A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6

APPLIED COMPLEX VARIABLES

TRINITY TERM 2018

TUESDAY, 29 May 2018, 2.30pm to 4.15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

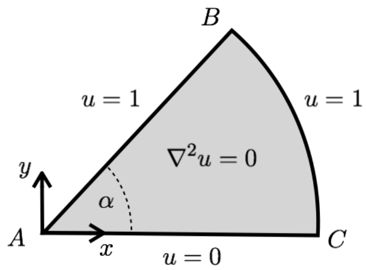
You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. (a) [8 marks] The steady-state temperature $u(x, y)$ in a circular wedge with angle α and radius 1 satisfies the following boundary-value problem:

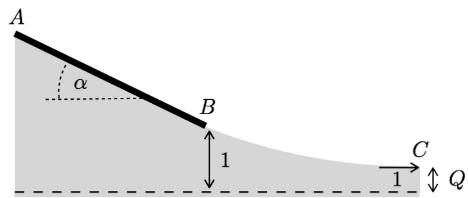


By using a conformal transformation to map this wedge to a half plane with A at the origin and C at infinity, show that the temperature is given by

$$u(x, y) = \frac{1}{\pi}\mathrm{Arg}\left(\frac{4z^{\pi/\alpha}}{(z^{\pi/\alpha} - 1)^2}\right),$$

where $z = x + \mathrm{i}y$, and $\mathrm{Arg}(\zeta)$ is the principal value of the argument of ζ (*i.e.* taking values in the range $-\pi < \mathrm{Arg}(\zeta) \leqslant \pi$).

(b) A two-dimensional fluid flow from a symmetric nozzle with angle α results in a free-surface flow as shown in the diagram below (only half of the domain is shown, with the dashed line being the axis of symmetry):



The nozzle has half-width 1 and the flow contracts far downstream to have speed 1 and half-width Q. The axis of symmetry is a streamline on which $\psi = 0$, while ABC is a streamline on which $\psi = Q$, where $w = \phi + \mathrm{i}\psi$ is the complex potential and $\phi = 0$ at the separation point B. On the free surface BC the complex velocity w' satisfies $|w'| = 1$.

(i) [7 marks] Concentrating only on the region above the dashed line, sketch the fluid domain in the potential and hodograph planes, clearly labelling the corresponding points A, B and C. By mapping the domain from each plane to an upper half plane, show that

$$e^{\pi w/Q} = F(w'),$$

where the function F should be determined.

(ii) [10 marks] By parameterising the free surface with $w' = e^{-\mathrm{i}\theta}$ and hence finding $\mathrm{d}y/\mathrm{d}\theta = f(\theta)$ on the free surface, show that the contraction ratio Q is given by

$$Q = \frac{1}{1 + J} \qquad \text{where} \quad J = \int_0^1 \frac{\sin \alpha\varphi}{\tan \frac{\pi}{2}\varphi}\mathrm{d}\varphi.$$

Hence calculate Q for the cases $\alpha = 0$, $\alpha = \pi/2$, and $\alpha = \pi$.

2. (a) [5 marks] The complex velocity describing flow past a thin aerofoil is represented by a function $w(z)$ which is holomorphic on $\mathbb{C} \setminus [0, c]$ and has limiting values $w_\pm$ on $y = 0\pm$, $0 < x < c$, that satisfy

$$w_+(x) + w_-(x) = 2\mathrm{i}g'(x),$$

where $g(x)$ is a smooth real-valued function describing the shape of the aerofoil.

By introducing a suitable auxiliary function $\tilde{w}(z)$, show that a solution for $w(z)$ can be written in the form

$$w(z) = \frac{\tilde{w}(z)}{\pi} \int_0^c \frac{g'(\xi)}{\tilde{w}_+(\xi)(\xi - z)} \mathrm{d}\xi + \tilde{w}(z)H(z),$$

where $H(z)$ is holomorphic on $\mathbb{C} \setminus \{0, c\}$, and the integral is evaluated on the real interval $0 < \xi < c$.

(b) [5 marks] In addition to the properties given above, w satisfies $w = O(z^{-1/2})$ at $z = 0$, $w = O(1)$ at $z = c$, and $w = O(1/z)$ as $z \to \infty$. Show that the solution in this case is

$$w(z) = \left(\frac{c - z}{z}\right)^{1/2} \frac{1}{\pi} \int_0^c \sqrt{\frac{\xi}{c - \xi}} \frac{g'(\xi)}{\xi - z} \mathrm{d}\xi.$$

Define carefully which branch of the leading square root term is used.

(c) [10 marks] By considering a contour that surrounds the branch cut and deforming the contour to a large circle, find an explicit expression for $w(z)$ in the case $g(x) = x^2 - x$. Show that the velocity at the leading edge $z = 0$ is finite if $c = 1$, and that the solution in that case reduces to

$$w(z) = \mathrm{i}(2z - 1) + 2z^{1/2}(1 - z)^{1/2}.$$

(d) [5 marks] Find a bounded solution $f(x)$ to the singular integral equation

$$\frac{1}{\pi} \fint_0^1 \frac{f(\xi)}{\xi - x} \mathrm{d}\xi + 4x - 2 = 0.$$

[*You may assume the Plemelj formulae*

$$W_\pm(x) = \pm\frac{1}{2}F(x) + \frac{1}{2\pi\mathrm{i}} \fint_0^c \frac{F(\xi)}{\xi - x} \,\mathrm{d}\xi,$$

for the function

$$W(z) = \frac{1}{2\pi\mathrm{i}} \int_0^c \frac{F(\xi)}{\xi - z} \,\mathrm{d}\xi.]$$

3. (a) [8 marks] Let $G(k)$ be holomorphic in the strip $0 < \mathrm{Im}(k) < 1$, with $G(k) \to 0$ as $k \to \infty$, and suppose $0 < \gamma_+ < \gamma_- < 1$. By applying Cauchy's Integral Formula to a contour surrounding k and deforming the contour, show that it is possible to write $G(k) = G_+(k) - G_-(k)$, where

$$G_\pm(k) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty+\mathrm{i}\gamma_\pm}^{\infty+\mathrm{i}\gamma_\pm} \frac{G(\zeta)}{\zeta - k} \mathrm{d}\zeta,$$

are functions holomorphic on $\mathrm{Im}(k) > \gamma_+$ and $\mathrm{Im}(k) < \gamma_-$ respectively.

Apply this decomposition to the function $G(k) = (k - \mathrm{i})^{1/2}/k^2$, with the branch cut for the square root taken to lie along $\mathrm{i}[1, \infty)$, to show that

$$G_+(k) = \frac{e^{-\mathrm{i}\pi/4}(1 + \frac{1}{2}\mathrm{i}k)}{k^2}.$$

(b) [9 marks] The function $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{in } y > 0,$$

with $u \to 0$ as $y \to \infty$, and with mixed boundary conditions on $y = 0$,

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{for } x < 0, \qquad u(x, 0) = x \quad \text{for } x > 0.$$

Define

$$f_-(x) = \begin{cases} u(x,0) & x < 0 \\ 0 & x > 0 \end{cases}, \qquad g_+(x) = \begin{cases} 0 & x < 0 \\ \partial u/\partial y(x, 0) & x > 0 \end{cases},$$

and assume that $f_-(x) = O(e^{\gamma_- x})$ as $x \to -\infty$ and $g_+(x) = O(e^{\gamma_+ x})$ as $x \to +\infty$.

By taking a Fourier transform in x (denoted by an overbar), show that

$$\overline{g}_+(k) + (k^2 - \mathrm{i}k)^{1/2}\overline{f}_-(k) = \frac{(k^2 - \mathrm{i}k)^{1/2}}{k^2},$$

for $\gamma_+ < \mathrm{Im}(k) < \gamma_-$, where you should specify an appropriate definition of $(k^2 - \mathrm{i}k)^{1/2}$.

