\zeta$,

$$0 = z\, w''(z) + 3w'(z) + (2\alpha+1-z)\, w(z)$$

$$= \int_\Gamma \left[z\zeta^2 + 3\zeta + (2\alpha+1-z) \right] g(\zeta)\, e^{z\zeta}\, d\zeta$$

$$= \left[(\zeta^2-1)\, g(\zeta)\, e^{z\zeta} \right]_\Gamma + \int_\Gamma \left[(3\zeta + 2\alpha+1)\, g(\zeta) - \frac{d}{d\zeta}\left((\zeta^2-1)\, g(\zeta) \right) \right] e^{z\zeta}\, d\zeta \quad ②$$

This must hold $\forall z$, and hence

$$\underline{\left[(\zeta^2-1)\, g(\zeta)\, e^{z\zeta} \right]_\Gamma = 0}$$ and

$$(3\zeta + 2\alpha + 1)\, g(\zeta) = \frac{d}{d\zeta}\left[(\zeta^2-1)\, g(\zeta) \right] = (\zeta^2-1)\, g'(\zeta) + 2\zeta\, g(\zeta)$$

$$\therefore \frac{g'(\zeta)}{g(\zeta)} = \frac{\zeta + 2\alpha + 1}{\zeta^2 - 1} = \frac{\alpha+1}{\zeta-1} - \frac{\alpha}{\zeta+1} \quad ③$$

$$\therefore \boxed{g(\zeta) = \frac{A\, (\zeta-1)^{\alpha+1}}{(\zeta+1)^{\alpha}}} \quad \text{where } A = \text{const.}$$

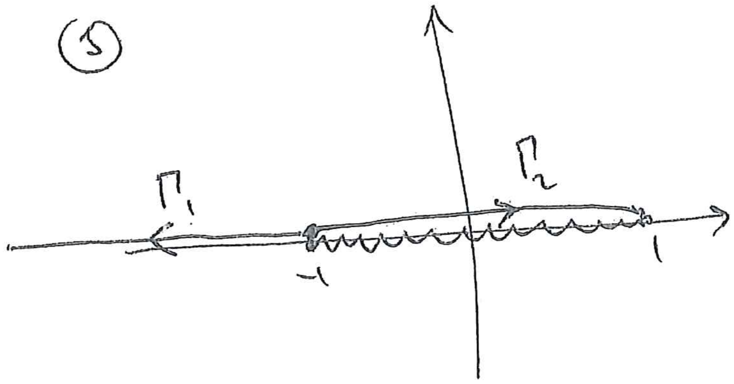
and

$$\boxed{\left[(\zeta-1)^{\alpha+2}\, (\zeta+1)^{1-\alpha}\, e^{z\zeta} \right]_\Gamma = 0} \quad ②$$

with $\alpha \in (0,1)$, the boundary term tends to zero at ②
$\zeta = -1, 1$, and as $\zeta \to \infty$ with $Re(z\zeta) < 0$.

Given $Re\, z > 0$, two contours leading to linearly independent solutions are

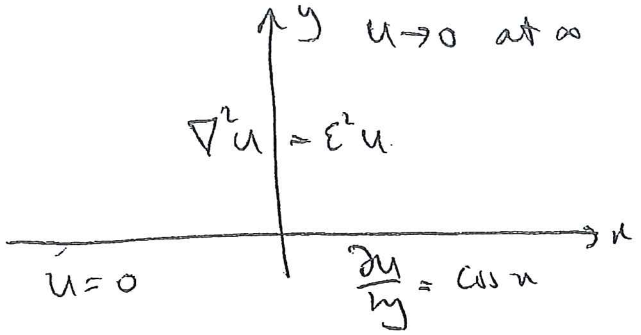
⑤



③ New example

[For Γ_1, solution is constant $\times\, e^{-z} \int_0^\infty \frac{(2+s)^{\alpha+1}}{s^\alpha} e^{-sz}\, ds$

