A12207H1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical Sciences: Paper C5.6
Master of Science in Mathematical and Theoretical Physics: Paper C5.6

Applied Complex Variables

TRINITY TERM 2025

Tuesday 03 June, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

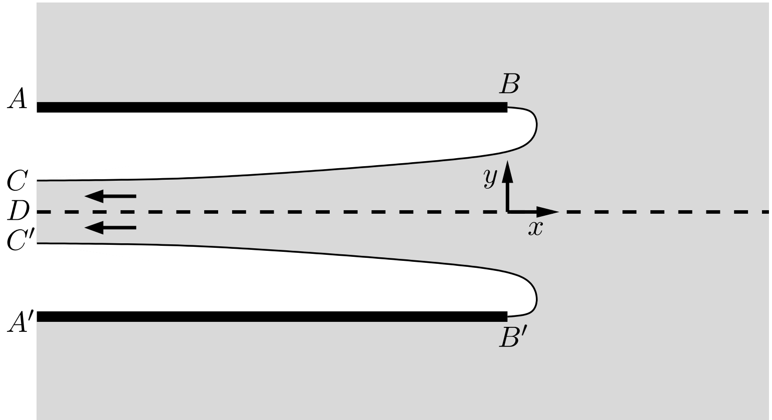
You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. An infinite two-dimensional, inviscid, irrotational, incompressible fluid is draining through a channel comprising two parallel semi-infinite plates AB and $A'B'$, as shown below. After nondimensionalisation, the separation of the plates $A'A$ is 2 and the velocity $\mathbf{u} \to (-1, 0)$ as $x \to -\infty$ at CDC'. DE is a symmetry line. Take the complex potential $w = \phi + \mathrm{i}\psi$ to vanish at B. Let the height of the jet $C'C$ be $2h$. Take the midpoint of $B'B$ to be the origin.



(a) [6 marks] Show that the images of the upper half of the shaded fluid domain (i.e. that bounded by $ABCDE$) in the potential w-plane and the hodograph w'-plane are a strip and a semi-circle, respectively; sketch these regions indicating clearly the image of each of the labelled points.

(b) [5 marks] Show that

$$\zeta = \left(\frac{1 + w'}{1 - w'}\right)^2$$

maps the image of the upper half of the fluid domain in the w'-plane to the upper half-plane. Indicate clearly the positions in the ζ-plane of the points A, B, C, D and E.

(c) [5 marks] Show that w satisfies the equation

$$\mathrm{e}^{\pi w/h} = \frac{4w'}{(1 + w')^2}.$$

(d) [9 marks] Show that the free surface BC is given by

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{h}{\pi}\left(\sin\theta - \tan(\theta/2)\right), \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{h}{\pi}\left(1 - \cos\theta\right),$$

where θ is the angle the tangent (in the flow direction) makes with the x-axis. By applying suitable boundary conditions determine x and y and deduce the value of h.

2. (a) [5 marks] Let Γ be a directed smooth contour in the complex plane and

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

where f is continuous on Γ and holomorphic in a neighbourhood of the point $t \in \Gamma$. Show that the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, where

$$w_\pm(t) = \pm\frac{f(t)}{2} + \frac{1}{2\pi \mathrm{i}} \fint_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - t},$$

and you should define the integral $\fint$ precisely; indicate with a sketch which side of the contour is + and which is −.

(b) [6 marks] Let $\Gamma = \{x + \mathrm{i}y : -1 < x < 1,\ y = 0\}$ and $\overline{\Gamma} = \{x + \mathrm{i}y : -1 \leqslant x \leqslant 1,\ y = 0\}$. Suppose $w(z)$ is holomorphic away from $\overline{\Gamma}$ and $w_+(x) + w_-(x) = g(x)$ on Γ for some known smooth complex-valued function $g(x)$. Suppose $\widetilde{w}(z)$ is holomorphic and non-zero away from $\overline{\Gamma}$ and $\widetilde{w}_+(x) = -\widetilde{w}_-(x) \neq 0$ on Γ. Determine the density $F(\xi)$ for which the solution for $w(z)$ is

$$\frac{w(z)}{\widetilde{w}(z)} = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{F(\xi)\mathrm{d}\xi}{\xi - z}.$$

Deduce that

$$f(x) = \frac{\widetilde{w}_+(x)}{\pi \mathrm{i}} \fint_{-1}^{1} \frac{g(\xi)\,\mathrm{d}\xi}{\widetilde{w}_+(\xi)(\xi - x)}$$

is a solution of the singular integral equation

$$\frac{1}{\pi \mathrm{i}} \fint_{-1}^{1} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} = g(x) \quad \text{for } -1 < x < 1.$$