Use the Wiener-Hopf method to deduce that

$$\overline{g}_+(k) = \frac{e^{-\mathrm{i}\pi/4}(1 + \frac{1}{2}\mathrm{i}k)k^{1/2}}{k^2}.$$

[*You may assume that* $\overline{f}_-(k) = O(k^{-3/2})$ *and* $\overline{g}_+(k) = O(k^{-1/2})$ *as* $k \to \infty$.]

(c) [8 marks] Write down the inversion integral for $\partial u/\partial y(x, 0)$, and describe a suitable inversion contour. Hence show that for $x > 0$,

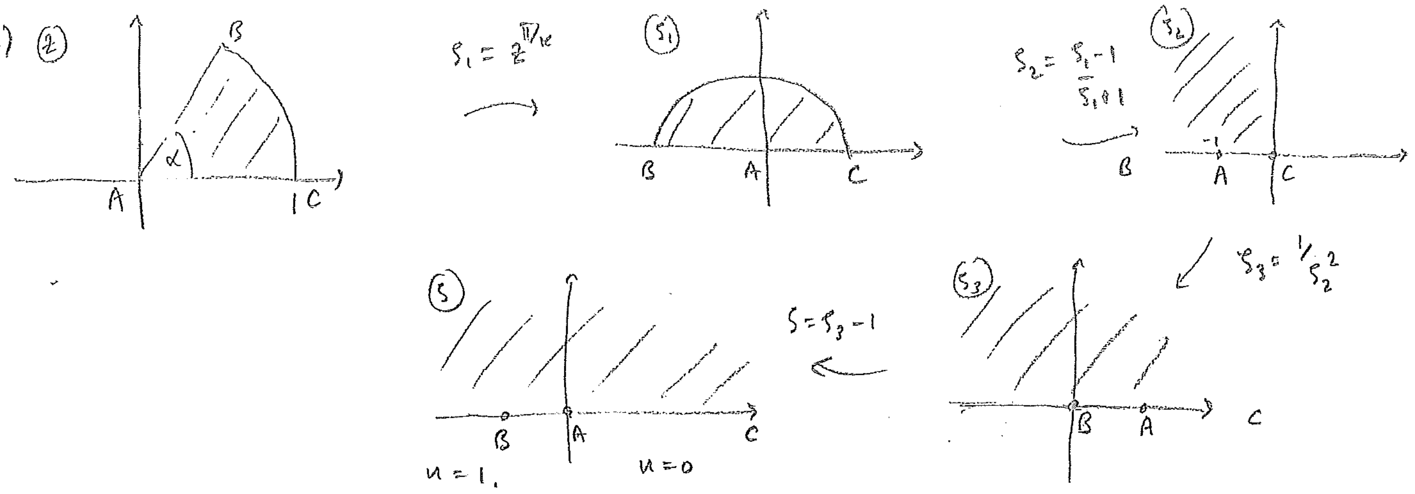
$$\frac{\partial u}{\partial y}(x, 0) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{x}} - 2\sqrt{x} \right).$$

[*You may assume that the relevant integrals around large circular arcs tend to zero, but should take care about the circular indentation required around the branch point at* $k = 0$. *You may make use of the result*

$$\int_0^\infty t^{-1/2} e^{-tx}\, \mathrm{d}s = \sqrt{\frac{\pi}{x}}.]$$

CS.6

1. (a)



Putting together, the required mapping is

$$\varsigma = \left(\frac{\varsigma_1+1}{\varsigma_1-1}\right)^2 - 1 = \frac{4\varsigma_1}{(\varsigma_1-1)^2} = \frac{4z^{\pi/\alpha}}{\left(z^{\pi/\alpha}-1\right)^2}.$$ [5]

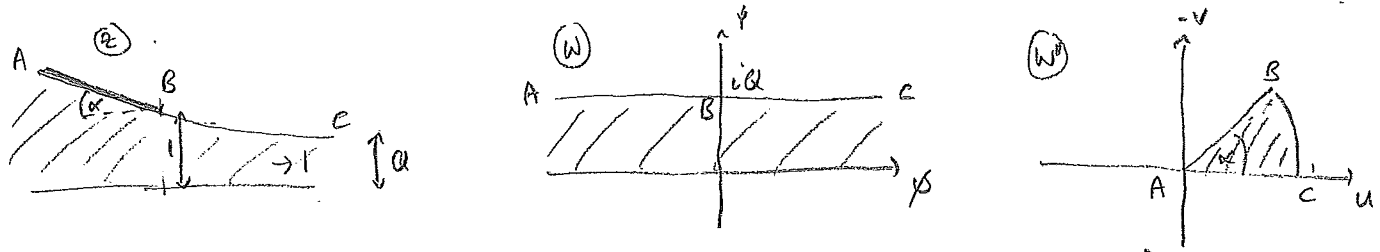
In the ς-plane, we must find u with u = 0 on +ve real axis, u = 1 on -ve real axis, and $\nabla^2 u = 0$.

By inspection, or using polar coordinates, the solution is $u = \frac{\theta}{\pi} = \frac{\mathrm{Arg}(\varsigma)}{\pi} = \mathrm{Re}\,\underbrace{\frac{-i\log(\varsigma)}{\pi}}_{W(\varsigma)\,=\,u+iv}$

Hence $u = \frac{1}{\pi}\mathrm{Arg}\left(\frac{4z^{\pi/\alpha}}{\left(z^{\pi/\alpha}-1\right)^2}\right)$. [3]

[Standard example]

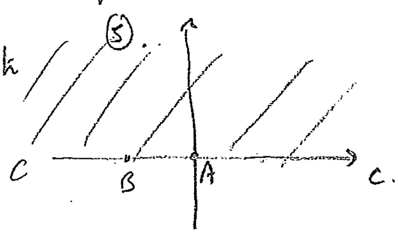
(b) (i).



Fluid is stagnant at ∞, so A at origin in hodograph plane. Free surface BC is part of unit circle with $u - iv = e^{+i\alpha}$ at B and u = 1 at C.

ABC has $\psi = Q$ since it is a streamline; the symmetry axis is also a streamline with $\psi = 0$. [4]

The map $\varsigma = e^{\pi w/Q}$ maps w domain to



The mapping from (a), $\varsigma = \frac{4(w')^{\pi/\alpha}}{\left((w')^{\pi/\alpha}-1\right)^2}$, maps the w' domain to this same UHP with the same mapping of the points A, B, C.

Hence $\boxed{e^{\frac{\pi w}{Q}} = \frac{4(w')^{\pi/\alpha}}{\left((w')^{\pi/\alpha}-1\right)^2}}$ [3]

[Bookwork / Standard]

ii) If $w' = e^{-i\theta}$ $\quad$ ($\theta = -\alpha$ at B, and $\theta = 0$ at C).

then $$e^{\frac{\pi w}{Q}} = \frac{4e^{-i\theta\pi/\alpha}}{\left(e^{-i\theta\pi/\alpha}-1\right)^2} = \frac{4}{\left(e^{\frac{-i\theta\pi}{2\alpha}} - e^{\frac{i\theta\pi}{2\alpha}}\right)^2} = \frac{4}{\left(2i\sin\left(\frac{\pi\theta}{2\alpha}\right)\right)^2} = -\frac{1}{\sin^2\frac{\pi\theta}{2\alpha}}.$$ [2]

Hence $$e^{\frac{\pi w}{Q}}\frac{\pi}{Q}\frac{dw}{d\theta} = \frac{2\pi}{2\alpha}\frac{1}{\sin^3\frac{\pi\theta}{2\alpha}}\cos\frac{\pi\theta}{2\alpha} = -\frac{\pi}{\alpha}e^{\frac{\pi w}{Q}}\cot\frac{\pi\theta}{2\alpha} \Rightarrow \boxed{\frac{dw}{d\theta} = -\frac{Q}{\alpha}\cot\frac{\pi\theta}{2\alpha}.}$$

But $$\frac{dz}{d\theta} = \frac{1}{w'}\frac{dw}{d\theta} = e^{i\theta}\frac{dw}{d\theta} = -\frac{Q}{\alpha}\cot\frac{\pi\theta}{2\alpha}\left(\cos\theta + i\sin\theta\right).$$