For Γ_2, solution is constant $\times \int_{-1}^{1} \frac{(1-s)^{\alpha+1}}{(1+s)^\alpha} e^{sz}\, ds$]



With u_+ and h_- as defined, we have

$$\begin{cases} u(x,0) = u_+(x) \\ \frac{\partial u}{\partial y}(x,0) = H(x)\cos x + h_-(x) \end{cases}$$

with H = Heaviside function

Now take Fourier transform. Note

$$\int_0^\infty \cos x \, e^{ikx} \, dx = \frac{1}{2}\int_0^\infty e^{i(1+k)x} + e^{i(k-1)x} \, dx$$

$$= -\frac{1}{2}\left[\frac{1}{i(k+1)} + \frac{1}{i(k-1)}\right] = \frac{i}{2}\left[\frac{1}{k+1} + \frac{1}{k-1}\right]$$

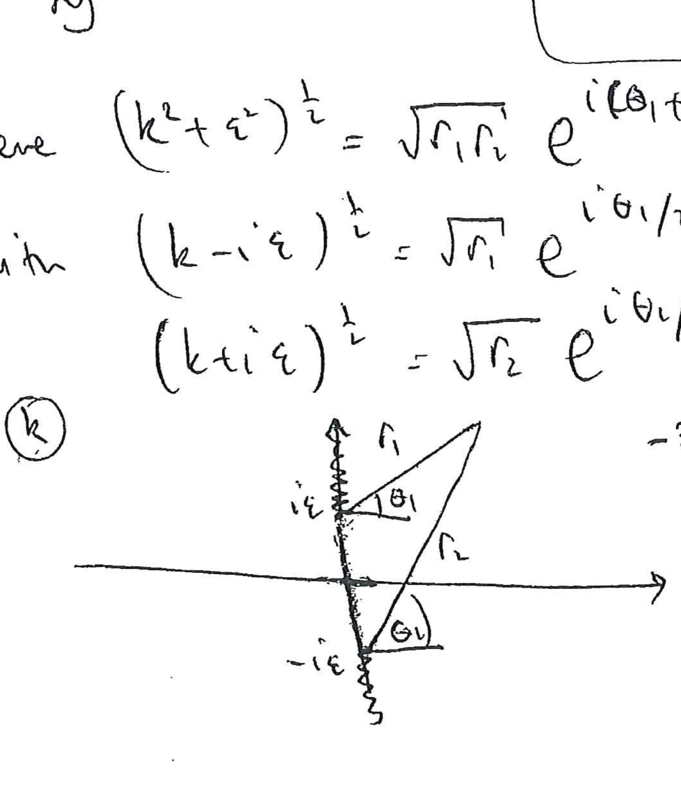
for **Im $k > 0$**. ①

$$\frac{\partial^2 \bar{u}}{\partial y^2} = (\varepsilon^2 + k^2)\bar{u} \Rightarrow \boxed{\bar{u} = A e^{-(k^2+\varepsilon^2)^{\frac{1}{2}} y}}$$

where $(k^2+\varepsilon^2)^{\frac{1}{2}} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} = (k+i\varepsilon)^{\frac{1}{2}}(k-i\varepsilon)^{\frac{1}{2}}$

with $(k-i\varepsilon)^{\frac{1}{2}} = \sqrt{r_1}\, e^{i\theta_1/2}$

$(k+i\varepsilon)^{\frac{1}{2}} = \sqrt{r_2}\, e^{i\theta_2/2}$



$-\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$

$-\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2}$

$\Rightarrow \mathrm{Re}\left[(k^2+\varepsilon^2)^{\frac{1}{2}}\right] > 0$ ①

Then BCs give

$$A = \bar{U}_+(k)$$

$$\& \quad -A\,(k^2+\varepsilon^2)^{\frac{1}{2}} = \frac{i}{2}\left[\frac{1}{k+1} + \frac{1}{k-1}\right] + \bar{h}_-(k)$$

$$\therefore \quad \boxed{(k^2+\varepsilon^2)^{\frac{1}{2}}\,\bar{U}_+(k) + \bar{h}_-(k) = -\frac{i}{2}\left[\frac{1}{k+1} + \frac{1}{k-1}\right]} \quad ⊛$$