(c) [7 marks] (i) Explain briefly why $(z^2 - 1)^{1/2}$ cannot be written as a Cauchy integral over $(-1, 1)$. By writing $(z^2 - 1)^{1/2} - z$ as a Cauchy integral show that

$$\fint_{-1}^{1} \frac{(1 - \xi^2)^{1/2}\,\mathrm{d}\xi}{\xi - x} = -\pi x \qquad \text{for } -1 < x < 1.$$

(ii) Given that the substitution $t = x/\sqrt{1 - x^2}$ allows

$$\int_{-1}^{1} \frac{x^2\,\mathrm{d}x}{(1 + 3x^2)\sqrt{1 - x^2}}$$

to be written as

$$\int_{-\infty}^{\infty} \frac{t^2\,\mathrm{d}t}{(1 + t^2)(1 + 4t^2)},$$

use contour integration to evaluate this expression.

(d) [7 marks] By taking $\int_{-1}^{1} \xi f(\xi)/(1 + 3\xi^2)\,\mathrm{d}\xi$ to be a constant that you should determine, show that

$$f(x) = \frac{6x}{(6 + \beta)\pi(1 - x^2)^{1/2}}$$

is a solution of the singular integral equation

$$\fint_{-1}^{1} \left(\frac{1}{\xi - x} + \frac{\beta\xi}{1 + 3\xi^2} \right) f(\xi)\,\mathrm{d}\xi = 1 \qquad \text{for } -1 < x < 1,$$

provided that $\beta \neq -6$.

3. (a) [3 marks] Let

$$g_- = \begin{cases} 0 & x > 0, \\ e^{2x} & x < 0. \end{cases}$$

For what values of $k \in \mathbb{C}$ is the Fourier transform $\bar{g}_-(k)$ defined? Evaluate $\bar{g}_-(k)$. To what region of the complex k-plane can it be analytically continued?

(b) [5 marks] The function $G(k)$ is holomorphic in a strip $\Omega = \{k \in \mathbb{C} : \alpha < \mathrm{Im}(k) < \beta\}$ and satisfies $G(k) \to 0$ as $k \to \infty$ in Ω. Show that $G(k)$ may be decomposed as

$$G(k) = G_+(k) - G_-(k),$$

where $G_+(k)$ is holomorphic in $\mathrm{Im}(k) > \gamma_+$ and $G_-(k)$ is holomorphic in $\mathrm{Im}(k) < \gamma_-$, for $\alpha < \gamma_+ < \gamma_- < \beta$. Give explicit formulae for $G_+(k)$ and $G_-(k)$, in terms of integrals along specified contours in the complex k-plane.

(c) [5 marks] Clearly define branches of the multifunctions $(k - \mathrm{i})^{1/2}$ and $(k + \mathrm{i})^{1/2}$ which are holomorphic and have positive real part in the strip $-1 < \mathrm{Im}(k) < 1$. By evaluating the integrals in part (b), or otherwise, express the function

$$G(k) = \frac{(k + \mathrm{i})^{1/2}}{k - 2\mathrm{i}}$$

as $G_+(k) - G_-(k)$ where $G_+(k)$ is holomorphic in $\mathrm{Im}(k) > -1$ and $G_-(k)$ is holomorphic in $\mathrm{Im}(k) < 2$.

Now consider the coupled integral equations

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\mathrm{i}kx} f(k)\, \mathrm{d}k &= 0 \qquad & x > 0, \\ \int_{-\infty}^{\infty} e^{-\mathrm{i}kx} (k^2+1)^{-1/2} f(k)\, \mathrm{d}k &= e^{2x} \qquad & x < 0, \end{aligned} \tag{$*$}$$

for the function $f : \mathbb{R} \to \mathbb{C}$, which you may assume is continuous and bounded.

(d) [5 marks] Show how to convert $(*)$ to the pair of equations

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}kx} f(k)\, \mathrm{d}k = u_-(x),$$

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}kx} (k^2+1)^{-1/2} f(k)\, \mathrm{d}k = v_+(x) + g_-(x),$$

defined on the full range $-\infty < x < \infty$, explaining how $u_-(x)$ and $v_+(x)$ are defined. Show that the Fourier transforms $\bar{u}_-$, $\bar{v}_+$ satisfy

$$\frac{\bar{u}_-}{(k^2+1)^{1/2}} = \bar{v}_+ - \frac{\mathrm{i}}{k - 2\mathrm{i}}.$$

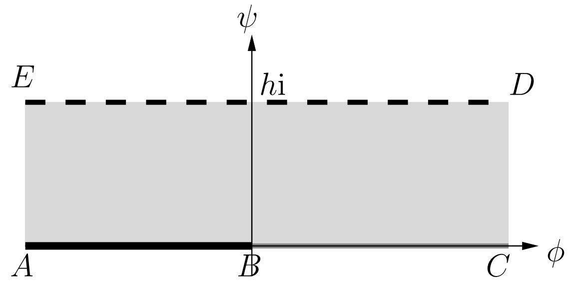
(e) [7 marks] Use the Wiener-Hopf technique to deduce that

$$f(x) = -\frac{3^{1/2} e^{3\mathrm{i}\pi/4} (x - \mathrm{i})^{1/2}}{2\pi (x - 2\mathrm{i})}.$$

Verify this answer by direct evaluation of the integrals $(*)$.