Hence $$\boxed{\frac{dy}{d\theta} = -\frac{Q}{\alpha}\cot\frac{\pi\theta}{2\alpha}\sin\theta}$$ [2]

Integrating from B to C $$\int_{-\alpha}^{0}\frac{dy}{d\theta}\,d\theta = \int_1^Q dy = Q-1$$

so $$1-Q = -\int_{-\alpha}^{0}\frac{dy}{d\theta}\,d\theta = \frac{Q}{\alpha}\int_{-\alpha}^{0}\cot\frac{\pi\theta}{2\alpha}\sin\theta\,d\theta = Q\underbrace{\int_0^1 \cot\frac{\pi}{2}\varphi\,\sin\alpha\varphi\,d\varphi}_{J}.$$

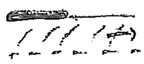
$\theta = -\alpha\varphi$

$d\theta = -\alpha\,d\varphi$

$\Rightarrow$ $1 - Q = JQ$

$\Rightarrow$ $\boxed{Q = \frac{1}{1+J}}$ where $\boxed{J = \int_0^1 \cot\frac{\pi}{2}\varphi\,\sin\alpha\varphi\,d\varphi.}$ [3]

For $\alpha = 0$, $J = 0 \Rightarrow \boxed{Q = 1}$



$\alpha = \frac{\pi}{2}$, $J = \int_0^1 \cos\frac{\pi}{2}\varphi\,d\varphi = \frac{2}{\pi} \Rightarrow \boxed{Q = \frac{\pi}{\pi+2}}$ [3]

$\alpha = \pi$, $J = \int_0^1 \cot\frac{\pi}{2}\varphi\,\sin\pi\varphi\,d\varphi = \int_0^1 2\cos^2\frac{\pi}{2}\varphi\,d\varphi = \int_0^1 \cos\pi\varphi + 1\,d\varphi = 1 \Rightarrow \boxed{Q = \frac{1}{2}.}$

[Standard technique, new example].

2.(a) Let $\tilde{w}(z)$ be holomorphic in $\mathbb{C}\setminus[0,c]$, and satisfying $\boxed{\tilde{w}_+(x)+\tilde{w}_-(x)=0}$ on $(0,c)$. [2]

- Then $W(z)=\dfrac{w(z)}{\tilde{w}(z)}$ is holomorphic in $\mathbb{C}\setminus[0,c]$ with $W_+(z)-W_-(z)=\dfrac{w_+(z)+w_-(z)}{\tilde{w}_+(z)}=\dfrac{2ig'(x)}{\tilde{w}_+(x)}$.

- Write $W(z)=\dfrac{1}{2\pi i}\displaystyle\int_0^c \frac{F(\xi)}{\xi-z}\,d\xi$, then by the Plemelj formulae, $W_+(x)-W_-(x)=F(x)$, so we can read off that $\boxed{F(x)=\dfrac{2ig'(x)}{\tilde{w}_+(x)}}$. Hence $W(z)=\dfrac{1}{\pi}\displaystyle\int_0^c \frac{g'(\xi)}{\tilde{w}_+(\xi)(\xi-z)}\,d\xi$ is a solution. [2]

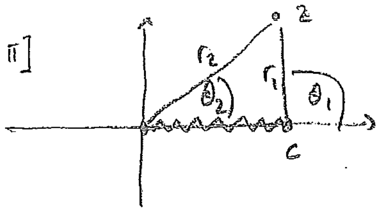
- The difference between this and any other solution must be holomorphic on $\mathbb{C}\setminus[0,c]$ and continuous across $(0,c)$, hence holomorphic on all $\mathbb{C}$ except possibly at 0 and c,

- So we can write $\boxed{w(z)=\dfrac{\tilde{w}(z)}{\pi}\displaystyle\int_0^c \frac{g'(\xi)}{\tilde{w}_+(\xi)(\xi-z)}\,d\xi+\tilde{w}(z)H(z)}$ [1]

where $H(z)$ is holomorphic on $\mathbb{C}\setminus\{0,c\}$.

[Bookwork]

(b) Let $\tilde{w}(z)=\left(\dfrac{c-z}{z}\right)^{1/2}$ which has the desired property $\tilde{w}_+(x)+\tilde{w}_-(x)=0$ if we take the branch cut on $[0,c]$, i.e. $\tilde{w}(z)=\left(\dfrac{r_1}{r_2}\right)^{1/2}e^{i(\theta_1-\theta_2-\pi)/2}$ with $\theta_1,\theta_2\in(-\pi,\pi]$



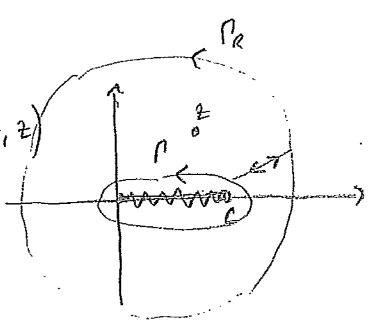
$\left(\text{so } \tilde{w}_+(x)=\left(\dfrac{c-x}{x}\right)^{1/2}\right)$
$\theta_1=\pi,\ \theta_2=0.$

Then from (a), $w(z)=\left(\dfrac{c-z}{z}\right)^{1/2}\left[\dfrac{1}{\pi}\displaystyle\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}}\frac{g'(\xi)}{\xi-z}\,d\xi\ \cancel{+H(z)}\right]$ [3]

But given singularities of $w(z)$ mean we need $H(z)$ finite at $z=0$ and at $z=c$, and satisfying $H\doteq O(\frac{1}{z})$ as $z\to\infty$, so $H(z)$ is entire, and bounded. Hence H is constant by Liouville's theorem, and since $H\to 0$ as $z\to\infty$, $\underline{H=0}$. [2]

[Standard]

(c) Consider $f(s)=\left(\dfrac{s}{c-s}\right)^{1/2}\dfrac{2s-1}{s-z}$ and $\left(\displaystyle\int_{\Gamma_R}-\int_{\Gamma}\right)f(s)\,ds=2\pi i\cdot\text{res}(f,z)$ (C.R.T). [2]



- $\displaystyle\int_\Gamma f(s)\,ds=\underset{\text{bottom}}{\int_0^c}-\underset{\text{top}}{\int_0^c}=-2\underset{\text{top}}{\int_0^c}f_+(\xi)\,d\xi=\underline{-2\int_0^c\frac{\xi^{1/2}}{(c-\xi)^{1/2}}\frac{2\xi-1}{\xi-z}\,d\xi}$. [2]

- $\text{res}(f,z)=\underline{\left(\dfrac{z}{c-z}\right)^{1/2}(2z-1)}$

As $\xi \to \infty$, $\left(\frac{c-\xi}{\xi}\right)^{1/2} \sim -i\left(\frac{\xi-c}{\xi}\right)^{1/2} = -i\left(1-\frac{c}{2\xi}+\dots\right)$,

so $f(\xi) \sim i\left(1+\frac{c}{2\xi}+\dots\right)\left(\frac{2\xi}{\xi}\right)\left(1-\frac{1}{2\xi}\right)\left(1+\frac{z}{\xi}+\dots\right) = 2i\left(1+\frac{1}{\xi}\left(\frac{c}{2}-\frac{1}{2}+z\right)+O\left(\frac{1}{\xi^2}\right)\right)$

Hence $\int_{\Gamma_R} f(\xi)\,d\xi = 2i\int_0^{2\pi} 1+\frac{1}{Re^{i\theta}}\left(\frac{c}{2}-\frac{1}{2}+z\right)+O\left(\frac{1}{R^2}\right) \; Rie^{i\theta}d\theta \to \underline{-4\pi\left(\frac{c}{2}-\frac{1}{2}+z\right)}$ as $R\to\infty$.