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New application of familiar approach

Given $U_+(x) = O(1)$ as $x \to +\infty$

$h_-(x) = O(e^{\varepsilon x})$ as $x \to -\infty$

$\Rightarrow$ $\begin{cases} \bar{U}_+(k) \text{ is holomorphic on } \operatorname{Im} k > 0 \\ \bar{h}_-(k) \text{ —— '' —— } \operatorname{Im} k < \varepsilon \end{cases}$

and ⊛ holds on $0 < \operatorname{Im} k < \varepsilon$ 1

$$\therefore \quad (k+i\varepsilon)^{\frac{1}{2}}\,\bar{U}_+(k) + \frac{\bar{h}_-(k)}{(k-i\varepsilon)^{\frac{1}{2}}} = -\frac{i}{2(k-i\varepsilon)^{\frac{1}{2}}}\left[\frac{1}{k+1} + \frac{1}{k-1}\right]$$

$$\text{RHS} = -\frac{i}{2}\left\{\frac{(-1-i\varepsilon)^{-\frac{1}{2}}}{k+1} + \frac{1}{k+1}\left[\frac{1}{(k-i\varepsilon)^{\frac{1}{2}}} - \frac{1}{(-1-i\varepsilon)^{\frac{1}{2}}}\right] + \frac{1}{(1-i\varepsilon)^{\frac{1}{2}}(k-1)} + \frac{1}{k-1}\left[\frac{1}{(k-i\varepsilon)^{\frac{1}{2}}} - \frac{1}{(1-i\varepsilon)^{\frac{1}{2}}}\right]\right\}$$

where $\dfrac{1}{(1-i\varepsilon)^{\frac{1}{2}}} = \dfrac{e^{+\frac{i}{2}\tan^{-1}\varepsilon}}{(1+\varepsilon^2)^{1/4}}$, $\dfrac{1}{(-1-i\varepsilon)^{\frac{1}{2}}} = \dfrac{i\,e^{-\frac{i}{2}\tan^{-1}\varepsilon}}{(1+\varepsilon^2)^{1/4}}$

$$\therefore\ (k+i\varepsilon)^{1/2}\,\bar{U}_+(k) + \frac{i}{2}\left\{ \frac{e^{\frac{i}{2}\tan^{-1}\varepsilon}}{(1+\varepsilon^2)^{1/4}(k-1)} + \frac{i e^{-\frac{i}{2}\tan^{-1}\varepsilon}}{(1+\varepsilon^2)^{1/4}(k+1)} \right\}$$

$$= -\frac{\bar{h}_-(k)}{(k-i\varepsilon)^{1/2}} - \frac{i}{2}\cdot\frac{1}{k-1}\cdot\left[\frac{1}{(k-i\varepsilon)^{\frac{1}{2}}} - \frac{i e^{-\frac{i}{2}\tan^{-1}\varepsilon}}{(1+\varepsilon^2)^{1/4}}\right]$$

$$-\frac{i}{2}\cdot\frac{1}{k+1}\cdot\left[\frac{1}{(k-i\varepsilon)^{1/2}} - \frac{e^{\frac{i}{2}\tan^{-1}\varepsilon}}{(1+\varepsilon^2)^{1/4}}\right] \qquad ③$$

where LHS is holomorphic in $\mathrm{Im}\,k > 0$

RHS is holomorphic in $\mathrm{Im}\,k < \varepsilon$

and they are both holomorphic on the overlap strip.

By analytic continuation, between them the LHS and RHS define an entire function $E(k)$.

