2. Delay Models

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t-T)}{K}\right)$$

linear growth with percapita growthrate r

delayed feedback (accounting for competition) with delay T, K is carrying cap.

Linear Stab analysis $\rightarrow$ transcendal eqn.

- Oscillatory solutions - critical values of T.

sol^{ns} $e^{\lambda t}$

Put $\lambda = \mu + i\omega$
equate real & imaginary parts.
want to find under what cond^{ns}
$\mu > 0$.

3. Structured Models.

Von-Foerster's Eqn: $dn(t,a) = \frac{\partial n}{\partial t}\delta t + \frac{\partial n}{\partial a}\delta a = -\mu(a)\, n(t,a)\,\delta t$

Conservation of mass

$$\Rightarrow \frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(a)\, n \qquad \text{since } \frac{\delta a}{\delta t} = 1.$$

I.C. $n(0,a) = f(a)$

$$n(t,0) = \int_0^\infty b(a)\, n(t,a)\, da$$

Characteristics: $n(t,a) = \hat{n}(s,r)$

$\frac{\partial t}{\partial s} = 1 \Rightarrow t = s + A(r)$

$\frac{\partial a}{\partial s} = 1 \Rightarrow a = s + B(r)$

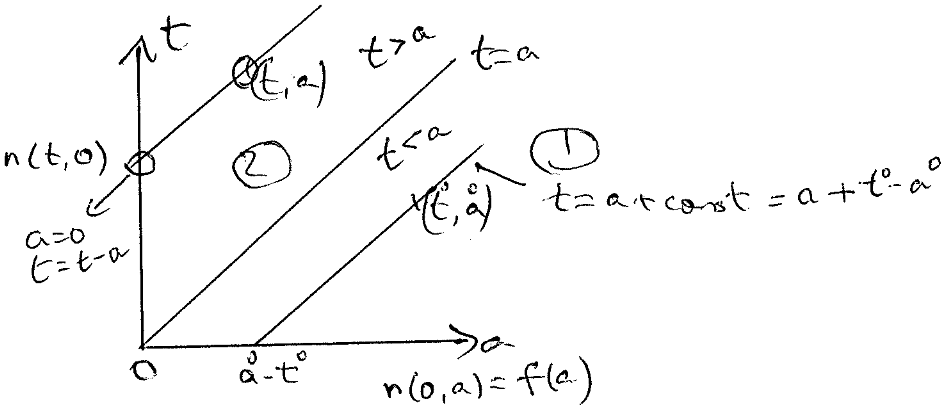
$\frac{\partial \hat{n}}{\partial s} = -\mu(a)\hat{n}$

$\frac{da}{dt} = 1$, on which $\frac{dn}{dt} = -\mu(a) n$

$a = t + \text{const.}$

$0 \le t < a$: $s = 0,\ t = 0 \Rightarrow t = s$

$a = r \quad \therefore a = s + r$



$$\therefore \frac{dn}{n} = -\mu(a)\,dt = -\mu(a)\,da \qquad \text{since } da = dt$$

$$\ln\left(\frac{n(t,a)}{n(0,a-t)}\right) = -\int_{a-t}^{a} \mu(\theta)\, d\theta$$

$$\therefore n(t,a) = n(0,a-t)\, e^{-\int_{a-t}^{a} \mu(\theta)\, d\theta}$$

$$= f(a-t)\, e^{-\int_{a-t}^{a} \mu(\theta)\, d\theta}$$

$t < a$ so this is saying that people born $a-t$ ago, die.

2. $t > a$

$$\ln\left(\frac{n(t,a)}{n(t-a,0)}\right) = -\int_0^a \mu(\theta)\,d\theta$$

$$\therefore\ n(t,a) = n(t-a,0)\, e^{-\int_0^a \mu(\theta)d\theta}$$

$$\cancel{n(t-a,0) = \int_0^\infty b(a)\, n(t-a,a)\, da}$$

$$\cancel{= \int_0^t b(a)}$$

$$n(t,0) = \int_0^t \underbrace{b(a)\, n(t-a,0)\, e^{-\int_0^a \mu(\theta)d\theta}}_{\text{region 2}}\, da$$

people born during that time

$$+ \int_t^\infty b(a)\, \underbrace{f(a-t)\, e^{-\int_{a-t}^a \mu(\theta)d\theta}}_{\text{region 1}}\, da$$

people age > t who gave birth in the time frame.

4. Space

Principle of mass balance:

$$\frac{\partial c}{\partial t} = -\nabla \cdot \underline{J} + f$$

flux

$$\underline{J} = -D\nabla c \quad \cancel{+ v\chi}$$

diffusion

If $\dfrac{\partial n}{\partial t} = -\nabla \cdot J + f$

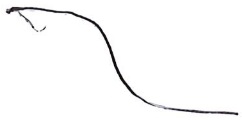
n = cells

$$\underline{J} = -D\nabla n + n\chi(a)\nabla(a)$$

chemoattractant.

