A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematics and Computer Science Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical Sciences: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6

Applied Complex Variables

TRINITY TERM 2024

Friday 31 May, 9:30am to 11:15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. Consider a two-dimensional porous-medium flow driven by a source of constant strength Q at the origin, in which the fluid occupies the region $D(t)$. The fluid velocity $\mathbf{u} = \nabla\phi$ satisfies $\nabla \cdot \mathbf{u} = 0$ in $D \setminus \{(0,0)\}$, with $\phi = 0$ and $\mathbf{u} \cdot \mathbf{n} = v_n$ on the free boundary ∂D, where $\mathbf{n}$ is the normal and v_n is the normal velocity of the boundary.

   (a) [2 marks] Show that ϕ satisfies

   $$\nabla^2\phi = 0 \qquad \text{in } D,$$

   with

   $$\phi = 0, \quad \frac{\partial\phi}{\partial t} + |\nabla\phi|^2 = 0 \qquad \text{on } \partial D,$$

   and

   $$\phi \sim \frac{Q}{2\pi}\log|z|$$

   as $z \to 0$, where $z = x + \mathrm{i}y$.

   (b) [8 marks] Let the complex potential be $w = \phi + \mathrm{i}\psi$, where ψ is the streamfunction. Suppose $D(t)$ is the image of the unit disc $|\zeta| < 1$ under the conformal map $\zeta \mapsto z = F(\zeta, t)$ satisfying $F(0,t) = 0$ and $\partial F/\partial\zeta(0,t) > 0$, and let $W(\zeta, t) = w(F(\zeta,t),t)$. Show that

   $$\mathrm{Re}\left[\frac{\partial W}{\partial t} - \frac{\partial W}{\partial \zeta}\frac{\partial F}{\partial t}\bigg/\frac{\partial F}{\partial \zeta}\right] + \left|\frac{\partial W}{\partial \zeta}\right|^2 \bigg/ \left|\frac{\partial F}{\partial \zeta}\right|^2 = 0,$$

   on $|\zeta| = 1$. By identifying the appropriate function $W(\zeta,t)$ deduce the Polubarinova-Galin equation

   $$\mathrm{Re}\left[\zeta\frac{\partial F}{\partial \zeta}\overline{\frac{\partial F}{\partial t}}\right] = \frac{Q}{2\pi} \qquad \text{on } |\zeta| = 1.$$

   (c) [6 marks] Suppose $F(\zeta, 0) = 6\zeta + \zeta^2 + \zeta^3/2$. Show that $F(\zeta,t) = a_1(t)\zeta + a_2(t)\zeta^2 + a_3(t)\zeta^3$ where

   $$a_1^2 + 2a_2^2 + 3a_3^2 = \frac{Qt}{\pi} + \frac{155}{4}, \qquad a_1^2a_2 + 3a_1a_2a_3 = 45, \qquad a_1^3a_3 = 108.$$

   (d) [9 marks] Deduce that when Q is negative (corresponding to a sink), F ceases to be conformal when $t = -93\pi/8Q$. What does the loss of conformality correspond to in terms of the free boundary?

   What fraction of fluid has been removed at this time?

   [You may assume that the loss of conformality occurs first at a complex value of ζ. You may use the identity

   $$\iint_D z^n \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\mathrm{i}}\oint_{\partial D} z^n\bar{z}\,\mathrm{d}z$$

   without proof.]

2. (a) [14 marks] (i) State the *Plemelj formulae*, making sure to define any notation that you use.

(ii) Clearly define a branch of the multifunction

$$W(z) = \frac{(z-1)^{1/2}z^{1/2}}{z^2+1}$$

that is holomorphic in $\mathbb{C} \setminus (\{x + \mathrm{i}y : y = 0, 0 \leqslant x \leqslant 1\} \cup \{-\mathrm{i}, \mathrm{i}\})$ such that $W(z)$ is real and positive when $z \in \{x + \mathrm{i}y : y = 0, x > 1\}$.

(iii) Determine the constants a and b such that

$$w(z) = W(z) + \frac{a}{z-\mathrm{i}} + \frac{b}{z+\mathrm{i}}$$

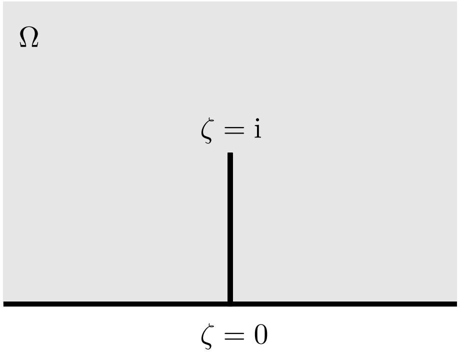
has only removeable singularities at $z = \pm\mathrm{i}$.