Given behaviour of $\bar{U}_+$ and $\bar{h}_-$ implies that $E(k) \to 0$ as $k \to \infty$ both in $\mathrm{Im}\,k > 0$ and in $\mathrm{Im}\,k < \varepsilon$

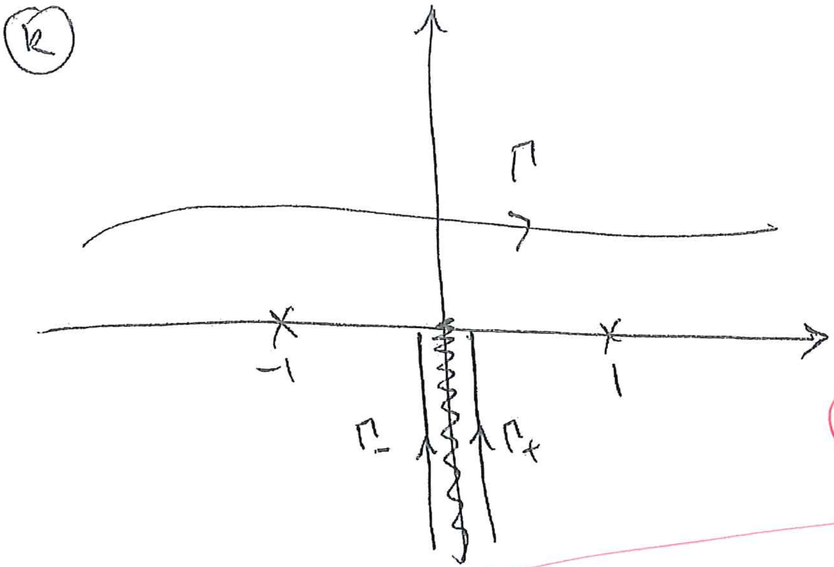
Therefore by Liouville, $E(k) \equiv 0$. New example ①

So, now taking limit $\varepsilon \to 0$

$$\bar{U}_+(k) = -\frac{i}{2k^{1/2}}\left[\frac{1}{k-1} + \frac{i}{k+1}\right]$$

i.e. $$\boxed{\bar{U}_+(k) = \frac{1}{2k^{1/2}}\left[\frac{1}{k+1} - \frac{i}{k-1}\right]} \qquad ②$$

Here $k^{1/2} = \lim_{\varepsilon\to 0}(k+i\varepsilon)^{\frac{1}{2}}$ has argument $\in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$



Inversion contour must pass <u>above</u> singularities in $\bar{u}_+(k)$ at $k=-1,0,1$

① New example

For $\underline{x>0}$, close the contour in the lower half-plane:

$$u_+(x) = \frac{1}{2\pi}\int_\Gamma \bar{u}_+(k)\,e^{-ikx}\,dk \quad \text{[NB clockwise orientation]}$$

$$= -i\,\mathrm{Res}\left(\bar{u}_+(k)e^{-ikx};\,1\right) - i\,\mathrm{Res}\left(\bar{u}_+(k)e^{-ikx};\,-1\right)$$

$$+\frac{1}{2\pi}\left(\int_{\Gamma_-} - \int_{\Gamma_+}\right)\bar{u}_+(k)\,e^{-ikx}\,dk \quad ②$$

On Γ_-, $k=-is$, $k^{1/2} = e^{3i\pi/4}\sqrt{s} = -e^{-i\pi/4}\sqrt{s}$

On Γ_+, $k=-is$, $k^{1/2} = e^{-i\pi/4}\sqrt{s}$

$$\therefore\ u_+(x) = \frac{-i}{2i}e^{ix} - \frac{i}{2}(-i)e^{-ix} \quad ①$$

$$+\frac{1}{\pi}\int_\infty^0 \frac{-e^{i\pi/4}}{2\sqrt{s}}\left[\frac{1}{1-is}+\frac{i}{1+is}\right]e^{-sx}(-i)\,ds$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{e^{-i\pi/4}}{\sqrt{s}}\left[\frac{(1+i)(1+s)}{1+s^2}\right]e^{-sx}\,ds - \cos x$$

$$\therefore\ \boxed{u_+(x) = -\cos x + \frac{1}{\sqrt{2}\,\pi}\int_0^\infty \frac{(1+s)\,e^{-sx}}{\sqrt{s}\,(1+s^2)}\,ds}$$

New example ①