Lie groups C3.5. HT 26 [P. Bousseau]

Lecture 14. [06/03/2026]

Thm (Weyl integration formula) $f \in \mathcal{C}(G)$ class function.

$$\int_G f = \frac{1}{|W|} \int_{t \in T} \det\left((\mathrm{Ad}(t^{-1}) - I)|_{\mathfrak{t}^\perp} \right) f(t)$$

Proof (Sketch) $F : G/T \times T \to G$ $\quad |W| : 1$ generically

$(g, t) \mapsto g t g^{-1}$

$T_{gT}(G/T) = \mathfrak{t}^\perp \quad \det dF_{(gT, t)} = \det\left((\mathrm{Ad}(t^{-1}) - I)|_{\mathfrak{t}^\perp} \right)$

$f \in \mathcal{C}(G) \qquad f \circ F : G/T \times T \to G$

$(g, t) \mapsto f(g t g^{-1}) = f(t)$

Change of variables formula: Jacobian. $\square$

Rem: $\mathfrak{t}^\perp = \bigoplus_a \mathfrak{g}_a \qquad 2\pi\theta_a$

$$\det\left((\mathrm{Ad}(t^{-1}) - I)|_{\mathfrak{t}^\perp} \right) = \prod_a \left(e^{-2i\pi\theta_a} - 1\right)\left(e^{2i\pi\theta_a} - 1\right)$$

$$= \prod_a 2\left(1 - \cos(2\pi\theta_a)\right)$$

Ex: $G = SU(2)$

$$T = \left\{ \begin{pmatrix} e^{2i\pi x} & 0 \\ 0 & e^{-2i\pi x} \end{pmatrix} : x \in \mathbb{R} \right\} \qquad W = \mathbb{Z}/2\mathbb{Z} \qquad |W| = 2$$

Roots are $\pm\theta_a = \pm 2x$ $\quad [x_1 - x_2 \ , \ x_1 + x_2 = 0 \Rightarrow 2x_1]$

$f \in \mathcal{C}(SU(2))$

$$\int_{SU(2)} f = \frac{1}{2}\int_0^1 2(1-\cos(4\pi x))\, f(e^{2i\pi x})\,dx$$

$$= \frac{1}{2}\int_0^1 4\sin^2(2\pi x)\, f(e^{2i\pi x})\,dx$$

$$= \frac{1}{2}\int_0^1 4\left(\frac{e^{2i\pi x} - e^{-2i\pi x}}{2i}\right)^2 f(e^{2i\pi x})\,dx$$

$t = e^{2i\pi x}$

$dt = 2i\pi e^{2i\pi x}\,dx$ $\qquad = -\frac{1}{2}\int_{S^1} (t - t^{-1})^2 \frac{f(t)}{2i\pi t}\,dt$

$\frac{1}{2i\pi}\frac{dt}{t} = dx$ $\qquad = -\frac{1}{2}\int_{S^1} (1-t^2)^2 \frac{f(t)}{2i\pi t^3}\,dt$

Take V_n = irrep of $SU(2)$ & consider $f = \chi_{V_m}\bar{\chi}_{V_n} \in \mathcal{C}(SU(2))$

$$\Rightarrow \int_{SU(2)} f = \langle \chi_{V_m}, \chi_{V_n} \rangle$$

<u>Note:</u> $\overline{\chi_{V_n}(t)} = \chi_{V_n}(t)$ $\quad [\mathrm{Inv}t/\ t \to t^{-1} = \bar{t} \quad \forall t \in S^1]$

$$\Rightarrow f(t) = \chi_{V_m}(t)\,\chi_{V_n}(t) = \frac{(1-t^{2m+2})(1-t^{2n+2})}{t^{m+n}(1-t^2)^2}$$

$$\Rightarrow \langle \chi_{V_m}, \chi_{V_n} \rangle = -\frac{1}{2} \int_{S^1} \frac{1}{2i\pi t^{m+n+3}} \left(1 - t^{2m+2}\right)\left(1 - t^{2n+2}\right) dt$$

$$= -\frac{1}{2} \operatorname{Res}\left\{ \frac{1 - z^{2m+2} - z^{2n+2} + z^{2m+2n+4}}{z^{m+n+3}} : z = 0 \right\}$$

$$= -\frac{1}{2} \operatorname{Res}\left\{ \frac{1}{z^{m+n+3}} - z^{m-n-1} - z^{n-m-1} + z^{m+n+1} \right\}$$

$$= \delta_{m,n}$$ Orthogonality of characters.