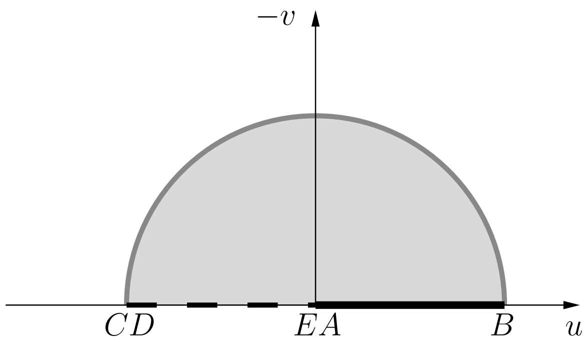
[You may assume $\bar{u}_-(k)$ and $\bar{v}_+(k)$ are holomorphic in $\mathrm{Im}(k) < 1$ and $\mathrm{Im}(k) > -1$, respectively, and that $f(k)$ is bounded as $|k| \to \infty$ for $k \in \mathbb{C}$.]

1. (a) [6 marks] Since $\psi = 0$ on ABC and at minus infinity the velocity is $(-1, 0) = (\psi_y, -\psi_x)$ with a jet height of $2h$ the stream function takes the value h on DE [1]. Thus the potential plane is the strip $0 < \psi < h$, $-\infty < \phi < \infty$, with $\phi(B) = 0$.



[1]

The velocity at C and D is $(-1, 0)$. Since $|w'|$ is constant on BC we must have $|w'| = 1$ on BC, and therefore the velocity at B is $(1, 0)$. [1] The speed at A and E tends to zero [1]. On AB and DE we know $v = 0$. Finally, we see that v is negative throughout. [1]

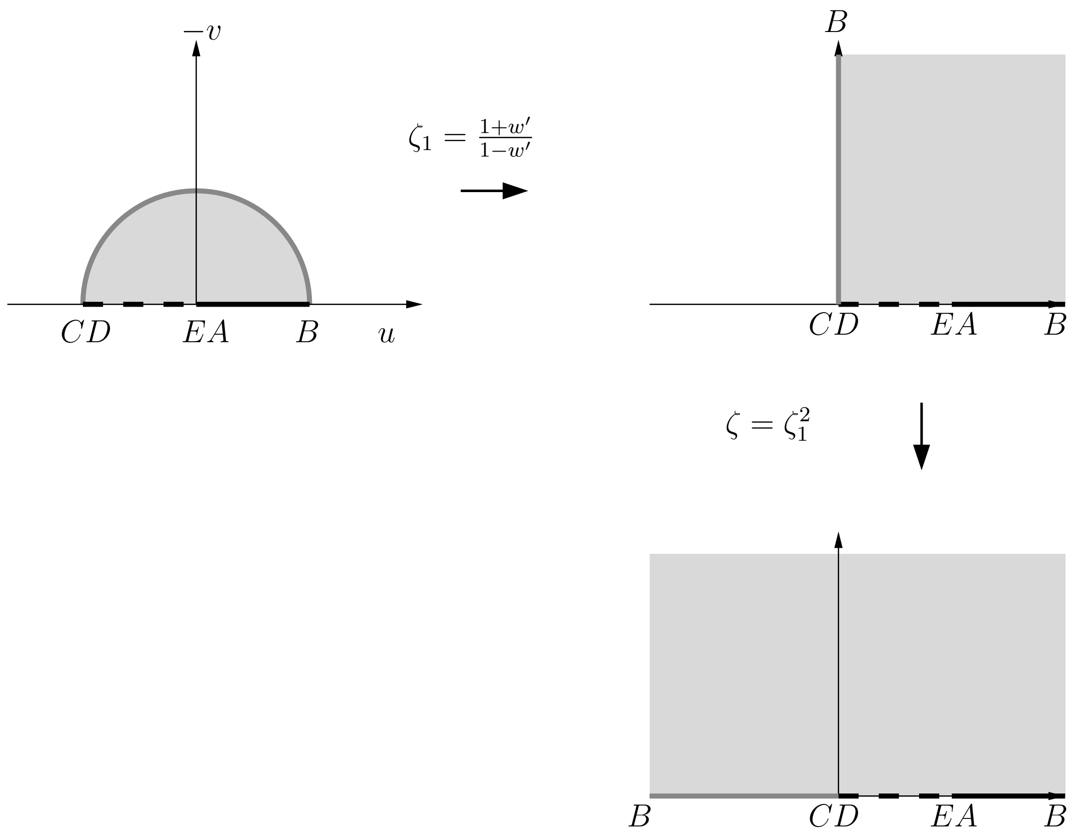


[1]