[2]

Putting together we have $-4\pi\left(\frac{c}{2}-\frac{1}{2}+z\right)+2\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}}\frac{2\xi-1}{\xi-z}\,d\xi = 2\pi i\left(\frac{z}{c-z}\right)^{1/2}(2z-1)$

Hence $\frac{1}{\pi}\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}}\frac{2\xi-1}{\xi-z}\,d\xi = i\left(\frac{z}{c-z}\right)^{1/2}(2z-1)+(c-1+2z)$

So $\boxed{W(z) = i(2z-1) + (c-1+2z)\Big/\left(\frac{c-z}{z}\right)^{1/2}.}$

[2]

To remove singularity at $z=0$, clearly need $\underline{c=1}$, in which case $\boxed{W(z) = i(2z-1)+2z^{1/2}(1-z)^{1/2}.}$ [2]

[New example, familiar approach].

If we write $W(z) = \frac{1}{2\pi i}\int_0^1 \frac{f(\xi)}{\xi-x}\,d\xi$ then Plemelj give $W_+ - W_- = f(x)$ [3]

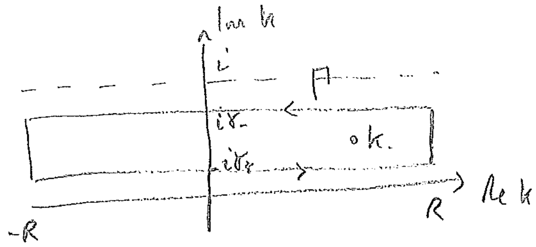
$$W_+ + W_- = \frac{1}{\pi i}\int_0^1 \frac{f(\xi)}{\xi-x}\,d\xi = i(4x-2)$$
$$= 2ig'(x) \text{ from earlier}$$

So we can find $f(x) = W_+ - W_-$ from the relation in (c).

The square-root terms double up to give, $\boxed{f(x) = 4x^{1/2}(1-x)^{1/2}}$. [2]

[New, though idea seen before - requires them to make a connection].

(a)



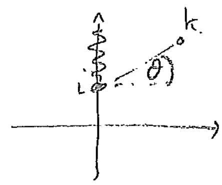
By Cauchy's Integral Formula, $G(k) = \frac{1}{2\pi i}\int_{\Gamma} \frac{G(s)}{s-k}\,ds$ for the contour Γ shown.

$$\int_{\Gamma} = \int_{-R+i\gamma_+}^{R+i\gamma_+} - \int_{-R+i\gamma_-}^{R+i\gamma_-} + \int_{\text{ends}}$$ and since $G(s)\to 0$ as $s\to\infty$, $\int_{\text{ends}} \to 0$ as $R\to\infty$

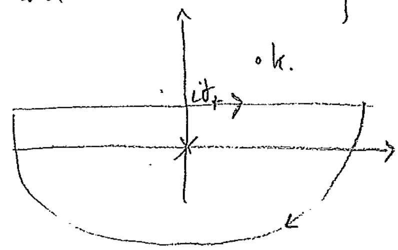
Hence, letting $R\to\infty$, $$G(k) = \underbrace{\frac{1}{2\pi i}\int_{-\infty+i\gamma_+}^{\infty+i\gamma_+} \frac{G(s)}{s-k}\,ds}_{G_+(k)} \underbrace{- \frac{1}{2\pi i}\int_{-\infty+i\gamma_-}^{\infty+i\gamma_-} \frac{G(s)}{s-k}\,ds}_{G_-(k).}$$ [4]

[Bookwork]

If $G(k) = \frac{(k-i)^{1/2}}{k^2}$ with $(k-i)^{1/2} = |k-i|^{1/2}e^{i\theta/2}$, $\theta\in\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right]$,



then $G_+(k)$ can be calculated by closing the contour in LHP and accounting for the pole at $k=0$.



i.e. $$\frac{1}{2\pi i}\int_{-\infty+i\gamma_+}^{\infty+i\gamma_+} \frac{(s-i)^{1/2}}{s^2(s-k)}\,ds = -\operatorname{res}\left(\frac{(s-i)^{1/2}}{s^2(s-k)}, s=0\right)$$

(minus since contour is clockwise)

The residue is $$\frac{d}{ds}\left(\frac{(s-i)^{1/2}}{s-k}\right)\Bigg|_{s=0} = \frac{1}{2(s-i)^{1/2}(s-k)} - \frac{(s-i)^{1/2}}{(s-k)^2}\Bigg|_{s=0} = \frac{1}{2(-i)^{1/2}(-k)} - \frac{(-i)^{1/2}}{k^2} = -\frac{e^{-\frac{i\pi}{4}}}{k^2}\left(1+\tfrac{1}{2}ik\right)$$

so $$G_+(k) = \frac{e^{-\frac{i\pi}{4}}\left(1+\frac{1}{2}ik\right)}{k^2}$$ $\left(\text{hence } G_-(k) = -\frac{(k-i)^{1/2}}{k^2} + \frac{e^{-i\pi/4}\left(1+\frac{1}{2}ik\right)}{k^2}\right)$ [4]

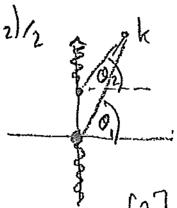
[Standard]

(b) Transforming $u_x = u_{xx} + u_{yy}$ gives $\bar{u}_{yy} - k^2\bar{u} = -ik\bar{u}$ $\left(\bar{u}(k,y) = \int_{-\infty}^{\infty} u(x,y)e^{ikx}\,dx\right)$

$$\Rightarrow \bar{u}_{yy} = (k^2-ik)\bar{u}$$

$$\Rightarrow \boxed{\bar{u}(k,y) = A(k)e^{-(k^2-ik)^{1/2}y}} + \cancel{B(k)e^{+(k^2-ik)^{1/2}y}}$$

where $(k^2-ik)^{1/2} = |k-i|^{1/2}|k|^{1/2}e^{i(\theta_1+\theta_2)/2}$ with $\theta_1\in\left(-\frac{\pi}{2},\frac{3\pi}{2}\right]$, $\theta_2\in\left(-\frac{3\pi}{2},\frac{\pi}{2}\right]$.



Since $\bar{u}(k,y)\to 0$ as $y\to\infty$, $B(k)=0$, [2]

Transforming $u(x,0) = xH(x) + f_-(x)$ gives $\boxed{\bar{u}(k,0) = -\frac{1}{k^2} + \bar{f}_-(k)}$ for $0 < \operatorname{Im} k < \gamma_-$

$$\left[\int_0^\infty xe^{ikx}\,dx = \frac{1}{ik}xe^{ikx} - \frac{1}{(ik)^2}e^{ikx}\Big|_0^\infty = -\frac{1}{k^2} \text{ for } \operatorname{Im}(k) > 0\right]$$

Transforming $\frac{\partial u}{\partial y}(x,0) = g_+(x)$ gives $\boxed{\frac{\partial\bar{u}}{\partial y}(k,0) = \bar{g}_+(k)}$ for $\operatorname{Im} k > \gamma_+$.

The boundary conditions imply $A(k) = -\frac{1}{k^2} + \bar f_-(k)$ and $-(k^2-ik)^{1/2}A(k) = \bar g_+(k)$

Eliminating $A(k) \Rightarrow$ $\boxed{\bar g_+(k) + (k^2-ik)^{1/2}\bar f_-(k) = \frac{(k^2-ik)^{1/2}}{k^2}}$ for $\gamma_+ < \operatorname{Im} k < \gamma_-$

Dividing by $k^{1/2}$, $\frac{\bar g_+(k)}{k^{1/2}} + (k-i)^{1/2}\bar f_-(k) = \frac{(k-i)^{1/2}}{k^2} = G_+(k) - G_-(k)$ (found in (a))

$$\Rightarrow \boxed{\frac{\bar g_+(k)}{k^{1/2}} - G_+(k) = -(k-i)^{1/2}\bar f_-(k) - G_-(k)} \quad \text{for } \gamma_+ < \operatorname{Im} k < \gamma_- \qquad [4].$$

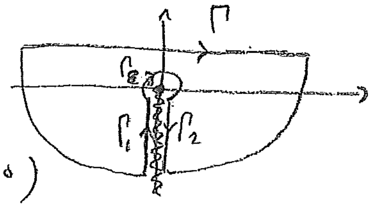
LHS is holomorphic in $\operatorname{Im}(k) > \gamma_+$, RHS is holomorphic in $\operatorname{Im}(k) < \gamma_-$, and since they are equal on the overlapping strip they must be analytic continuations of each other and hence define an entire function, $E(k)$ say.

