Final Honour School of Mathematics Part B

B2.2 Commutative Algebra
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Checked by:

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dd/mm/yyyy

Do not turn this page until you are told that you may do so

All rings are commutative, associative, and unital; ring homomorphisms preserve the units.

1. *Throughout this question, standard results about posets and nilradicals may be used without proof, provided that they are clearly stated.*

   Let R be a ring.

   (a) [4 marks] Prove that every proper ideal of R is contained in a prime ideal.

   (b) [3 marks] Define the spectrum $\mathrm{Spec}(R)$. Show that $\mathrm{Spec}(R) = \varnothing$ if and only if R is the zero ring.

   (c) [6 marks] Define the Zarisky topology on $\mathrm{Spec}(R)$ (you do *not* need to check that it is a topology). Show that $\mathrm{Spec}(R)$ contains at most one dense point, and that it contains a dense point if and only if the nilradical is prime. (By a *dense point* we mean a point x such that $\{x\}$ is a dense subset.)

   (d) [7 marks] Justifying your answer briefly, give an example of a ring whose spectrum

   (i) is infinite countable;

   (ii) is uncountable;

   (iii) has exactly n elements for a given positive natural number n.

   (e) [5 marks] Let S and T be non-zero rings, and let $\phi\colon R \to S \times T$ be a surjective ring homomorphism. Show that R cannot be a local ring.

2. Let R be a subring of a ring S.

   (a) [2 marks] When is an element of S *integral* over R?

   (b) [5 marks] For $s \in S$, state (without proof) the relationship between finite generation of subrings of S containing R and s, and integrality of s over R.

   Define the subring $R[s]$ of S. Prove that when s is integral over R then $R[s]$ is finitely generated as an R-module.

   (c) [5 marks] Show that the set of elements of S integral over R forms a subring of S. (You may use the result stated in (b).)

   (d) [3 marks] State (without proof) the Going-Up Theorem.

   (e) [10 marks] Suppose that all elements of S are integral over R. Let $\phi\colon R \to K$ be a surjective ring homomorphism to an algebraically closed field. Show that there exists a ring homomorphism $\psi\colon S \to K$ such that $\psi(r) = \phi(r)$ for every $r \in R$.

3. Let R be a ring.

   (a) [3 marks] Give the definition of R being *noetherian*.

   (b) [7 marks] Let R be noetherian, and let M be a finitely generated R-module. Prove that all submodules of M are finitely generated.

   (c) [3 marks] State (without proof) Krull's theorem describing $\bigcap_n I^n M$ for an ideal I of a noetherian ring R and a finitely generated module M.

   (d) [8 marks] Let V be a non-zero finite-dimensional vector space over a field K, and let A be a linear transformation $A\colon V \to V$. Show that if $\bigcap_n \mathrm{im}(A^n) \neq \{0\}$ then there is some polynomial $p(x) \in K[x]$ with constant term zero such that 1 is an eigenvalue of $p(A)$. (*Noetherianity of standard rings should be briefly justified.*)

   (e) [4 marks] Show that the previous statement fails if V is not finite dimensional.