□ **New example, but familiar method.**

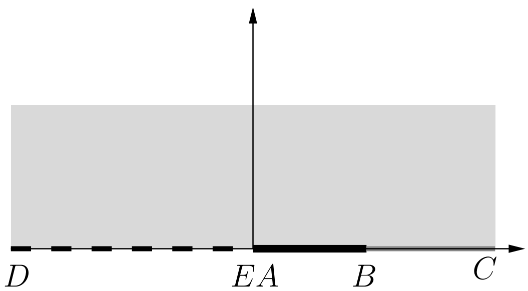
(b) [5 marks] We have:

 Turn Over



$\boxed{5}$ **New example, but familiar method.**

(c) [5 marks] We can map the potential plane to the upper half plane with the map $\zeta_1 = e^{\pi w/h}$.



$\boxed{1}$

The points are in the wrong places though. To match with part (b) we need to send $0 \to 1$, $1 \to \infty$, $\infty \to 0$ $\boxed{1}$. Thus we need to apply the Möbius map

$$\zeta = \frac{1}{1-\zeta_1} = \frac{1}{1-e^{\pi w/h}} \quad \boxed{1}.$$

Equating the two upper half planes,

$$\frac{1}{1-e^{\pi w/h}} = \left(\frac{1+w'}{1-w'}\right)^2 \quad \boxed{1}.$$

Thus

$$e^{\pi w/h} = 1 - \left(\frac{1-w'}{1+w'}\right)^2 = \frac{4w'}{(1+w')^2} \quad \boxed{1}.$$

□ **New example, but familiar method.**

(d) [9 marks] On the free surface $w' = \mathrm{e}^{-\mathrm{i}\theta}$ where θ is the angle the free surface makes with the x-axis $\boxed{1}$. This gives

$$\mathrm{e}^{\pi w/h} = \frac{4\mathrm{e}^{-\mathrm{i}\theta}}{(1+\mathrm{e}^{-\mathrm{i}\theta})^2} = \frac{4}{(\mathrm{e}^{\mathrm{i}\theta/2}+\mathrm{e}^{-\mathrm{i}\theta/2})^2} = \frac{1}{\cos^2(\theta/2)} = \sec^2(\theta/2) \quad \boxed{1}.$$

Now differentiate with respect the θ to get

$$\frac{\pi}{h}\mathrm{e}^{\pi w/h}\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\theta} = \sec^2(\theta/2)\tan(\theta/2). \quad \boxed{1}$$

Substituting

$$\mathrm{e}^{\pi w/h} = \sec^2(\theta/2), \qquad \frac{\mathrm{d}w}{\mathrm{d}z} = \mathrm{e}^{-\mathrm{i}\theta}$$

gives

$$\frac{\pi}{h}\sec^2(\theta/2)\mathrm{e}^{-\mathrm{i}\theta}\frac{\mathrm{d}z}{\mathrm{d}\theta} = \sec^2(\theta/2)\tan(\theta/2),$$

i.e.

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{h}{\pi}\mathrm{e}^{\mathrm{i}\theta}\tan(\theta/2). \quad \boxed{1}$$

Equating real and imaginary parts gives

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{h}{\pi}\cos\theta\tan(\theta/2), \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{h}{\pi}\sin\theta\tan(\theta/2). \quad \boxed{1}$$

Writing

$$\cos\theta = 2\cos^2(\theta/2) - 1, \qquad \sin\theta = 2\cos(\theta/2)\sin(\theta/2)$$

gives

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{h}{\pi}\left(2\cos(\theta/2)\sin(\theta/2) - \tan(\theta/2)\right) = \frac{h}{\pi}\left(\sin\theta - \tan(\theta/2)\right),\\ \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{2h}{\pi}\sin^2(\theta/2) = \frac{h}{\pi}\left(1-\cos\theta\right). \quad \boxed{1}\end{aligned}$$

Choosing the midpoint of $B'B$ to be the origin the boundary conditions are

$$x(0) = 0, \quad y(0) = 1, \qquad x(-\pi) = -\infty, \quad y(-\pi) = h. \quad \boxed{1}$$

Integrating gives

$$x = \frac{h}{\pi}\left(A - \cos\theta + 2\log(\cos(\theta/2)))\right), \qquad y = \frac{h}{\pi}\left(B + \theta - \sin\theta\right).$$

