Lie groups. C3.5. HT26. [P. Boutleau]

Lecture 16 [13/03/2026]

G_2 $\qquad \mathbb{R}^7$ $\langle -,- \rangle$ Euclidian inner product

$\{e_1, \dots, e_7\}$ oriented orthonormal basis of $\mathbb{R}^7 \Rightarrow \mathbb{R}^7 \simeq \operatorname{Im} \mathbb{O}$

$\rightsquigarrow$ Cross-product $\times$ on $\mathbb{R}^7$ $\qquad \alpha \times \beta := \alpha \cdot \beta |_{\mathbb{R}^7}$

$\alpha \times \beta = -\beta \times \alpha \qquad \alpha \times \beta \perp \alpha, \beta \qquad$ Eg: $e_3 \times e_1 = e_2$

Def: $\varphi \in \Lambda^3 (\mathbb{R}^7)^* \qquad \varphi(u,v,w) = \langle u, v \times w \rangle \qquad \forall u,v,w \in \mathbb{R}^7$

totally skew-symmetric

Let $e^1, \dots, e^7$ be the dual basis.

Let $e^{ij\dots k} = e^i \wedge e^j \wedge \dots \wedge e^k$

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{356} - e^{347}$$

Fact: φ determines $\langle -,- \rangle$ and hence $\times$.

Proof: $\forall u, v \in \mathbb{R}^7 \qquad \underbrace{i_u \varphi}_{2} \wedge \underbrace{i_v \varphi}_{2} \wedge \underbrace{\varphi}_{3} = 6 \langle u,v \rangle e^{12\dots 7}$

$\square$

Induced rep ρ of $GL(7,\mathbb{R})$ on $\Lambda^3 (\mathbb{R}^7)^*$ is: $\qquad \forall A \in GL(7,\mathbb{R})$

$$(\rho(A)\varphi)(u,v,w) = \varphi(Au, Av, Aw) \qquad \forall u,v,w \in \mathbb{R}^7$$

Derivative σ of ρ at I: $\qquad \forall X \in \mathfrak{gl}(7,\mathbb{R})$

$$\sigma(X)\varphi(u,v,w) = \varphi(Xu,v,w) + \varphi(u,Xv,w) + \varphi(u,v,Xw)$$

Def: $G_2 = \{A \in GL(7,\mathbb{R}) \mid \rho(A)\varphi = \varphi\} = \mathrm{Stab}(\varphi)$

$\to G_2 = \mathrm{Aut}(\mathrm{Im}\,\mathbb{O}) = \mathrm{Aut}(\mathbb{O})$

$\mathfrak{g}_2 = \{X \in \mathfrak{gl}(7,\mathbb{R}) = M(7,\mathbb{R}) \mid \sigma(X)\varphi = 0\}$

G_2 group. Claim (without proof): G_2 connected

Prop: G_2 is a compact embedded Lie subgroup of $SO(7)$.

Proof: $A \in G_2 \Rightarrow A$ preserves $\varphi \Rightarrow A$ preserves $\langle -,-\rangle$

$\Rightarrow A \in O(7)$ connected $G_2 \subset SO(7)$

G_2 closed (stabilizer) & $SO(7)$ compact $\Rightarrow G_2$ compact embedded Lie subgroup. $\square$

Prop: Let $SO(6) = \mathrm{Stab}(e_1) \subset SO(7)$ $\qquad \mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$

Let $\mathbb{C}^3 = \mathrm{Span}\{e_2 + ie_3, e_4 + ie_5, e_6 + ie_7\}$

$\Rightarrow SO(6) \cap G_2 = SU(3)$.

Proof: Let $\omega = e^{23} + e^{45} + e^{67}$

$$\Omega = (e^2 + ie^3) \wedge (e^4 + ie^5) \wedge (e^6 + ie^7)$$

$\Rightarrow \varphi = e^1 \wedge \omega + \mathrm{Re}\,\Omega$

J multiplication by i on $\mathbb{C}^3$: $\omega(u,v) = \langle Ju, v\rangle$

$A \in SO(6) \cap G_2 \Rightarrow A$ preserves ω and $\mathrm{Re}\,\Omega$

A preserves ω $\Rightarrow$ A preserves J $\Leftrightarrow$ $A \in U(3)$
| also $\langle -,- \rangle$

$A \in U(3)$ $\quad \rho(A)\Omega = \det(A)\Omega$
| A preserves $\operatorname{Re}\Omega \Leftrightarrow \det(A) = 1.$

$\square$

<u>Prop:</u> $\mathfrak{g}_2 = \mathbb{R}^6 \oplus \mathfrak{su}(3)$ $\quad$ In particular: $\dim G_2 = 14.$

<u>Proof:</u> Let $E_{ij} \in \mathfrak{so}(7)$ $\quad (E_{ij})_{kl} = \begin{cases} 1 & (k,l) = (i,j) \\ -1 & (k,l) = (j,i) \\ 0 & \text{else} \end{cases}$

$$\sigma(E_{12})\varphi = e^{245} + e^{267} - e^{146} + e^{157}$$

Send $1 \to 2$, $2 \to 1$ $\quad \sigma(E_{45} + E_{67})\varphi = 2(e^{146} - e^{157} - e^{245} - e^{267})$

so $2E_{12} + E_{45} + E_{67} \in \mathfrak{g}_2$

Similarly $E_{1j} \to$ unique element in $\mathfrak{g}_2$ $\forall\, 2 \leq j \leq 7$

$$\Rightarrow \mathbb{R}^6 \subset \mathfrak{g}_2$$

Restrict to $\operatorname{span}\{E_{ij} \mid i,j \in \{2,\dots,7\}\} = \mathfrak{so}(6) = \operatorname{Lie}(SO(6))$

$SO(6) \cap G_2 = SU(3) \Rightarrow$ Result.

