Lie groups. C3.5. HT26 [P. Boussean]

Lecture 15 [08/03/2026]

Recall from last time:

$\mathfrak{g}$ Lie algebra <u>simple</u> if $\left|\begin{array}{l} \mathfrak{g} \text{ non-abelian and} \\ \forall \mathfrak{h} \subset \mathfrak{g} \text{ ideal: } \mathfrak{h} = \{0\} \text{ or } \mathfrak{h} = \{\mathfrak{g}\} \end{array}\right.$

$\mathfrak{g}$ Lie algebra <u>semi-simple</u> if $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ ← simple.

G Lie group <u>simple</u> if $\left|\begin{array}{l} \forall H \subset G \text{ connected normal Lie subgroup} \quad H = \{e\} \text{ or } H = G \\ G \text{ non-abelian} \end{array}\right.$

(G connected)

G <u>semi-simple</u> if $\mathfrak{g}$ is semi-simple.

<u>Lemma:</u> G simple $\Longleftrightarrow$ $\mathfrak{g}$ simple.

<u>Proof:</u> <u>PS</u> sheet $\mathfrak{h} \subset \mathfrak{g}$ ideal $\Longleftrightarrow$ $H \subset G$ connected normal subgroup

$\mathfrak{g}$ non-abelian $\Longleftrightarrow$ G non-abelian. $\square$

Warning: G semi-simple $\not\Longrightarrow$ $G = \prod_i G_i$ ← simple.

<u>Ex:</u> $\mathbb{Z}/n\mathbb{Z} = Z(SU(n)) \xhookrightarrow[\text{diagonal}]{} SU(n) \times SU(n)$

$$G = (SU(n) \times SU(n)) / (\mathbb{Z}/n\mathbb{Z})$$

$\mathfrak{g} = \mathfrak{su}(n) \oplus \mathfrak{su}(n)$ but can show G not a product.

But True for G simply connected.

$\mathfrak{g}$ Lie algebra? How to determine if $\mathfrak{g}$ is semi-simple or not?

<u>Thm:</u> $\mathfrak{g}$ semi-simple $\iff$ The Killing form $(-,-)$ is non-degenerate
(no proof)

($(X,Y) := tr(ad_X \, ad_Y)$ ad-invt symmetry bilinear form)

$\forall X \in \mathfrak{g} \setminus \{0\}$, $\exists Y \in \mathfrak{g}$ s.t. $(X, Y) \neq 0$.

<u>Cor:</u> • If $\mathfrak{g} = \text{Lie}(G)$ G compact connected Lie group.

$\mathfrak{g}$ semi-simple $\iff$ $Z(\mathfrak{g}) = 0$.

• G connected compact Lie group.

G semi-simple $\iff$ $\dim Z(\mathfrak{g}) = 0$.

<u>Proof:</u>

$$\mathfrak{g} = \underbrace{Z(\mathfrak{g})}_{(-,-)=0} \oplus \underbrace{Z(\mathfrak{g})^{\perp}}_{(-,-)<0}$$

so non-degenerate. $\square$

<u>Ex:</u> • $U(n)$ not semi-simple (compact)

• $SU(n)$, $SO(n)$ simple (compact)

• $SL(n, \mathbb{R})$ simple (non-compact)

• $\mathbb{R}^n$ not semi-simple (non-compact)

Thm [Classification of compact simple simply connected Lie groups]
A compact simple simply connected Lie group is exactly one of the following:

a) $Sp(n)$ $\quad n \geq 1 \quad$ $\dim = n(2n+1)$ $\quad$ rank $= n$
$\quad \shortparallel \{A \in GL(n, \mathbb{H}) \mid \bar{A}^T A = I\}$

b) $SU(n)$ $\quad n \geq 3$
$\quad \shortparallel \{A \in GL(n, \mathbb{C}) \mid \bar{A}^T A = I, \det A = 1\}$
$\dim = n^2 - 1 \quad$ rank $= n-1$

c) $Spin(n) =$ double cover of $\quad n \geq 7$
$SO(n) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I, \det A = 1\}$

d) Exceptional cases:

G_2 $\quad \dim = 14 \quad rk = 2$

F_4 $\quad \dim = 52 \quad rk = 4$

E_6 $\quad \dim = 78 \quad rk = 6$

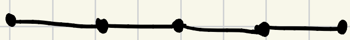
E_7 $\quad \dim = 133 \quad rk = 7$

E_8 $\quad \dim = 248 \quad rk = 8$

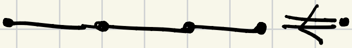
Low dimensions:

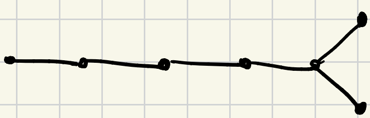
- $\dim = 1$ $\quad SO(2) = S^1 = U(1)$? abelian not simple
- $\dim = 2$ $\quad T^2 = S^1 \times S^1$? abelian not simple
- $\dim = 3$ $\quad SU(2) = \mathrm{Spin}(3) = Sp(1) \simeq S^3$
- $\mathrm{Spin}(4) = SU(2) \times SU(2)$ semi-simple not simple.
- $\mathrm{Spin}(5) = Sp(2)$
- $\mathrm{Spin}(6) = SU(4)$

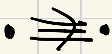
Classification via Dynkin diagram:

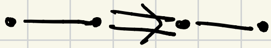
A_n  n dots $\leftrightarrow SU(n+1)$

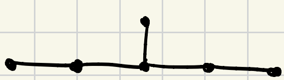
B_n  $\mathrm{Spin}(2n+1)$

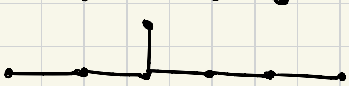
C_n  $Sp(n)$

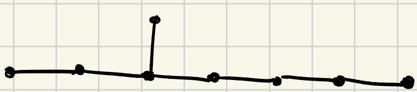
D_n  $\mathrm{Spin}(2n)$

G_2 

F_4 

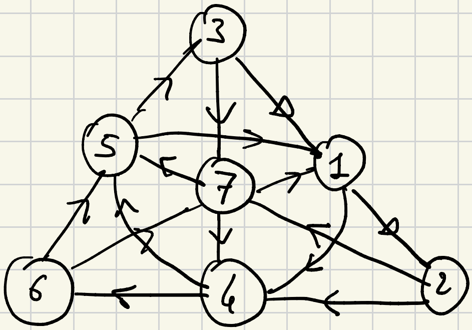
E_6 

E_7 

E_8 

Exceptional groups are related to octonions:

$$\mathbb{O} = \mathrm{Span}\{1, e_1, \dots, e_7\} \simeq \mathbb{R}^8 \qquad e_i^2 = -1$$



$e_3 \times e_1 = e_2$

$e_2 \times e_4 = e_6$

$e_6 \times e_5 = e_3$

$e_3 \times e_7 = e_4$

$e_6 \times e_7 = e_1$

$e_2 \times e_7 = e_5$

$e_1 \times e_4 = e_5$

+ Cyclic permutations.

$$e_1 \times (e_2 \times e_4) = e_1 \times e_6 = e_7$$

$$(e_1 \times e_2) \times e_4 = e_3 \times e_4 = -e_7$$

Not associative!

Facts: F_4 = Isometry group of a 16-dim Riem manifold $\mathbb{O}P^2$

Weyl group = symmetry group of 24-cell

$$|W| = 1152$$

E_6, E_7, E_8

$|W_{E_6}| = 51840 \qquad |W_{E_7}| = 290340 \qquad |W_{E_8}| = 2^{14} 3^5 5^2 7$

$= 696\,729\,600$