(iv) By applying the Plemelj formulae to $w(z)$ show that

$$⨍_0^1 \frac{(1-\xi)^{1/2}\xi^{1/2}}{1+\xi^2}\frac{\mathrm{d}\xi}{\xi - x} = \frac{2^{1/4}\pi(\sin(\pi/8) - \cos(\pi/8)x)}{(x^2+1)} \qquad \text{for } 0 < x < 1.$$

(b) [11 marks] (i) State the *Schwarz-Christoffel* formula for mapping the upper-half z-plane to a polygon in the ζ-plane.

(ii) Use the Schwarz-Christoffel formula to show that $\zeta = \sqrt{z^2 - 1}$ maps the upper half z-plane to the slit ζ-plane Ω shown below.



(iii) A point charge of strength Q sits at position $\zeta = 2\mathrm{i}$ in this slit ζ-plane. The electric potential satisfies

$$\nabla^2\phi = 0 \text{ in } \Omega \setminus \{(0,2)\}, \qquad \phi = 0 \text{ on } \partial\Omega,$$

with

$$\phi \sim \frac{Q}{2\pi}\log\sqrt{\xi^2 + (\eta-2)^2} \text{ as } (\xi,\eta) \to (0,2),$$

where $\zeta = \xi + \mathrm{i}\eta$ and $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. Show that

$$\phi = \frac{Q}{2\pi}\log\left|\frac{\sqrt{\zeta^2+1} - \sqrt{3}\mathrm{i}}{\sqrt{\zeta^2+1} + \sqrt{3}\mathrm{i}}\right|.$$

3. Let

$$K(x) = \mathrm{e}^{-|x|}, \qquad g_+(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0. \end{cases}$$

(a) [5 marks] Compute the Fourier transforms

$$\bar{K}(k) = \int_{-\infty}^{\infty} K(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x, \qquad \bar{g}_+(k) = \int_{-\infty}^{\infty} g_+(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x,$$

specifying for which values of k each is defined, and into what regions of the complex k-plane they can be analytically continued.

(b) [4 marks] Consider the integro-differential equation

$$\frac{2}{3}f'(x) - f(x) + \frac{1}{2}\int_0^{\infty} K(x-t)f(t)\,\mathrm{d}t = 1, \qquad x > 0,$$

with the boundary condition $f(0) = 0$. Give appropriate definitions of $f_+(x)$ and $h_-(x)$, such that the equation can be extended to hold for $x \in \mathbb{R} \setminus \{0\}$ as

$$\frac{2}{3}f_+'(x) - f_+(x) + \frac{1}{2}\int_{-\infty}^{\infty} K(x-t)f_+(t)\,\mathrm{d}t = g_+(x) + h_-(x).$$

Assuming that $f(x) = O(\mathrm{e}^{\gamma x})$ as $x \to \infty$ for some real $\gamma < 1$, state the regions of the complex k-plane in which the Fourier transforms $\bar{f}_+(k)$ and $\bar{h}_-(k)$ are defined and holomorphic.

(c) [10 marks] Show that

$$-\frac{\mathrm{i}k(2k+\mathrm{i})}{3(k+\mathrm{i})}\bar{f}_+ - \frac{\mathrm{i}}{2k} = \frac{\mathrm{i}}{2(k-2\mathrm{i})} + \frac{(k-\mathrm{i})\bar{h}_-}{k-2\mathrm{i}},$$

for k in a strip $\alpha < \mathrm{Im}(k) < \beta$, specifying suitable values of α and β. Explain clearly why the left- and right-hand sides of this equation must be zero.

(d) [6 marks] Solve for $\bar{f}_+(k)$ and describe a suitable inversion contour in the complex plane. Hence show that

$$f(x) = ax + b + c\mathrm{e}^{-x/2},$$

for some constants a, b and c which you should evaluate.