$x(0) = 0$ implies $A = 1$; $y(0) = 1$ implies $B = \pi/h$. Thus

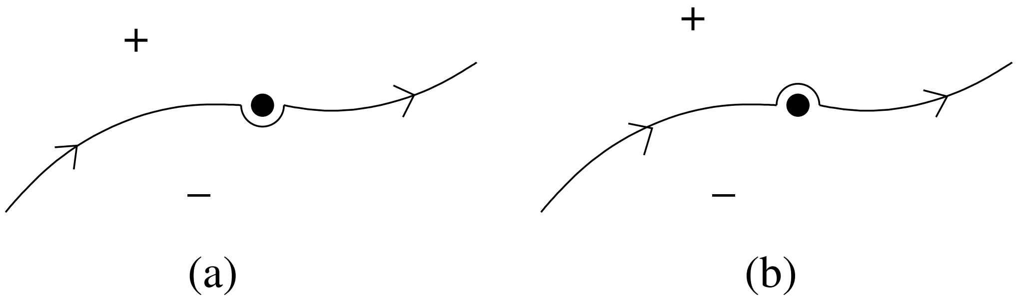
$$x = \frac{h}{\pi}\left(1 - \cos\theta + 2\log(\cos(\theta/2))\right), \qquad y = 1 + \frac{h}{\pi}\left(\theta - \sin\theta\right). \quad \boxed{1}$$

Now imposing the condition $y(-\pi) = h$ gives

$$h = 1 - h \qquad \Rightarrow \qquad h = \frac{1}{2}. \quad \boxed{1}$$

☐ **New example, but familiar method. Will require a bit of thought.**

2. (a) [5 marks] Label the left-hand side of Γ as "+" and the right-hand side as "-" $\boxed{1}$. As $z \to t \in \Gamma$ from the plus side indent the contour with a small semi-circle around t as shown in (a) $\boxed{1}$.



Then

$$w_+(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi \mathrm{i}} \left(\int_{\text{one end}}^{t-\epsilon} \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - t} + \int_{\text{semicircle}|\zeta - t| = \epsilon} \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - t} + \int_{\text{other end}}^{t-\epsilon} \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - t} \right).$$

As $\epsilon \to 0$ the semicircle gives a contribution $\frac{1}{2\pi \mathrm{i}} \times \pi \mathrm{i} f(t)$ $\boxed{1}$. Thus

$$w_+(t) = \frac{f(t)}{2} + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - t},$$

where

$$⨍ = \lim_{\epsilon \to 0} \left(\int^{t-\epsilon} + \int_{t+\epsilon} \right). \quad \boxed{1}$$

As $z \to t \in \Gamma$ from the minus side we need to indent the contour on the other side with a small semi-circle around t as shown in (b). The semicircle now gives a contribution $-\frac{1}{2\pi \mathrm{i}} \times \pi \mathrm{i} f(t)$ as $\epsilon \to 0$, $\boxed{1}$ so that

$$w_-(t) = -\frac{f(t)}{2} + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - t}.$$

□ **Bookwork**

(b) [6 marks] Let $W = w/\tilde{w}$. Then

$$W_+ - W_- = \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-} = \frac{w_+}{\tilde{w}_+} + \frac{w_-}{\tilde{w}_+} = \frac{g}{\tilde{w}_+}. \quad \boxed{1}$$

Seek F for which

$$W(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{F(\xi)\, \mathrm{d}\xi}{\xi - z}.$$

Then by Plemelj

$$F(x) = W_+(x) - W_-(x) = \frac{g(x)}{\tilde{w}_+(x)}. \quad \boxed{1}$$

Thus

$$W(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{g(\xi)}{\tilde{w}_+(\xi)} \frac{\mathrm{d}\xi}{\xi - z}.$$

Now let

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(\xi)\, \mathrm{d}\xi}{\xi - z}. \quad \boxed{1}$$

Then, by Plemelj

$$
\begin{aligned}
w_+(x) + w_-(x) &= \left(\frac{f}{2} + \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} \right) + \left(-\frac{f}{2} + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} \right) \\
&= \frac{1}{\pi \mathrm{i}} \int_{0}^{1} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} = g(x), \quad \boxed{1}
\end{aligned}
$$

and the solution is

$$
\begin{aligned}
f(x) &= w_+(x) - w_-(x) = \tilde{w}_+(x) W_+(x) - \tilde{w}_-(x) W_-(x) = \tilde{w}_+(x) W_+(x) + \tilde{w}_+(x) W_-(x) \\
&= \tilde{w}_+(x) \left(\frac{F(x)}{2} + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1} \frac{F(\xi)\,\mathrm{d}\xi}{\xi - x} - \frac{F(x)}{2} + \frac{1}{2\pi \mathrm{i}} \int_{0}^{1} \frac{F(\xi)\,\mathrm{d}\xi}{\xi - x} \right) \\
&= \frac{\tilde{w}_+(x)}{\pi \mathrm{i}} \int_{-1}^{1} \frac{F(\xi)\,\mathrm{d}\xi}{\xi - x} = \frac{\tilde{w}_+(x)}{\pi \mathrm{i}} \int_{-1}^{1} \frac{g(\xi)\,\mathrm{d}\xi}{\tilde{w}_+(\xi)(\xi - x)}. \quad \boxed{2}
\end{aligned}
$$