End of the term feedback form

Simple Lie groups.

G n-dim Lie group $\Rightarrow$ Lie algebra $\mathfrak{g} \simeq \mathbb{R}^n$

Suppose $\langle -, - \rangle$ is an Ad-invt inner product on $\mathfrak{g}$ (always exist if G compact)

$$\langle \mathrm{Ad}(g) X, \mathrm{Ad}(g) Y \rangle = \langle X, Y \rangle \quad \forall g \in G, \; X, Y \in \mathfrak{g}$$

$$\mathrm{Ad} : G \longrightarrow O(n)$$

$\Big\downarrow$ differentiate

$$\langle \mathrm{ad}(Z) X, Y \rangle = -\langle X, \mathrm{ad}(Z) Y \rangle \quad \forall X, Y, Z \in \mathfrak{g}$$

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{o}(n) = \{ A \in M(n, \mathbb{R}) \mid A^T = -A \}$$

$A, B \in \mathfrak{o}(n) \rightsquigarrow (A,B) = \mathrm{tr}(AB)$ symmetric bilinear pairing

Invt/ conjugation by $O(n)$

and $\underbrace{\mathrm{tr}(A^2)}_{``(A,A)"} = -\mathrm{tr}(A^T A) \leq 0$ and $= 0 \Leftrightarrow A = 0.$

<u>Def</u>: G Lie group, $\mathfrak{g}$ Lie algebra. <u>Killing form</u> on $\mathfrak{g}$:

$$(X,Y) := \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)) \quad \forall X, Y \in \mathfrak{g}$$

<u>Note</u>: $(\cdot,\cdot)$ is Ad-invariant, symmetric bilinear.

<u>Lemma</u>: G compact $\Rightarrow (X,X) \leq 0 \ \forall X$

and $(X,X) = 0 \Leftrightarrow \mathrm{ad}(X) = 0 \Leftrightarrow X \in Z(\mathfrak{g})$

<u>Ex</u>: $G = SO(n)$: $Z(\mathfrak{g}) = 0 \Rightarrow$ Killing form is negative definite

<u>Ex</u>: $G = U(n)$: $Z(\mathfrak{g}) = \mathbb{R} \Rightarrow$ ——— is <u>not</u> ———

<u>Ex</u>: $G = SU(n)$: $Z(\mathfrak{g}) = 0 \Rightarrow$ ——— <u>is</u> ———

<u>Ex</u>: $G = SL(2,\mathbb{R}) \Rightarrow$ Killing form has mixed signature.

<u>Cor</u>: If G is compact, $\mathfrak{g} = Z(\mathfrak{g}) \oplus Z(\mathfrak{g})^\perp$ with Killing form negative definite on $Z(\mathfrak{g})^\perp$.

Def: $\mathfrak{g}$ a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if

$\forall X \in \mathfrak{g},\ Y \in \mathfrak{h},\ [X,Y] \in \mathfrak{h}$.

Rem: Ideal $\Rightarrow$ Lie subalgebra.

Def: $\mathfrak{g}$ simple if $\mathfrak{g}$ non-abelian & $\mathfrak{h} \subset \mathfrak{g}$ ideal $\Rightarrow \mathfrak{h} = \{0\}$ or $\mathfrak{g}$

$\mathfrak{g}$ semi-simple if $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ $\quad \mathfrak{g}_i$ simple.

Def: G connected Lie group is simple if G non-abelian and $H \subset G$ connected normal Lie subgroup $\Rightarrow H = \{e\}$ or $H = G$.

G connected Lie group is semi-simple if $\mathfrak{g} = \mathrm{Lie}(G)$ semi-simple.

Ex: $SU(n)$ has $Z(SU(n)) = \mathbb{Z}/n\mathbb{Z}$ normal Lie subgroup but (as we will see) $SU(n)$ is simple ($\mathbb{Z}/n\mathbb{Z}$ disconnected!)