1. (a) [2 marks]

$$\mathbf{u} = \nabla\phi, \quad \nabla\cdot\mathbf{u} = 0 \qquad \Rightarrow \qquad \nabla^2\phi = 0. \quad \boxed{1}$$

The singularity corresponding to a source/sink is

$$\phi \sim \frac{Q}{2\pi}\log|z| \quad \text{as } z \to 0.$$

We are given $\phi = 0$ on ∂D. Differentiating this with respect to time gives

$$0 = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = \frac{\partial\phi}{\partial t} + \nabla\phi\cdot\nabla\phi = \frac{\partial\phi}{\partial t} + |\nabla\phi|^2, \quad \boxed{1}$$

where we have used the fact that the velocity of the boundary is the velocity of the fluid.

□ **Bookwork**

(b) [8 marks] We have w holomorphic in $D \setminus \{0\}$,

$$w \sim \frac{Q}{2\pi}\log z \quad \text{as } z \to 0 \tag{1}$$

and

$$\mathrm{Re}(w) = 0, \qquad \mathrm{Re}\left(\frac{\partial w}{\partial t}\right) + \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = 0 \qquad \text{on } \partial D. \quad \boxed{1}$$

With $W(\zeta,t) = w(F(\zeta,t),t)$,

$$\frac{\partial W}{\partial\zeta} = \frac{\partial w}{\partial z}\frac{\partial F}{\partial\zeta}, \qquad \frac{\partial W}{\partial t} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial F}{\partial t},$$

so that

$$\frac{\partial w}{\partial t} = \frac{\partial W}{\partial t} - \frac{\partial W}{\partial\zeta}\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial\zeta} \quad \boxed{1}$$

Thus the second boundary condition is

$$\mathrm{Re}\left[\frac{\partial W}{\partial t} - \frac{\partial W}{\partial\zeta}\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial\zeta}\right] + \left|\frac{\partial W}{\partial\zeta}\right|^2\Big/\left|\frac{\partial F}{\partial\zeta}\right|^2 = 0 \qquad \text{on } |\zeta| = 1. \quad \boxed{1} \tag{2}$$

The remaining boundary condition is

$$\mathrm{Re}(W) = 0 \text{ on } |\zeta| = 1, \quad \boxed{1} \tag{3}$$

while because F maps the origin to the origin and $\partial F/\partial\zeta(0,t) \neq 0$, (1) implies

$$W \sim \frac{Q}{2\pi}\log\zeta \qquad \text{as } \zeta \to 0. \quad \boxed{1} \tag{4}$$

Thus we have to identify W holomorphic in the punctured unit disc, satisfying (1) and (4). The solution is simply

$$W = \frac{Q}{2\pi}\log\zeta. \quad \boxed{1}$$

Thus

$$\frac{\partial W}{\partial t} = 0, \qquad \frac{\partial W}{\partial\zeta} = \frac{Q}{2\pi\zeta}. \quad \boxed{1}$$

Substitituting into (2) gives

$$\mathrm{Re}\left[-\frac{Q}{2\pi\zeta}\frac{\partial F}{\partial t}\Big/\frac{\partial F}{\partial\zeta}\right] + \frac{Q^2}{4\pi^2|\zeta|^2}\Big/\left|\frac{\partial F}{\partial\zeta}\right|^2 = 0 \qquad \text{on } |\zeta| = 1. \quad \boxed{1}$$

Multiplying by $2\pi|\zeta|^2\,|\partial F/\partial\zeta|^2\,/Q$ and conjugating gives

$$\mathrm{Re}\left[\zeta\frac{\overline{\partial F}}{\partial t}\frac{\partial F}{\partial \zeta}\right]=\frac{Q}{2\pi}\qquad \text{on } |\zeta|=1,$$

as required.

□ **Bookwork**

(c) [6 marks] With the ansatz $F(\zeta,t)=a_1(t)\zeta+a_2(t)\zeta^2+a_3(t)\zeta^3$ with a_i real the P-G equation is

$$\mathrm{Re}\left[\zeta\left(\dot a_1(t)\overline{\zeta}+\dot a_2(t)\overline{\zeta}^2+\dot a_3(t)\overline{\zeta}^3\right)\left(a_1(t)+2a_2(t)\zeta+3a_3(t)\zeta^2\right)\right]=\frac{Q}{2\pi},\quad \boxed{1}$$