Given the behaviour as $k\to\infty$, $E(k)$ must $\to 0$ as $k \to \infty$, so by Liouville's theorem $E(k) = 0$.

Hence $\boxed{\bar g_+(k) = k^{1/2}G_+(k) = e^{-i\pi/4}\frac{(1+\frac{1}{2}ik)k^{1/2}}{k^2}}$ [3]

[Variant on example seen in lectures].

$$\frac{\partial u}{\partial y}(x,0) = \frac{1}{2\pi}\int_\Gamma \bar g_+(k)e^{-ikx}\,dk. \qquad [2]$$



Inversion contour Γ as shown, (passing above $k=0$)

For $x>0$, close in LHP as shown, noting integrals around large circular arcs tend to zero by Jordan's lemma

So $\int_\Gamma \to \int_{\Gamma_1} + \int_{\Gamma_\varepsilon} + \int_{\Gamma_2} = \int_{\Gamma_\varepsilon} + 2\int_{\Gamma_2}$

Now, $$\int_{\Gamma_\varepsilon} \underset{k=\varepsilon e^{i\theta}}{=} \underset{\text{since clockwise}}{-}e^{-i\pi/4}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\varepsilon^{1/2}e^{i\theta/2}}{\varepsilon^2 e^{2i\theta}}(1+O(\varepsilon))(\varepsilon i e^{i\theta}d\theta) = -\frac{e^{-i\pi/4}}{\varepsilon^{1/2}}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} ie^{-\frac{i\theta}{2}}d\theta + O(\varepsilon^{1/2})$$

$$= -\frac{e^{-i\pi/4}}{\varepsilon^{1/2}}\left[-2e^{-i\theta/2}\right]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} + O(\varepsilon^{1/2}) = \boxed{-\frac{4}{\varepsilon^{1/2}} + O(\varepsilon^{1/2})} \quad [2$$

Also, $$\int_{\Gamma_2} \underset{k=-it}{=} e^{-i\pi/4}\int_\varepsilon^\infty \frac{(1+\frac{1}{2}t)}{-t^2}e^{-\frac{i\pi}{4}}t^{1/2}e^{-tx}(-i\,dt) = \int_\varepsilon^\infty \left(t^{-3/2} + \tfrac{1}{2}t^{-1/2}\right)e^{-tx}\,dt.$$

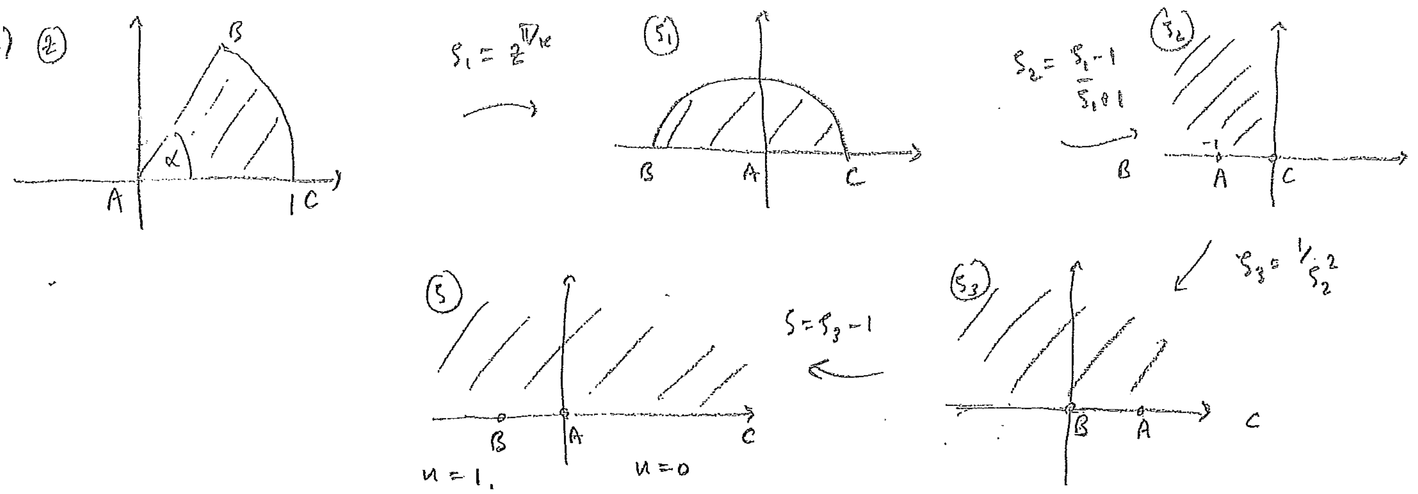
Note $$\int_\varepsilon^\infty t^{-3/2}e^{-tx}\,dt = \underbrace{-2t^{-1/2}e^{-tx}\Big|_\varepsilon^\infty}_{\frac{2}{\varepsilon^{1/2}}+O(\varepsilon)} - 2x\int_\varepsilon^\infty t^{-1/2}e^{-tx}\,dt \quad \Rightarrow \quad = \boxed{\frac{2}{\varepsilon^{1/2}} + \left(\tfrac{1}{2}-2x\right)\int_\varepsilon^\infty t^{-1/2}e^{-tx}\,dt + O(\varepsilon^{1/2})}$$

[2]

Hence, $$\frac{\partial u}{\partial y}(x,0) = \frac{1}{2\pi}\left[-\frac{4}{\varepsilon^{1/2}} + \frac{4}{\varepsilon^{1/2}} + (1-4x)\int_\varepsilon^\infty t^{-1/2}e^{-tx}\,dt + O(\varepsilon^{1/2})\right] \to \frac{1}{\pi}\left(\tfrac{1}{2}-2x\right)\int_0^\infty t^{-1/2}e^{-tx}\,dt \quad \text{as } \varepsilon\to 0$$

But $$\int_0^\infty t^{-1/2}e^{-tx}\,dt \underset{tx=s^2}{=} \int_0^\infty \frac{x^{1/2}}{s}e^{-s^2}\frac{2s\,ds}{x} = \frac{2}{x^{1/2}}\int_0^\infty e^{-s^2}\,ds = \frac{\sqrt{\pi}}{x^{1/2}},$$ so this is $\boxed{\frac{1}{\sqrt{\pi}}\left(\frac{1}{2x^{1/2}} - 2x^{1/2}\right)}$ [2]

1. (a)



Putting together, the required mapping is

$$\zeta = \left(\frac{\zeta_1+1}{\zeta_1-1}\right)^2 - 1 = \frac{4\zeta_1}{(\zeta_1-1)^2} = \frac{4z^{\pi/\alpha}}{(z^{\pi/\alpha}-1)^2}.$$ [5]

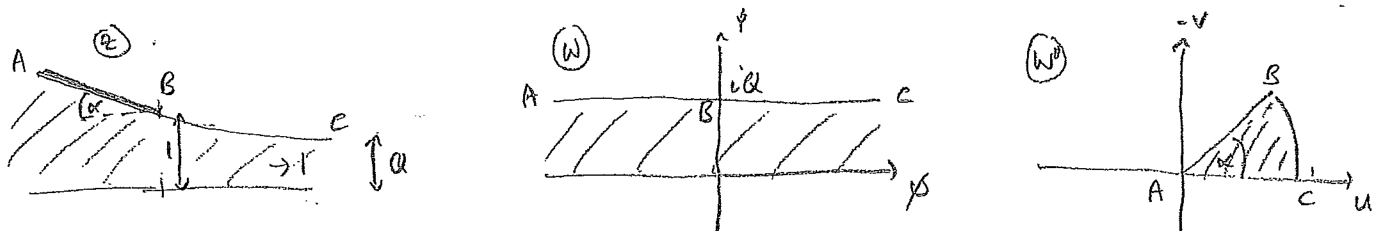
In the ζ-plane, we must find W with $u=0$ on +ve real axis, $u=1$ on −ve real axis, and $\nabla^2 u = 0$.