□ **Seen before in lectures**

(c) [7 marks]

(i) A Cauchy integral

$$
W(z) = \int_{-1}^{1} \frac{F(\xi)\,\mathrm{d}\xi}{\xi - z}
$$

is $O(|z|^{-1})$ as $|z| \to \infty$ $\boxed{1}$. Thus before we can write $(z^2-1)^{1/2}$ as a Cauchy integral we must subtract the dominant behaviour at infinity. Since $(z^2-1)^{1/2} \sim z + O(|z|^{-1})$ as $|z| \to \infty$ we can write

$$
W(z) = (z^2 - 1)^{1/2} - z = \int_{-1}^{1} \frac{F(\xi)\,\mathrm{d}\xi}{\xi - z},
$$

where we take the branch cut to lie on $[-1, 1]$. Then, by Plemelj,

$$
F = W_+ - W_- = 2(x^2-1)^{1/2}, \quad \boxed{1} \qquad \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{F(\xi)\,\mathrm{d}\xi}{\xi - x} = W_+ + W_- = -2x. \quad \boxed{1}
$$

Thus

$$
\frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{2(\xi^2-1)^{1/2}\,\mathrm{d}\xi}{\xi - x} = -2x \qquad \text{i.e.} \qquad \int_{-1}^{1} \frac{(1-\xi^2)^{1/2}\,\mathrm{d}\xi}{\xi - x} = -\pi x, \quad \boxed{1}
$$

as required.

□ **Unseen. Will require some thought**

(ii) Closing the contour in the upper half plane we pick up $2\pi\mathrm{i} \times$ the residues at $t = \mathrm{i}$ and $t = \mathrm{i}/2$, $\boxed{1}$ so that

$$
\int_{-\infty}^{\infty} \frac{t^2}{(1+t^2)} \frac{\mathrm{d}t}{1+4t^2} = 2\pi\mathrm{i} \left(\frac{\mathrm{i}^2}{2\mathrm{i}(1+4\mathrm{i}^2)} + \frac{(\mathrm{i}/2)^2}{(1+(\mathrm{i}/2)^2)4\mathrm{i}} \right) = 2\pi\mathrm{i} \left(\frac{1}{6\mathrm{i}} - \frac{1}{12\mathrm{i}} \right) = \frac{\pi}{6}. \quad \boxed{2}
$$

□ **Unseen but familiar method**

(d) [7 marks] With

$$\int_0^1 \frac{\xi f(\xi)\,\mathrm{d}\xi}{1+3\xi^2} = \alpha,$$

the equation is

$$⨍_0^1 \frac{f(\xi)}{\xi - x}\,\mathrm{d}\xi = 1 - \alpha\beta \qquad \text{for } 0 < x < 1. \quad \boxed{1}$$

This corresponds to the equation from part (b) if we set $\pi \mathrm{i} g = 1 - \alpha\beta$. $\boxed{1}$ If we choose

$$\tilde{w}(z) = (z^2 - 1)^{-1/2}$$

then, using (b),

$$\begin{aligned} f(x) &= \frac{(1-\alpha\beta)}{(\pi \mathrm{i})^2 (x^2-1)^{1/2}} ⨍_0^1 \frac{(\xi^2-1)^{1/2}\,\mathrm{d}\xi}{\xi - x} = -\frac{(1-\alpha\beta)}{\pi^2(1-x^2)^{1/2}} ⨍_0^1 \frac{(1-\xi^2)^{1/2}\,\mathrm{d}\xi}{\xi - x} \quad \boxed{1} \\ &= \frac{(1-\alpha\beta)x}{\pi(1-x^2)^{1/2}}, \quad \boxed{1} \end{aligned}$$

using (c)(i). Now

$$\alpha = \int_{-1}^1 \frac{\xi f(\xi)\,\mathrm{d}\xi}{1+3\xi^2} = \int_{-1}^1 \frac{(1-\alpha\beta)\xi^2}{\pi(1+3\xi^2)(1-\xi^2)^{1/2}}\,\mathrm{d}x = \frac{1-\alpha\beta}{6} \quad \boxed{1}$$

using (c)(ii). Thus

$$\alpha\left(1 + \frac{\beta}{6}\right) = \frac{1}{6} \qquad \Rightarrow \qquad \alpha = \frac{1}{6+\beta} \quad \boxed{1} \qquad \Rightarrow \qquad 1 - \alpha\beta = \frac{6}{6+\beta},$$

so that

$$f(x) = \frac{6x}{(6+\beta)\pi(1-x^2)^{1/2}} \quad \boxed{1}$$

as required.