on $|\zeta|=1$. Thus

$$\mathrm{Re}\left[\dot a_1a_1+2\dot a_1a_2\zeta+3\dot a_1a_3\zeta^2+\dot a_2a_1\overline{\zeta}+2\dot a_2a_2+3\dot a_2a_3\zeta+\dot a_3a_1\overline{\zeta}^2+2\dot a_3a_2\overline{\zeta}+3\dot a_3a_3\right]=\frac{Q}{2\pi},\quad \boxed{1}$$

giving

$$\begin{aligned}\dot a_1a_1+2\dot a_2a_2+3\dot a_3a_3&=\frac{Q}{2\pi}, &(5)\\ 2\dot a_1a_2+\dot a_2a_1+3\dot a_2a_3+2\dot a_3a_2&=0, &(6)\\ 3\dot a_1a_3+\dot a_3a_1&=0.\quad\boxed{1} &(7)\end{aligned}$$

Eqn (5) can be integrated immediately to give

$$a_1^2+2a_2^2+3a_3^2=\frac{Qt}{\pi}+6^2+2+\frac{3}{2^2}=\frac{Qt}{\pi}+38\frac{3}{4}=\frac{Qt}{\pi}+\frac{155}{4},\quad\boxed{1}$$

where we have used the given initial condition $a_1(0)=6$, $a_2(0)=1$, $a_3(0)=1/2$. Multiplying (7) by a_1^2 turns it into an exact differential, which can be integrated to give

$$a_1^3a_3=\text{const.}=\frac{6^3}{2}=108.\quad\boxed{1}$$

Multiplying (6) by a_1 gives

$$\begin{aligned}0&=2a_1\dot a_1a_2+\dot a_2a_1^2+3\dot a_2a_1a_3+2\dot a_3a_1a_2\\ &=\frac{\mathrm{d}}{\mathrm{d}t}(a_1^2a_2)+3\frac{\mathrm{d}}{\mathrm{d}t}(a_1a_2a_3)-\dot a_3a_1a_2-3\dot a_1a_2a_3=\frac{\mathrm{d}}{\mathrm{d}t}(a_1^2a_2+3a_1a_2a_3),\end{aligned}$$

using (7). Thus

$$a_1^2a_2+3a_1a_2a_3=\text{const.}=6^2+\frac{3\times 6}{2}=36+9=45.\quad\boxed{1}$$

□ **Has been seen with $a_3\equiv 0$, so the approach should be familiar but the details are new. In particular (6) is harder to solve in this case.**

(d) [9 marks] Now F ceases to be conformal when $F'(\zeta)=0$ for some $|\zeta|=1$ $\boxed{1}$. Solving $F'(\zeta)=0$ gives

$$a_1+2a_2\zeta+3a_3\zeta^2=0\qquad\Rightarrow\qquad \zeta=-\frac{a_2}{3a_3}\pm\frac{\sqrt{a_2^2-3a_1a_3}}{3a_3}.\quad\boxed{1}$$

We are given that ζ is not real at the point of breakdown. Then the discriminant must be negative. Imposing $|\zeta| = 1$ gives

$$\frac{a_2^2}{9a_3^2} + \frac{3a_1a_3 - a_2^2}{9a_3^2} = 1 \qquad \Rightarrow \qquad a_1 = 3a_3. \quad \boxed{1}$$

Thus

$$27a_3^4 = 108 \qquad \Rightarrow \qquad a_3 = \sqrt{2} \qquad \Rightarrow \qquad a_1 = 3\sqrt{2}. \quad \boxed{1}$$

Then

$$a_2 = \frac{45}{a_1^2 + 3a_1a_3} = \frac{45}{18+18} = \frac{5}{4}.$$

Then

$$\frac{Qt}{\pi} = 18 + \frac{25}{8} + 6 - \frac{155}{4} = \frac{192 + 25 - 310}{8} = -\frac{93}{8}.$$

Thus the breakdown happens when

$$t = t_* = -\frac{93\pi}{8Q}. \quad \boxed{1}$$

The non-conformality corresponds to two cusps forming in the free boundary $\boxed{1}$. We are given that the fluid volume

$$V = \iint_D \mathrm{d}x\,\mathrm{d}y = \frac{1}{2\mathrm{i}} \oint_{\partial D} \bar{z}\ \mathrm{d}z.$$