By inspection, or using polar coordinates, the solution is $u = \frac{\theta}{\pi} = \frac{\operatorname{Arg}(\zeta)}{\pi} = \operatorname{Re}\underbrace{\frac{-i\log(\zeta)}{\pi}}_{W(\zeta)\,=\,u+iv}$

Hence $u = \frac{1}{\pi}\operatorname{Arg}\left(\frac{4z^{\pi/\alpha}}{(z^{\pi/\alpha}-1)^2}\right).$ [3]

[Standard example].

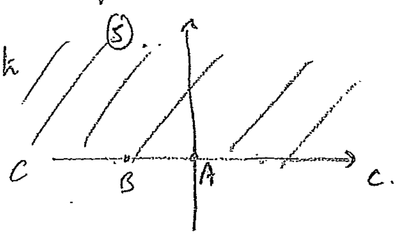
(b) (i)



Fluid is stagnant at ∞, so A at origin in hodograph plane. Free surface BC is part of unit circle with $u - iv = e^{+i\alpha}$ at B and $u=1$ at C.

ABC has $\psi = a$ since it is a streamline; the symmetry axis is also a streamline with $\psi = 0$. [4]

The map $\zeta = e^{\pi W/a}$ maps W domain to



The mapping from (a), $\zeta = \frac{4(w')^{\pi/\alpha}}{((w')^{\pi/\alpha}-1)^2}$, maps the w' domain to this same UHP with the same mapping of the points A, B, C.

Hence $\boxed{e^{\frac{\pi W}{a}} = \frac{4(w')^{\pi/\alpha}}{((w')^{\pi/\alpha}-1)^2}}$ [3]

[Bookwork / Standard]

ii) If $w' = e^{-i\theta}$ $\quad$ ($\theta = -\alpha$ at B, and $\theta = 0$ at C).

Then $$e^{\frac{\pi w}{Q}} = \frac{4e^{-i\theta\pi/\alpha}}{\left(e^{-i\theta\pi/\alpha}-1\right)^2} = \frac{4}{\left(e^{-\frac{i\theta\pi}{2\alpha}} - e^{\frac{i\theta\pi}{2\alpha}}\right)^2} = \frac{4}{\left(2i\sin\left(\frac{\pi\theta}{2\alpha}\right)\right)^2} = -\frac{1}{\sin^2\frac{\pi\theta}{2\alpha}}.$$ [2]

Hence $$e^{\frac{\pi w}{Q}}\frac{\pi}{Q}\frac{dw}{d\theta} = \frac{2\pi}{2\alpha}\frac{1}{\sin^3\frac{\pi\theta}{2\alpha}}\cos\frac{\pi\theta}{2\alpha} = -\frac{\pi}{\alpha}e^{\frac{\pi w}{Q}}\cot\frac{\pi\theta}{2\alpha} \Rightarrow \boxed{\frac{dw}{d\theta} = -\frac{Q}{\alpha}\cot\frac{\pi\theta}{2\alpha}.}$$

But $$\frac{dz}{d\theta} = \frac{1}{w'}\frac{dw}{d\theta} = e^{i\theta}\frac{dw}{d\theta} = -\frac{Q}{\alpha}\cot\frac{\pi\theta}{2\alpha}\left(\cos\theta + i\sin\theta\right):$$

Hence $\boxed{\frac{dy}{d\theta} = -\frac{Q}{\alpha}\cot\frac{\pi\theta}{2\alpha}\sin\theta}$ [2]

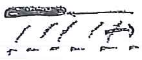
Integrating from B to C $$\int_{-\alpha}^{0}\frac{dy}{d\theta}\,d\theta = \int_1^Q dy = Q-1$$

so $$1-Q = -\int_{-\alpha}^{0}\frac{dy}{d\theta}\,d\theta = \frac{Q}{\alpha}\int_{-\alpha}^{0}\cot\frac{\pi\theta}{2\alpha}\sin\theta\,d\theta = Q\underbrace{\int_0^1 \cot\frac{\pi}{2}\varphi\,\sin\alpha\varphi\,d\varphi}_{J}.$$
$\theta = -\alpha\varphi$
$d\theta = -\alpha\,d\varphi$

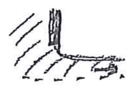
$\Rightarrow$ $1-Q = JQ$

$\Rightarrow$ $\boxed{Q = \frac{1}{1+J}}$ where $\boxed{J = \int_0^1 \cot\frac{\pi}{2}\varphi\,\sin\alpha\varphi\,d\varphi.}$ [3]

For $\alpha = 0$, $J = 0$ $\Rightarrow$ $\boxed{Q = 1}$



$\alpha = \frac{\pi}{2}$, $J = \int_0^1 \cos\frac{\pi}{2}\varphi\,d\varphi = \frac{2}{\pi}$ $\Rightarrow$ $\boxed{Q = \frac{\pi}{\pi+2}}$



[3]

$\alpha = \pi$, $J = \int_0^1 \cot\frac{\pi}{2}\varphi\,\sin\pi\varphi\,d\varphi = \int_0^1 2\cos^2\frac{\pi}{2}\varphi\,d\varphi = \int_0^1 \cos\pi\varphi + 1\,d\varphi = 1$ $\Rightarrow$ $\boxed{Q = \frac{1}{2}.}$

[Standard technique, new example]

(a) Let $\tilde{\omega}(z)$ be holomorphic in $\mathbb{C}\setminus[0,c]$, and satisfying $\boxed{\tilde{\omega}_+(x)+\tilde{\omega}_-(x)=0}$ on $(0,c)$. [2]

- Then $W(z)=\dfrac{w(z)}{\tilde{\omega}(z)}$ is holomorphic in $\mathbb{C}\setminus[0,c]$ with $W_+(z)-W_-(z)=\dfrac{w_+(z)+w_-(z)}{\tilde{\omega}_+(z)}=\dfrac{2ig'(x)}{\tilde{\omega}_+(x)}$.

- Write $W(z)=\dfrac{1}{2\pi i}\displaystyle\int_0^c \frac{F(\xi)}{\xi-z}\,d\xi$, then by the Plemelj formulae, $W_+(x)-W_-(x)=F(x)$, so we can read off that $\boxed{F(x)=\dfrac{2ig'(x)}{\tilde{\omega}_+(x)}}$. Hence $W(z)=\dfrac{1}{\pi}\displaystyle\int_0^c \frac{g'(\xi)}{\tilde{\omega}_+(\xi)(\xi-z)}\,d\xi$ is a solution. [2]

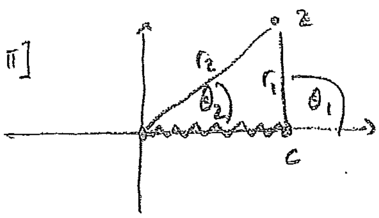
- The difference between this and any other solution must be holomorphic on $\mathbb{C}\setminus[0,c]$ and continuous across $(0,c)$, hence holomorphic on all $\mathbb{C}$ except possibly at 0 and c,

- So we can write $\boxed{w(z)=\dfrac{\tilde{\omega}(z)}{\pi}\displaystyle\int_0^c \frac{g'(\xi)}{\tilde{\omega}_+(\xi)(\xi-z)}\,d\xi+\tilde{\omega}(z)H(z)}$ [1]

where $H(z)$ is holomorphic on $\mathbb{C}\setminus\{0,c\}$.