□ **Unseen. Will require some thought**

3. (a) [3 marks] $\bar{g}_-(k)$ is defined for $\mathrm{Im}(k) < 2$. $\boxed{1}$

$$\bar{g}_-(k) = \int_0^\infty \mathrm{e}^{2x+\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{2x+\mathrm{i}kx}}{2+\mathrm{i}k}\right]_{-\infty}^0 = \frac{1}{2+\mathrm{i}k} = -\frac{\mathrm{i}}{k-2\mathrm{i}}. \quad \boxed{1}$$

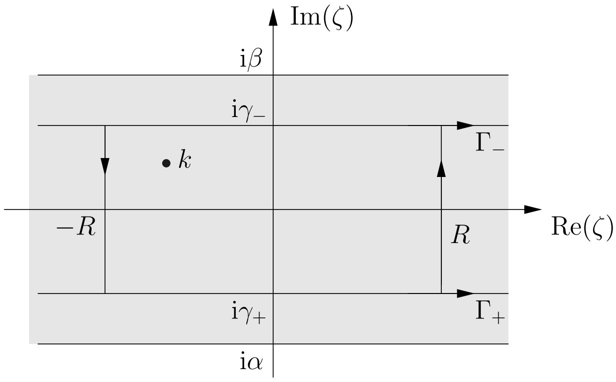
It may be analytically continued to $\mathbb{C}\backslash\{2\mathrm{i}\}$. $\boxed{1}$

□ **Bookwork/Straightforward calculation**

(b) [5 marks] By Cauchy

$$G(k) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{G(\zeta)}{\zeta - k}\,\mathrm{d}\zeta, \quad \boxed{1}$$

where $\Gamma \subset \Omega$ is the rectangular contour with sides $\mathrm{Re}(\zeta) = \pm R$, $\mathrm{Im}(\zeta) = \gamma_\pm$ where $-R < \mathrm{Re}(k) < R$, $\alpha < \gamma_+ < \mathrm{Im}(k) < \gamma_- < \beta$. $\boxed{1}$



Since $G(\zeta) \to 0$ as $|\zeta| \to \infty$ in Ω the contribution from the vertical sides $\to 0$ as $R \to \infty$, [1] giving $G(k) = G_+(k) - G_-(k)$ for $\gamma_+ < \mathrm{Im}(k) < \gamma_-$ where

$$G_\pm(k) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\pm} \frac{G(\zeta)}{\zeta - k}\, \mathrm{d}\zeta,$$

and $\Gamma_\pm = \{\xi + \mathrm{i}\gamma_\pm : -\infty < \xi < \infty\}$. [1] Note that Γ_+ passes underneath k, while Γ_- passes above k. Since $G_\pm(k)$ is holomorphic everywhere except on $\Gamma_\pm$, we deduce that $G_+(k)$ is holomorphic in $\mathrm{Im}(k) > \gamma_+$ and $G_-(k)$ is holomorphic in $\mathrm{Im}(k) < \gamma_-$. [1]

□ **Bookwork**

(c) [5 marks] Put the branch cuts up the the imaginary axis from i and down the imaginary axis from −i, so that $-3\pi/2 < \arg(k - \mathrm{i}) < \pi/2$ and $-\pi/2 < \arg(k + \mathrm{i}) < 3\pi/2$. Then $-\pi/2 < \arg(k - \mathrm{i})^{1/2} < 0$ and $0 < \arg(k + \mathrm{i})^{1/2} < \pi/2$ for $-1 < \mathrm{Im}(k) < 1$. [2]

$$G_+(k) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_+} \frac{(\zeta + \mathrm{i})^{1/2}}{(\zeta - 2\mathrm{i})(\zeta - k)}\, \mathrm{d}\zeta, \qquad -1 < \gamma_+ < \mathrm{Im}(k)$$

Since there is a branch point at $\zeta = -\mathrm{i}$ close at $\mathrm{i}\infty$ to give $-2\pi\mathrm{i}\times$ residue at $\zeta = 2\mathrm{i}$ and the residue at $\zeta = k$. [1] Thus

$$G_+(k) = \frac{(3\mathrm{i})^{1/2}}{k - 2\mathrm{i}} - \frac{(k + \mathrm{i})^{1/2}}{k - 2\mathrm{i}}. \quad \boxed{1}$$

$$G_-(k) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_-} \frac{(\zeta + \mathrm{i})^{1/2}}{(\zeta - 2\mathrm{i})(\zeta - k)}\, \mathrm{d}\zeta, \qquad \mathrm{Im}(k) < \gamma_- < 2$$