Under tha mapping,

$$V = \frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} \overline{F}\frac{\partial F}{\partial \zeta}\ \mathrm{d}\zeta = \frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} (a_1\overline{\zeta} + a_2\overline{\zeta}^2 + a_3\overline{\zeta}^3)(a_1 + 2a_2\zeta + 3a_3\zeta^2)\ \mathrm{d}\zeta \quad \boxed{1}$$

$$= \frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} \frac{a_1^2 + 2a_2^2 + 3a_3^2}{\zeta}\ \mathrm{d}\zeta = (a_1^2 + 2a_2^2 + 3a_3^2)\pi. \quad \boxed{1}$$

Thus, using (c),

$$V(t) = Qt + \frac{155\pi}{4}.$$

At the time of breakdown,

$$V(t_*) = -\frac{93\pi}{8} + \frac{155\pi}{4} = \frac{(310-93)\pi}{8} = \frac{217\pi}{8}.$$

Thus the fraction of fluid removed is $93/310 = 3/10$. $\boxed{1}$

☐ **Unseen. Will require some thought.**

2. (a) (i) [3 marks] For

$$w(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - z},$$

denote $w_+(t)$ as the limit as $z \to t \in \Gamma$ from the left (relative to the direction of Γ) and $w_-(t)$ as the limit as $z \to t \in \Gamma$ from the right. Then

$$w_\pm(t) = \pm\frac{f(t)}{2} + \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t},$$

where, with $\gamma_\epsilon = \Gamma \cap D(t;\epsilon)$,

$$\fint_\Gamma \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t} = \lim_{\epsilon\to 0}\int_{\Gamma\setminus\gamma_\epsilon} \frac{f(\zeta)\ \mathrm{d}\zeta}{\zeta - t}.$$

$\boxed{3}$ **Bookwork**

(ii) [2 marks] With $z-1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1}$, $z = r_2\mathrm{e}^{\mathrm{i}\theta_2}$

$$W(z) = \frac{(z-1)^{1/2}z^{1/2}}{z^2+1} = \frac{r_1^{1/2}\mathrm{e}^{\mathrm{i}\theta_1/2}r_2^{1/2}\mathrm{e}^{\mathrm{i}\theta_2/2}}{1+z^2}.$$

If we put both branch cuts to the left so that $-\pi < \theta_1 < \pi$, $-\pi < \theta_2 < \pi$ then $W(z)$ is holomorphic on $\mathbb{C}\setminus[0,1]$ apart from the poles at $z = \pm\mathrm{i}$, and is real and positive on $[1,\infty)$.

$\boxed{2}$ **Standard approach**

(iii) [4 marks] The residues at $\pm\mathrm{i}$ are

$$\mathrm{res}(\mathrm{i}) = \frac{2^{1/4}\mathrm{e}^{3\mathrm{i}\pi/8}\mathrm{e}^{\mathrm{i}\pi/4}}{2\mathrm{i}} = \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{2^{3/4}} \qquad (r_1 = 2^{1/2}, \theta_1 = 3\pi/4, r_2 = 1, \theta_2 = \pi/2)$$

and

$$\mathrm{res}(-\mathrm{i}) = \frac{\mathrm{e}^{-\mathrm{i}\pi/8}}{2^{3/4}} \qquad (r_1 = 2^{1/2}, \theta_1 = -3\pi/4, r_2 = 1, \theta_2 = -\pi/2).$$

Thus

$$w(z) = W(z) - \frac{\mathrm{e}^{\mathrm{i}\pi/8}}{2^{3/4}(z-\mathrm{i})} - \frac{\mathrm{e}^{-\mathrm{i}\pi/8}}{2^{3/4}(z+\mathrm{i})} = W(z) - \frac{(z+\mathrm{i})\mathrm{e}^{\mathrm{i}\pi/8} + (z-\mathrm{i})\mathrm{e}^{-\mathrm{i}\pi/8}}{2^{3/4}(1+z^2)}$$

has only removeable singularities, as so is holomorphic in $\mathbb{C}\setminus[0,1]$.