[Bookwork]

(b) Let $\tilde{\omega}(z)=\left(\dfrac{c-z}{z}\right)^{1/2}$ which has the desired property $\tilde{\omega}_+(x)+\tilde{\omega}_-(x)=0$ if we take the branch cut on $[0,c]$, i.e. $\tilde{\omega}(z)=\left(\dfrac{r_1}{r_2}\right)^{1/2}e^{i(\theta_1-\theta_2-\pi)/2}$ with $\theta_1,\theta_2\in(-\pi,\pi]$



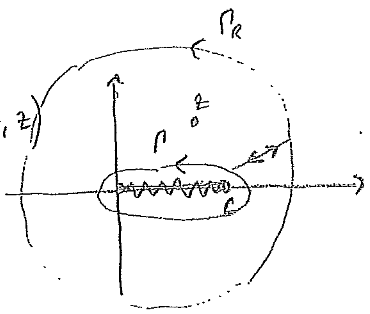
$\left(\text{so } \underset{\theta_1=\pi,\ \theta_2=0.}{\tilde{\omega}_+(x)} = \left(\dfrac{c-x}{x}\right)^{1/2}\right)$

Then from (a), $w(z)=\left(\dfrac{c-z}{z}\right)^{1/2}\left[\dfrac{1}{\pi}\displaystyle\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}}\frac{g'(\xi)}{\xi-z}\,d\xi\ +\cancel{H(z)}\right]$ [3]

But given singularities of $w(z)$ mean we need $H(z)$ finite at $z=0$ and at $z=c$, and satisfying $H=O(\frac{1}{z})$ as $z\to\infty$, so $H(z)$ is entire, and bounded. Hence H is constant by Liouville's theorem, and since $H\to 0$ as $z\to\infty$, $\underline{H=0}$. [2]

[Standard]

(c) Consider $f(s)=\left(\dfrac{\xi}{c-\xi}\right)^{1/2}\dfrac{2\xi-1}{\xi-z}$ and $\left(\displaystyle\int_{\Gamma_R}-\int_{\Gamma}\right)f(s)\,d\xi=2\pi i.\operatorname{res}(f,z)$ (C.R.T). [2]



- $\displaystyle\int_\Gamma f(s)\,ds=\underset{\text{bottom}}{\int_0^c}-\underset{\text{top}}{\int_0^c}=-2\underset{\text{top}}{\int_0^c}f_+(\xi)\,d\xi=\underline{-2\int_0^c\frac{\xi^{1/2}}{(c-\xi)^{1/2}}\frac{2\xi-1}{\xi-z}\,d\xi}.$ [2]

- $\operatorname{res}(f,z)=\underline{\left(\dfrac{z}{c-z}\right)^{1/2}(2z-1)}$

As $\xi \to \infty$, $\left(\frac{c-\xi}{\xi}\right)^{1/2} \sim -i\left(\frac{\xi-c}{\xi}\right)^{1/2} = -i\left(1 - \frac{c}{2\xi} + \dots\right)$,

so $f(\xi) \sim i\left(1 + \frac{c}{2\xi} + \dots\right)\left(\frac{2\xi}{\xi}\right)\left(1 - \frac{1}{2\xi}\right)\left(1 + \frac{z}{\xi} + \dots\right) = 2i\left(1 + \frac{1}{\xi}\left(\frac{c}{2} - \frac{1}{2} + z\right) + O\left(\frac{1}{\xi^2}\right)\right)$

Hence $\int_{\Gamma_R} f(\xi)\,d\xi = 2i \int_0^{2\pi} 1 + \frac{1}{Re^{i\theta}}\left(\frac{c}{2} - \frac{1}{2} + z\right) + O\left(\frac{1}{R^2}\right) \; Rie^{i\theta}\,d\theta \to \underline{-4\pi\left(\frac{c}{2} - \frac{1}{2} + z\right)}$ as $R \to \infty$.

[2]

Putting together we have $-4\pi\left(\frac{c}{2} - \frac{1}{2} + z\right) + 2\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}} \frac{2\xi - 1}{\xi - z}\,d\xi = 2\pi i\left(\frac{z}{c-z}\right)^{1/2}(2z-1)$

Hence $\frac{1}{\pi}\int_0^c \frac{\xi^{1/2}}{(c-\xi)^{1/2}} \frac{2\xi-1}{\xi - z}\,d\xi = i\left(\frac{z}{c-z}\right)^{1/2}(2z-1) + (c - 1 + 2z)$

So $\boxed{w(z) = i(2z-1) + (c-1+2z)\left(\frac{c-z}{z}\right)^{1/2}}$.

[2]

To remove singularity at $z=0$, clearly need $\underline{c=1}$, in which case $\boxed{w(z) = i(2z-1) + 2z^{1/2}(1-z)^{1/2}}$. [2]

[New example, familiar approach].

If we write $w(z) = \frac{1}{2\pi i}\int_0^1 \frac{f(\xi)}{\xi - x}\,d\xi$ then Plemelj gives $w_+ - w_- = f(x)$ [3]

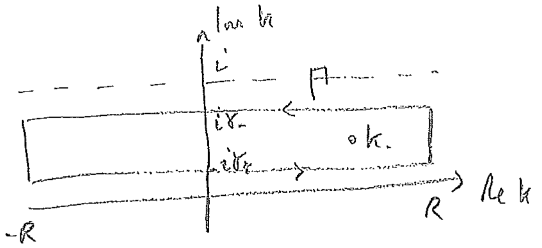
$$w_+ + w_- = \frac{1}{\pi i}⨍_0^1 \frac{f(\xi)}{\xi - x}\,d\xi = i(4x-2)$$
$$= 2ig'(x) \text{ from earlier}$$

So we can find $f(x) = w_+ - w_-$ from the solution in (c).

The square-root terms double up to give $\boxed{f(x) = 4x^{1/2}(1-x)^{1/2}}$. [2]

[New, though idea seen before - requires them to make a connection].

(a)



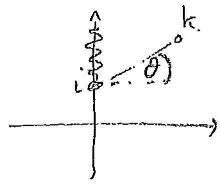
By Cauchy's Integral Formula, $G(k) = \frac{1}{2\pi i}\int_\Gamma \frac{G(s)}{s-k}\,ds$ for the contour Γ shown.

$$\int_\Gamma = \int_{-R+i\gamma_+}^{R+i\gamma_+} - \int_{-R+i\gamma_-}^{R+i\gamma_-} + \int_{\text{ends}}$$ and since $G(s) \to 0$ as $s \to \infty$, $\int_{\text{ends}} \to 0$ as $R \to \infty$

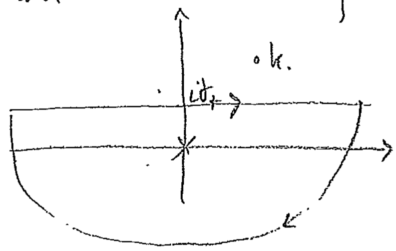
Hence, letting $R \to \infty$, $$G(k) = \underbrace{\frac{1}{2\pi i}\int_{-\infty+i\gamma_+}^{\infty+i\gamma_+} \frac{G(s)}{s-k}\,ds}_{G_+(k)} \underbrace{- \frac{1}{2\pi i}\int_{-\infty+i\gamma_-}^{\infty+i\gamma_-} \frac{G(s)}{s-k}\,ds}_{G_-(k).}$$ [4]

[Bookwork]

If $G(k) = \frac{(k-i)^{1/2}}{k^2}$ with $(k-i)^{1/2} = |k-i|^{1/2} e^{i\theta/2}$, $\theta \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right]$,



then $G_+(k)$ can be calculated by closing the contour in Lhp and accounting for the pole at $k=0$.



i.e. $$\frac{1}{2\pi i}\int_{-\infty+i\gamma_+}^{\infty+i\gamma_+} \frac{(s-i)^{1/2}}{s^2(s-k)}\,ds = -\operatorname{res}\left(\frac{(s-i)^{1/2}}{s^2(s-k)}, s=0\right)$$

(minus since contour is clockwise)