$\square$

$$\operatorname{Orb}(\varphi) = GL(7,\mathbb{R}) / \operatorname{Stab}(\varphi) = GL(7,\mathbb{R})/G_2$$

$$\dim \operatorname{Orb}(\varphi) = \dim GL(7,\mathbb{R}) - \dim G_2 = 49 - 14 = 35 = \binom{7}{3}$$

$= \dim \Lambda^3(\mathbb{R}^7)^*$ $\quad \operatorname{Orb}(\varphi)$ open in $\Lambda^3(\mathbb{R}^7)^*$

$\forall \gamma \in \mathrm{Orb}(\varphi)$, $\mathrm{Stab}(\gamma) \simeq G_2$.

<u>Lemma</u>: $\mathrm{rk}(G_2) = 2$. Roots are $\pm(x_1 - x_2)$ $\pm(x_2 - x_3)$ $\pm(x_3 - x_1)$

$\pm x_1$ $\pm x_2$ $\pm x_3$

with $x_1 + x_2 + x_3 = 0$.

<u>Proof</u>: T max torus in $SU(3)$

$T \subset G_2$ $\quad \mathfrak{g}_2 = \underset{\underset{\mathbb{C}^3}{\wr\wr}}{\mathbb{R}^6} \oplus \mathfrak{su}(3)$ weights of T acting on $\mathbb{R}^6$: $\pm x_1, \pm x_2, \pm x_3$

$$\mathfrak{g}_2 = \mathfrak{t} \oplus \mathbb{C}^3 \oplus \bigoplus \mathfrak{g}_a$$

$\Rightarrow$ T max torus in G_2 & Roots ... $\square$

<u>Lem</u>: $W(G_2) = D_6$.

<u>Proof</u>: $W \hookrightarrow O(2)$ generated / 6 root hyperplanes

$\Rightarrow$ Symmetries of regular hexagon $\Rightarrow D_6$. $\square$

<u>Prop</u>: G_2 is simple.

<u>Proof</u>: $\mathfrak{h} \subset \mathfrak{g}_2$ proper ideal $\Rightarrow \mathfrak{h}, \mathfrak{h}^\perp$ proper ideals & rep of $SU(3)$

$$\underset{\underset{SU(3) \text{ simple}}{\uparrow}}{\Longrightarrow} \quad \mathfrak{h}, \mathfrak{h}^\perp = \mathfrak{su}(3) \text{ or } \underbrace{\mathbb{R}^6}_{\text{not a Lie subalgebra}}. \quad \square$$

<u>Claim</u> (no proof) G_2 is simply connected.

Prop: $R(G_2) = \mathbb{Z}[\sigma_1+\sigma_2,\ \sigma_1\sigma_2]$ $\quad \sigma_1 = \cdots \quad \sigma_2 = \cdots \quad t_1, t_2, t_3 \quad t_1t_2t_3 = 1$

Proof:

$$R(G_2) \hookrightarrow R(T)^W$$

$$R(T)^{S_3} = R(SU(3)) = \mathbb{Z}[\sigma_1, \sigma_2]$$

W contains an extra reflection : $t_j \longmapsto t_j^{-1}$

so $\sigma_1 = t_1 + t_2 + t_3$

$$\longmapsto t_1^{-1} + t_2^{-1} + t_3^{-1} \underset{\substack{\uparrow \\ t_1t_2t_3 = 1.}}{=} \sigma_2$$

$\square$

$$\Rightarrow R(T)^W = \mathbb{Z}[\sigma_1+\sigma_2,\ \sigma_1\sigma_2]$$

Standard rep of G_2 on $\mathbb{R}^7 \quad \otimes \mathbb{C}$:

$V_7 = \mathbb{R}^7 \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}^3 \oplus \overline{\mathbb{C}^3}$ as irrep of $SU(3)$

$$\chi_{V_7}|_T = 1 + t_1 + t_2 + t_3 + t_1^{-1} + t_2^{-1} + t_3^{-1} = 1 + \sigma_1 + \sigma_2$$

$$V_{14} = \mathfrak{g}_2 \otimes \mathbb{C} = \mathbb{C}^3 \oplus (\mathfrak{su}(3) \otimes \mathbb{C}) \oplus \overline{\mathbb{C}}^3$$

$$\chi_{V_{14}}|_T = t_1 + t_2 + t_3 + t_1^{-1} + t_2^{-1} + t_3^{-1} + \chi_{SU(3)}|_T = \sigma_1 + \sigma_2 + \sigma_1\sigma_2 - 1.$$

$\square$

$$su(3) \otimes \mathbb{C} = \mathbb{C}^2 \oplus \bigoplus \mathfrak{g}_\alpha$$

$$\chi = 2 + t_1 t_2^{-1} + t_1^{-1} t_2 + t_2 t_3^{-1} + t_1^{-1} t_3 + t_2 t_3^{-1} + t_2^{-1} t_3$$

$$\| $$

$$6_1 6_2 - 1.$$

$$6_1 6_2 = (t_1 + t_2 + t_3)(t_1^{-1} + t_2^{-1} + t_3^{-1})$$

$$= \cdots\cdots + 3$$