$\boxed{4}$ **Unseen but the approach is familiar in other contexts.**

(iv) [5 marks] On top of the branch cut $\theta_1 = \pi$, $\theta_2 = 0$, while underneath the branch cut $\theta_1 = -\pi$, $\theta_2 = 0$. Thus

$$w_+(x) = \frac{\mathrm{i}(1-x)^{1/2}x^{1/2}}{1+x^2} - \frac{2^{1/4}(\cos(\pi/8)x - \sin(\pi/8))}{(x^2+1)},$$

$$w_-(x) = \frac{-\mathrm{i}(1-x)^{1/2}x^{1/2}}{1+x^2} - \frac{2^{1/4}(\cos(\pi/8)x - \sin(\pi/8))}{(x^2+1)},$$

Applying the Plemelj formulae

$$w_+(x) - w_-(x) = f(x) = \frac{2\mathrm{i}(1-x)^{1/2}x^{1/2}}{1+x^2},$$

$$w_+(x) + w_-(x) = \frac{1}{\pi\mathrm{i}}\fint_0^1 \frac{f(\xi)\ \mathrm{d}\xi}{\xi - x} = -\frac{2^{5/4}(\cos(\pi/8)x - \sin(\pi/8))}{(x^2+1)}.$$

thus

$$\fint_0^1 \frac{(1-\xi)^{1/2}\xi^{1/2}}{1+\xi^2}\frac{\mathrm{d}\xi}{\xi - x} = \frac{2^{1/4}\pi(\sin(\pi/8) - \cos(\pi/8)x)}{(x^2+1)} \qquad \text{for } 0 < x < 1.$$

$\boxed{5}$ **Unfamiliar (but straightforward) application and new example**

(b) (i) [2 marks]

$$\zeta = A + C \int^z \prod_{j=1}^{n} (t - x_j)^{-\beta_j} \, \mathrm{d}t,$$

where $\pi\beta_j$ are the exterior angles at the vertices ζ_j, which have pre-images x_j, and A and C are constants.

$\boxed{2}$ **Bookwork**

(ii) [4 marks] We have four vertices $\zeta_1 = \zeta_3 = 0$, $\zeta_2 = \mathrm{i}$, $\zeta_4 = \infty$ $\boxed{1}$. The corresponding exterior angles give $\beta_1 = 1/2$, $\beta_2 = -1$, $\beta_3 = 1/2$, $\beta_4 = 2$ $\boxed{1}$. In general we can only choose three locations for 3 of the x_j, but here symmetry allows us to take $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = \infty$. Thus

$$\zeta = A + C \int^z \frac{t}{\sqrt{t^2 - 1}} \, \mathrm{d}t = A + C\sqrt{z^2 - 1}. \quad \boxed{1}$$

To fix A and C we must ensure that the vertices end up in the right places:

$$\zeta_1 = \zeta_3 = 0 \quad \Rightarrow \quad \zeta = 0 \text{ when } z = \pm 1 \quad \Rightarrow \quad A = 0,$$

$$\zeta_2 = \mathrm{i} \quad \Rightarrow \quad \zeta = \mathrm{i} \text{ when } z = 0 \quad \Rightarrow \quad C = 1. \quad \boxed{1}$$

Thus

$$\zeta = \sqrt{z^2 - 1}.$$

☐ **Previously seen example**

(iii) [5 marks] Introducing the complex potential w such that $\phi = \mathrm{Re}(w)$ we have w holomorphic in the slit ζ-plane (excluding the point charge), $\mathrm{Re}(w) = 0$ on the boundary, and $w \sim (Q/2\pi)\log(\zeta - 2\mathrm{i})$ as $\zeta \to 2\mathrm{i}$ $\boxed{1}$. Under the mapping $\zeta \mapsto z$, defining $W(z) = w(\zeta(z))$ we have W holomorphic in the upper half $z = x + \mathrm{i}y$ plane (excluding the point charge) and $\mathrm{Re}(W) = 0$ for $y = 0$ $\boxed{1}$. The point charge is mapped to $z = \sqrt{\zeta^2 + 1} = \sqrt{-4 + 1} = \mathrm{i}\sqrt{3}$, so that

$$W \sim \frac{Q}{2\pi} \log(z - \mathrm{i}\sqrt{3}) \qquad \text{as } z \to \mathrm{i}\sqrt{3}. \quad \boxed{1}$$

The solution for W is simply

$$W = \frac{Q}{2\pi} \log(z - \mathrm{i}\sqrt{3}) - \frac{Q}{2\pi} \log(z + \mathrm{i}\sqrt{3}) = \frac{Q}{2\pi} \log\left(\frac{z - \mathrm{i}\sqrt{3}}{z + \mathrm{i}\sqrt{3}}\right). \quad \boxed{1}$$