5. Travelling Waves

Fisher-KPP : $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$ (1)



constant profile tw $u(x,t) = U(x-ct)$

N.B (1) : $u=0$ unstable :
in (x,t) coords.

$$\left[\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{u} \qquad \hat{u} = A e^{ikx+\lambda t} \qquad \lambda = -k^2+1 \right]$$

$u=1$ stable :

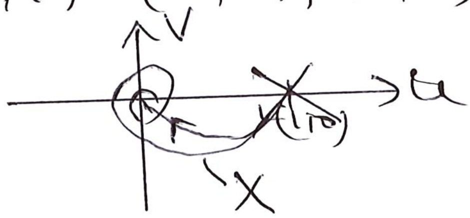
$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial^2 \tilde{u}}{\partial x^2} - \tilde{u} \qquad \lambda = -k^2-1 \quad]$$

But in TW coords, switch stability :

$$-cU' = U'' + U(1-U) \qquad ' \equiv \frac{d}{dz} \qquad z = x-ct$$

$$U' = V$$
$$V' = -cV - U(1-U)$$

st.st. $(U,V) = (0,0), (1,0)$



6. Pattern formation.

$$\frac{\partial \underline{u}}{\partial t} = \underline{\underline{D}}\nabla^2 \underline{u} + \underline{F}(\underline{u}) \qquad \underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\underline{F}(\underline{u}) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

Zero flux BCs $\Rightarrow$ spatially uniform st. st
$\underline{F}(\underline{u}^*) = \underline{0}$. (must satisfy BCs)

Linearise: $\underline{w} = \underline{u} - \underline{u}^*$ $\quad \|\underline{w}\| \ll 1$.

$$\Rightarrow \frac{\partial \underline{w}}{\partial t} = \underline{\underline{D}}\nabla^2 \underline{w} + \underline{\underline{J}}\,\underline{w} \qquad \underline{\underline{J}} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{(\underline{u}^*)}$$

$$\underline{w} = A e^{i\underline{k}.\underline{x} + \lambda t}$$

[① Sep. sol^{n}: $\underline{w} = \underline{X}(\underline{x})T(t)$

Try $\nabla^2 \underline{X} = -k^2 \underline{X}$ $\quad \underline{n}.\nabla \underline{X} = 0$

efns of Laplacian complete]

$$\Rightarrow \begin{vmatrix} \lambda - f_u + D_u k^2 & -f_v \\ -g_u & \lambda - g_v - D_v k^2 \end{vmatrix} = \underline{0}$$

$$\Rightarrow \lambda^2 + \left[(D_u + D_v)k^2 - (f_u + g_v)\right]\lambda + h(k^2) = 0$$

$$h(k^2) = D_u D_v k^4 - (D_v f_u + D_u g_v)k^2 + f_u g_v - f_v g_u.$$

DDI. $\lambda(0) < 0 \Rightarrow$ $\quad f_u + g_v < 0$

$f_u g_v - f_v g_u > 0$

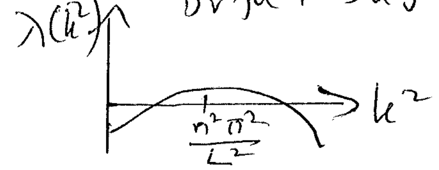
$\lambda(k^2) > 0$ $\; k^2 \neq 0$ (admissible mode):

Need $h(k^2) > 0$

$\Rightarrow D_v f_u + D_u g_v > 0$

$D_v f_u + D_u g_v > 2\sqrt{D_u D_v (f_u g_v - f_v g_u)}$

$\Rightarrow$



7. Moving Bdy Problems.

Domain moves

$\therefore \quad x = [-R(t), R(t)]$

$$\frac{dR}{dt} = \int_0^{R(t)} P(c)\,dx \qquad (*)$$

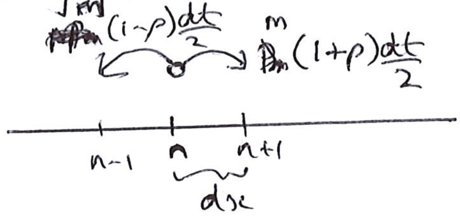
proliferation rate

c = nutrient

Solve P.D.E to c, sub into $(*)$ to find R.

8. Discrete → Continuum.

Moment-generating fns:



Prob moves in time $dt = \overset{m}{\cancel{p_m}}\, dt$

m = prob move / unit time

(Mistake in notes $P_m \frac{dt}{2}$)

$p > 0$ = biase.

$$p_n(t+dt) = \underbrace{\tfrac{m}{2}(1+p)dt\, p_{n-1}}_{\text{at } n-1 \xrightarrow{} \text{Right}} + \underbrace{\tfrac{m}{2}(1-p)dt\, p_{n+1}}_{\text{at } n+1 \xleftarrow{} \text{Left.}}$$

$$+ p_n \underbrace{(1 - \overset{m}{\cancel{p}}\, dt)}_{\text{did not move.}}$$

Let $P(x,t)$ be prob density fn that particle is at pos^n x at time t:

$$P(\underbrace{n\,dx}_{x}, t+dt) = \frac{m}{2}(1+p)dt\, P(\underbrace{n\,dx}_{x} - dx, t) + \frac{m}{2}(1-p)dt\, P(x+dx, t) + P(x,t) - m\, P(x,t)\, dt$$

$$\Rightarrow \quad \frac{P(x,t+dt) - P(x,t)}{dt} = \frac{m}{2}\Big[(1+p)P(x,t) - (1+p)\frac{\partial P}{\partial x}dx + \frac{(1+p)}{2}\frac{\partial^2 P}{\partial x^2}dx^2 + \; - \; + (1-p) \; - \; - \; \Big] - mP$$

$$\underset{dt \to 0}{Lt} \quad \frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} - v\frac{\partial P}{\partial x}$$

$$D = \underset{dx \to 0}{Lt}\, \frac{m \cancel{P_m}\, dx^2}{2}$$

$$v = \underset{dx \to 0}{Lt}\, \cancel{P_m}\, \overset{m}{} p\, dx$$

OR Prob moves is M $\therefore$ $D = \underset{\substack{dx,\\ dt \to 0}}{Lt} \frac{M\, dx^2}{2\, dt}$ ie $dx^2 \sim dt$

$\langle dx^2 \rangle \sim \langle dt \rangle$