Again close at $\mathrm{i}\infty$ to give now only the contribution from $\zeta = 2\mathrm{i}$,

$$G_-(k) = \frac{(3\mathrm{i})^{1/2}}{k - 2\mathrm{i}}. \quad \boxed{1}$$

□ **Bookwork/standard calculation.**

(d) [5 marks] Define

$$u_-(x) = \begin{cases} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(k)\, \mathrm{d}k & x < 0, \\ 0 & x > 0 \end{cases}, \quad \boxed{1}$$

$$v_+(x) = \begin{cases} 0 & x > 0, \\ \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} (k^2 + 1)^{-1/2} f(k)\, \mathrm{d}k & x > 0. \end{cases} \quad \boxed{1}$$

 Turn Over

Then, with $g_-(x)$ as defined in (a),

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(k)\ \mathrm{d}k = u_-(x),$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} (k^2+1)^{-1/2} f(k)\ \mathrm{d}k = v_+(x) + g_-(x),$$

on $-\infty < x < \infty$ as required.
Taking a Fourier transform, and using the Fourier inversion theorem,

$$\bar{u}_- = 2\pi f, \quad \boxed{1} \qquad \bar{v}_+ + \bar{g}_- = \frac{2\pi f}{(k^2+1)^{1/2}}. \quad \boxed{1}$$

Eliminating f, and using (a) gives

$$\frac{\bar{u}_-}{(k^2+1)^{1/2}} = \bar{v}_+ - \frac{\mathrm{i}}{k-2\mathrm{i}}. \quad \boxed{1}$$

□ **Standard approach, but coupled equations is new. Will require some thought.**

(e) [7 marks] Performing the product decomposition

$$\frac{\bar{u}_-}{(k-\mathrm{i})^{1/2}} = (k+\mathrm{i})^{1/2}\bar{v}_+ - \frac{\mathrm{i}(k+\mathrm{i})^{1/2}}{(k-2\mathrm{i})}. \quad \boxed{1}$$

Now performing the sum decomposition of the last term using (c) gives

$$(k-\mathrm{i})^{-1/2}\bar{u}_- + \frac{\mathrm{i}(3\mathrm{i})^{1/2}}{(k-2\mathrm{i})} = \bar{v}_+(k+\mathrm{i})^{1/2} - \frac{\mathrm{i}\left((k+\mathrm{i})^{1/2} - (3\mathrm{i})^{1/2}\right)}{(k-2\mathrm{i})} = E(k), \quad \boxed{1}$$

say. The LHS is holomorphic in $\mathrm{Im}(k) < 1$, while the RHS is holomorphic in $\mathrm{Im}(k) > -1$. Thus together they define an entire function $E(k)$. $\boxed{1}$ Now $\bar{u}_- = 2\pi f$ is bounded as $k \to \pm\infty$. Assuming $f(k)$ is bounded as $|k| \to \infty$ we have $E \to 0$ as $|k| \to \infty$, so that by Liouville $E \equiv 0$. $\boxed{1}$ Thus

$$\bar{u}_- = 2\pi f = -\frac{3^{1/2}\mathrm{e}^{3\mathrm{i}\pi/4}(k-\mathrm{i})^{1/2}}{(k-2\mathrm{i})}.$$

To verify directly that

$$-\frac{3^{1/2}\mathrm{e}^{3\mathrm{i}\pi/4}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\frac{(k-\mathrm{i})^{1/2}}{(k-2\mathrm{i})}\ \mathrm{d}k = 0$$

when $x > 0$ simply close the contour in the lower half plane. $\boxed{1}$ To evalute

$$-\frac{3^{1/2}\mathrm{e}^{3\mathrm{i}\pi/4}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}(k^2+1)^{-1/2}\frac{(k-\mathrm{i})^{1/2}}{(k-2\mathrm{i})}\ \mathrm{d}k = -\frac{3^{1/2}\mathrm{e}^{3\mathrm{i}\pi/4}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\frac{1}{(k+\mathrm{i})^{1/2}(k-2\mathrm{i})}\ \mathrm{d}k$$

for $x < 0$ close the contour in the upper half plane $\boxed{1}$ to pick up $2\pi\mathrm{i}\times$ the residue at $k = 2\mathrm{i}$, $\boxed{1}$ which is

$$2\pi\mathrm{i}\left(-\frac{3^{1/2}\mathrm{e}^{3\mathrm{i}\pi/4}}{2\pi}\mathrm{e}^{2x}\frac{1}{(3\mathrm{i})^{1/2}}\right) = \mathrm{e}^{2x}.$$

□ **New example.**