Returning to the ζ-plane,

$$w = \frac{Q}{2\pi} \log\left(\frac{\sqrt{\zeta^2 + 1} - \mathrm{i}\sqrt{3}}{\sqrt{\zeta^2 + 1} + \mathrm{i}\sqrt{3}}\right) \qquad \Rightarrow \qquad \phi = \mathrm{Re}(w) = \frac{Q}{2\pi} \log\left|\frac{\sqrt{\zeta^2 + 1} - \mathrm{i}\sqrt{3}}{\sqrt{\zeta^2 + 1} + \mathrm{i}\sqrt{3}}\right|. \quad \boxed{1}$$

☐ **New example. Will require some thought.**

3. (a) [5 marks]

$$\bar{K}(k) = \int_{-\infty}^{0} \mathrm{e}^{x}\mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-x}\mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x = \left[\frac{\mathrm{e}^{x}\mathrm{e}^{\mathrm{i}kx}}{1 + \mathrm{i}k}\right]_{-\infty}^{0} + \left[\frac{\mathrm{e}^{-x}\mathrm{e}^{\mathrm{i}kx}}{-1 + \mathrm{i}k}\right]_{0}^{\infty}$$

$$= \frac{1}{1 + \mathrm{i}k} + \frac{1}{1 - \mathrm{i}k} = \frac{2}{1 + k^2}. \quad \boxed{1}$$

with $-1 < \mathrm{Im}(k) < 1$. $\boxed{1}$

$$\bar{g}_+(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\ \mathrm{d}x = \left[\frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\right]_0^{\infty} = -\frac{1}{\mathrm{i}k} = \frac{\mathrm{i}}{k},$$

with $0 < \mathrm{Im}(k)$. $\boxed{1}$ $\bar{K}(k)$ can be analytically continued into $\mathbb{C} \setminus \{\mathrm{i}, -\mathrm{i}\}$ $\boxed{1}$, while $\bar{g}(k)$ can be analytically continued into $\mathbb{C} \setminus \{0\}$ $\boxed{1}$.

☐ **Easy bookwork**

(b) [4 marks] Define

$$f_+ = \begin{cases} 0 & x < 0, \\ f(x) & x \geqslant 0, \end{cases} \qquad h_-(x) = \begin{cases} \frac{1}{2}\int_0^\infty K(x-t)f(t)\ \mathrm{d}t & x < 0, \\ 0 & x \geqslant 0. \end{cases} \quad \boxed{2}$$

Then

$$\frac{2}{3}f_+'(x) - f_+(x) + \frac{1}{2}\int_{-\infty}^{\infty} K(x-t)f_+(t)\ \mathrm{d}t = g_+(x) + h_-(x), \qquad x \in \mathbb{R} \setminus \{0\}.$$

Note that $f_+'(x)$ might be discontinuous at the origin, but $f_+(x)$ is continuous.
If $f(x) = O(\mathrm{e}^{\gamma x})$ as $x \to \infty$ for some real γ, then $\bar{f}_+(k)$ is defined and holomorphic for $\mathrm{Im}(k) > \gamma$ $\boxed{1}$. We have

$$h_-(x) = \frac{1}{2}\int_0^\infty \mathrm{e}^{x-t} f(t)\ \mathrm{d}t = \frac{\mathrm{e}^x}{2}\int_0^\infty \mathrm{e}^{-t} f(t)\ \mathrm{d}t$$

for $x < 0$. Thus $\bar{h}_-(k)$ is defined and holomorphic for $\mathrm{Im}(k) < 1$ $\boxed{1}$.

☐ **Seen before without the derivative term. This makes no difference to the method, apart from the fact that the origin is excluded because f' jumps there, but might cause some anxiety.**

(c) [10 marks] Taking a Fourier transform

$$-\frac{2}{3}\mathrm{i}k\bar{f}_+ - \bar{f}_+ + \frac{1}{2}\bar{K}\bar{f}_+ = \bar{g}_+ + \bar{h}_-. \quad \boxed{1}$$

Note that f_+ is continuous so that $\bar{f_+'} = -\mathrm{i}k\bar{f}_+$, even though f_+' might be discontinuous (if we split the range of integration, since f_+ is continuous the boundary terms will vanish). $\boxed{1}$