Residue is $$\frac{d}{ds}\left(\frac{(s-i)^{1/2}}{s-k}\right)\Bigg|_{s=0} = \frac{1}{2(s-i)^{1/2}(s-k)} - \frac{(s-i)^{1/2}}{(s-k)^2}\Bigg|_{s=0} = \frac{1}{2(-i)^{1/2}(-k)} - \frac{(-i)^{1/2}}{k^2} = -\frac{e^{-\frac{i\pi}{4}}}{k^2}\left(1+\tfrac{1}{2}ik\right)$$

so $$G_+(k) = \frac{e^{-\frac{i\pi}{4}}\left(1+\frac{1}{2}ik\right)}{k^2}$$ $\left(\text{\& hence } G_-(k) = -\frac{(k-i)^{1/2}}{k^2} + \frac{e^{-i\pi/4}\left(1+\frac{1}{2}ik\right)}{k^2}\right)$ [4]

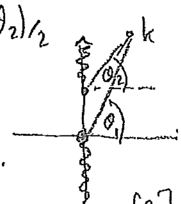
[Standard]

(b) Transforming $u_x = u_{xx} + u_{yy}$ gives $\bar{u}_{yy} - k^2\bar{u} = -ik\bar{u}$ $\left(\bar{u}(k,y) = \int_{-\infty}^{\infty} u(x,y)e^{ikx}\,dx\right)$

$\Rightarrow \bar{u}_{yy} = (k^2-ik)\bar{u}$

$\Rightarrow \boxed{\bar{u}(k,y) = A(k)e^{-(k^2-ik)^{1/2}y}} + \cancel{B(k)e^{+(k^2-ik)^{1/2}y}}$

where $(k^2-ik)^{1/2} = |k-i|^{1/2}|k|^{1/2}e^{i(\theta_1+\theta_2)/2}$ with $\theta_1 \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\theta_2 \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right]$.



Since $\bar{u}(k,y) \to 0$ as $y \to \infty$, $B(k) = 0$. [2]

Transforming $u(x,0) = xH(x) + f_-(x)$ gives $\boxed{\bar{u}(k,0) = -\frac{1}{k^2} + \bar{f}_-(k).}$ for $0 < \operatorname{Im} k < \gamma_-$

$\left[\int_0^\infty xe^{ikx}\,dx = \frac{1}{ik}xe^{ikx} - \frac{1}{(ik)^2}e^{ikx}\Big|_0^\infty = -\frac{1}{k^2} \text{ for } \operatorname{Im}(k) > 0\right]$

Transforming $\frac{\partial u}{\partial y}(x,0) = g_+(x)$ gives $\boxed{\frac{\partial \bar{u}}{\partial y}(k,0) = \bar{g}_+(k)}$ for $\operatorname{Im} k > \gamma_+$.

The boundary conditions imply $A(k) = -\frac{1}{k^2} + \bar f_-(k)$ and $-(k^2-ik)^{1/2}A(k) = \bar g_+(k)$

Eliminating $A(k) \Rightarrow$ $\boxed{\bar g_+(k) + (k^2-ik)^{1/2}\bar f_-(k) = \frac{(k^2-ik)^{1/2}}{k^2}}$ for $\gamma_+ < \mathrm{Im}\, k < \gamma_-$

Dividing by $k^{1/2}$, $\frac{\bar g_+(k)}{k^{1/2}} + (k-i)^{1/2}\bar f_-(k) = \frac{(k-i)^{1/2}}{k^2} = G_+(k) - G_-(k)$ (found in (a))

$$\Rightarrow \boxed{\frac{\bar g_+(k)}{k^{1/2}} - G_+(k) = -(k-i)^{1/2}\bar f_-(k) - G_-(k).} \quad \text{for } \gamma_+ < \mathrm{Im}\, k < \gamma_- \qquad [4].$$

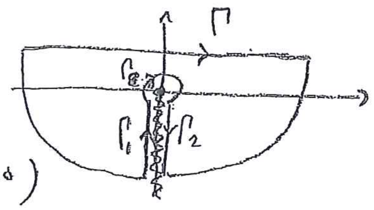
LHS is holomorphic in $\mathrm{Im}(k) > \gamma_+$; RHS is holomorphic in $\mathrm{Im}(k) < \gamma_-$, and since they are equal on the overlapping strip they must be analytic continuations of each other and hence define an entire function, $E(k)$ say.

Given the behaviour as $k \to \infty$, $E(k)$ must $\to 0$ as $k \to \infty$, so by Liouville's theorem $E(k) = 0$.

Hence $\boxed{\bar g_+(k) = k^{1/2}G_+(k) = e^{-i\pi/4}\frac{(1+\frac{1}{2}ik)k^{1/2}}{k^2}}$ [3]

[Variant on example seen in lectures].

$$\frac{\partial u}{\partial y}(x,0) = \frac{1}{2\pi}\int_\Gamma \bar g_+(k)e^{-ikx}\,dk.$$



Inversion contour Γ as shown, (passing above $k=0$) [2]

For $x>0$, close in LHP as shown, noting integrals around large circular arcs tend to zero by Jordan's lemma

So $\int_\Gamma \to \int_{\Gamma_1} + \int_{\Gamma_\varepsilon} + \int_{\Gamma_2} = \int_{\Gamma_\varepsilon} + 2\int_{\Gamma_2}$

Now, $\int_{\Gamma_\varepsilon} \underset{k=\varepsilon e^{i\theta}}{=} \underset{\text{since clockwise}}{-}e^{-i\pi/4}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\varepsilon^{1/2}e^{i\theta/2}}{\varepsilon^2 e^{2i\theta}}(1+O(\varepsilon))(\varepsilon i e^{i\theta}\,d\theta) = -\frac{e^{-i\pi/4}}{\varepsilon^{1/2}}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} ie^{-\frac{i\theta}{2}}\,d\theta + O(\varepsilon^{1/2})$

$= -\frac{e^{-i\pi/4}}{\varepsilon^{1/2}}\left[-2e^{-i\theta/2}\right]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} + O(\varepsilon^{1/2}) = \boxed{-\frac{4}{\varepsilon^{1/2}} + O(\varepsilon^{1/2})}$ [2]

Also, $\int_{\Gamma_2} \underset{k=-it}{=} e^{-i\pi/4}\int_\varepsilon^\infty \frac{(1+\frac{1}{2}t)e^{-\frac{i\pi}{4}}t^{1/2}e^{-tx}}{-t^2}(-i\,dt) = \int_\varepsilon^\infty \left(t^{-3/2} + \frac{1}{2}t^{-1/2}\right)e^{-tx}\,dt.$

Note $\int_\varepsilon^\infty t^{-3/2}e^{-tx}\,dt = \underbrace{-2t^{-1/2}e^{-tx}\Big|_\varepsilon^\infty}_{\frac{2}{\varepsilon^{1/2}}+O(\varepsilon)} - 2x\int_\varepsilon^\infty t^{-1/2}e^{-tx}\,dt \Rightarrow$ $= \boxed{\frac{2}{\varepsilon^{1/2}} + \left(\frac{1}{2}-2x\right)\int_\varepsilon^\infty t^{-1/2}e^{-tx}\,dt + O(\varepsilon^{1/2})}$

[2]

Hence, $\frac{\partial u}{\partial y}(x,0) = \frac{1}{2\pi}\left[-\frac{4}{\varepsilon^{1/2}} + \frac{4}{\varepsilon^{1/2}} + (1-4x)\int_\varepsilon^\infty t^{-1/2}e^{-tx}\,dt + O(\varepsilon^{1/2})\right] \to \frac{1}{\pi}\left(\frac{1}{2}-2x\right)\int_0^\infty t^{-1/2}e^{-tx}\,dt$ as $\varepsilon \to 0$

But $\int_0^\infty t^{-1/2}e^{-tx}\,dt \underset{tx=s^2}{=} \int_0^\infty \frac{x^{1/2}}{s}e^{-s^2}\frac{2s\,ds}{x} = \frac{2}{x^{1/2}}\int_0^\infty e^{-s^2}\,ds = \frac{\sqrt{\pi}}{x^{1/2}}$, so this is $\boxed{\frac{1}{\sqrt{\pi}}\left(\frac{1}{2x^{1/2}} - 2x^{1/2}\right)}$ [2]