Using the results from (a) for $\bar{K}$ the LHS is

$$\left(-\frac{2}{3}\mathrm{i}k - 1 + \frac{1}{1+k^2}\right)\bar{f}_+ = \left(\frac{-(3+2\mathrm{i}k)(1+k^2)+3}{3(1+k^2)}\right)\bar{f}_+ = -\left(\frac{3k^2 + 2\mathrm{i}k + 2\mathrm{i}k^3}{3(1+k^2)}\right)\bar{f}_+ \quad \boxed{1}$$

$$= -\frac{\mathrm{i}k(k-2\mathrm{i})(2k+\mathrm{i})}{3(k+\mathrm{i})(k-\mathrm{i})}\bar{f}_+. \quad \boxed{1}$$

Substituting for $\bar{g}_+(k)$ and rearranging gives

$$-\frac{\mathrm{i}k(2k+\mathrm{i})}{3(k+\mathrm{i})}\bar{f}_+ = \frac{\mathrm{i}(k-\mathrm{i})}{k(k-2\mathrm{i})} + \frac{(k-\mathrm{i})\bar{h}_-}{(k-2\mathrm{i})}. \quad \boxed{1}$$

We need an additive decomposition of the first term on the RHS. Writing

$$\frac{\mathrm{i}(k-\mathrm{i})}{k(k-2\mathrm{i})} = \frac{\mathrm{i}}{2k} + \frac{\mathrm{i}}{2(k-2\mathrm{i})}. \quad \boxed{1}$$

we find

$$-\frac{\mathrm{i}k(2k+\mathrm{i})}{3(k+\mathrm{i})}\bar{f}_{+}-\frac{\mathrm{i}}{2k}=\frac{\mathrm{i}}{2(k-2\mathrm{i})}+\frac{(k-\mathrm{i})\bar{h}_{-}}{k-2\mathrm{i}}.$$

The LHS is holomorphic in $\mathrm{Im}(k) > \max(0,\gamma)$ $\boxed{1}$ while the RHS is holomorphic in $\mathrm{Im}(k) < 1$ $\boxed{1}$. Since there is an overlap region, together they define an entire function. Since $\bar{h}_{-} = O(k^{-1})$ as $|k| \to \infty$ the RHS tends to zero at infinity $\boxed{1}$. Thus this entire function must be zero by Liouville's theorem $\boxed{1}$.

☐ **Standard method but new problem. The inclusion of f' makes it different to problems they have seen before, and will require some thought.**

(d) [6 marks] Solving gives

$$\bar{f}_{+}=-\frac{3(k+\mathrm{i})}{2k^2(2k+\mathrm{i})}. \quad \boxed{1}$$

Inverting

$$f(x)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{3(k+\mathrm{i})}{2k^2(2k+\mathrm{i})}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k$$

where the inversion contour lies above the poles $\boxed{1}$. Closing the contour in the lower-half plane for $x > 0$ we pick up minus residue contributions from the poles at $k = -\mathrm{i}/2$ and $k = 0$ $\boxed{1}$.

$$\begin{aligned}
\mathrm{res}(0) &= -\frac{1}{2\pi}\frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}k}\left.\frac{(k+\mathrm{i})}{(2k+\mathrm{i})}\mathrm{e}^{-\mathrm{i}kx}\right|_{k=0}\\
&= -\frac{1}{2\pi}\frac{3}{2}\left.\left(\frac{1}{(2k+\mathrm{i})}\mathrm{e}^{-\mathrm{i}kx}-\frac{2(k+\mathrm{i})}{(2k+\mathrm{i})^2}\mathrm{e}^{-\mathrm{i}kx}-\mathrm{i}x\frac{(k+\mathrm{i})}{(2k+\mathrm{i})}\mathrm{e}^{-\mathrm{i}kx}\right)\right|_{k=0}\\
&= -\frac{1}{2\pi}\frac{3}{2}\left(-\mathrm{i}+2\mathrm{i}-\mathrm{i}x\right) = -\frac{1}{2\pi}\frac{3\mathrm{i}}{2}\left(1-x\right),\\
\mathrm{res}(-\mathrm{i}/2) &= -\frac{1}{2\pi}\frac{3\mathrm{i}/2}{4(\mathrm{i}/2)^2}\mathrm{e}^{-x/2} = \frac{1}{2\pi}\frac{3\mathrm{i}}{2}\mathrm{e}^{-x/2}.
\end{aligned}$$

Thus

$$f(x)=\frac{3}{2}\left(x-1+\mathrm{e}^{-x/2}\right). \quad \boxed{3}$$

☐ **Standard method but new problem. The double pole will require some thought.**