A12213H1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.11
Honour School of Mathematics and Computer Science Part C: Paper C5.11
Honour School of Mathematics and Statistics Part C: Paper C5.11
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.11
Honour School of Physics and Philosophy Part C: Paper C5.11
Master of Science in Mathematical Sciences: Paper C5.11

Mathematical Geoscience

TRINITY TERM 2025

Friday 06 June, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. (a) [9 marks] A radiative energy balance model for the average temperature $T(t)$ of Earth's atmosphere is given by

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma T^4,$$

in which C, σ, and γ are constants, and the incoming solar radiation Q is assumed to oscillate around a typical value Q_0.

(i) Briefly outline the derivation of this model, including the physical interpretation of a and γ.

(ii) Suppose the albedo $a(t)$ evolves according to the equation,

$$t_i\frac{\mathrm{d}a}{\mathrm{d}t} = a_0(T) - a, \qquad a_0(T) = \frac{1}{2} - \frac{1}{4}\tanh\left[\lambda\left(\frac{T^4}{T_0^4} - \frac{1}{2}\right)\right],$$

in which $T_0 = (Q_0/4\sigma\gamma)^{1/4}$, and λ and t_i are constants.
Sketch a graph of $a_0(T)$, and briefly explain why it might be expected to depend on temperature in this way, and what processes might influence the timescale t_i.

(iii) Writing $T = T_0(\frac{1}{2}+\theta)^{1/4}$, $Q = Q_0\hat{Q}$, show how to non-dimensionalise the model to obtain the equations

$$\frac{\mu}{\left(\frac{1}{2}+\theta\right)^{3/4}}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \hat{Q}(1-a) - \frac{1}{2} - \theta, \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = a_*(\theta) - a,$$

where t is now dimensionless, $a_*(\theta) = \frac{1}{2} - \frac{1}{4}\tanh\lambda\theta$ is the dimensionless version of $a_0(T)$, and you should give a definition of the dimensionless parameter μ.

(b) [4 marks] Suppose that $\hat{Q} = 1$. Graphically or otherwise, show that there are multiple steady states if $\lambda > \lambda_c$, and a unique steady state if $\lambda < \lambda_c$, where λ_c is a critical value which you should find.

(c) [12 marks] Now suppose that $\hat{Q}$ can vary, but consider the limit in which $\lambda \gg 1$ so that $a_*(\theta)$ undergoes a rapid jump around $\theta = 0$.

(i) Show that multiple steady states are now possible provided $\hat{Q}$ lies in the range $\frac{2}{3} < \hat{Q} < 2$, and construct a bifurcation diagram showing the steady-state (scaled) temperature θ as a function of $\hat{Q}$. [*You may assume that when there are three possible steady states, the one with intermediate temperature is unstable.*]

(ii) Supposing that $\hat{Q}$ oscillates slowly according to $\hat{Q}(t) = 1 + \Delta\sin\omega t$, carefully sketch the possible quasi-steady evolution of $\theta(t)$ over time, noting that there may be more than one possibility, and ensuring to distinguish between the behaviour for different values of the amplitude Δ.

2. (a) [5 marks] The dimensionless St Venant equations for river flow are given by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \qquad \delta F^2\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 1 - \frac{u^2}{h} - \delta\frac{\partial h}{\partial x},$$

in which δ and F are constants.

(i) Briefly describe the meaning of the terms in these equations, and the physical principles on which they are based.

(ii) Assuming $\delta \ll 1$, derive the reduced model for the water depth $h(x,t)$,

$$\frac{\partial h}{\partial t} + \frac{3}{2}h^{1/2}\frac{\partial h}{\partial x} = 0,$$

and give an expression for the water flux $q = hu$ in terms of h. This reduced model should be used for the rest of the question.

(b) [12 marks] The lower reach of a river occupies the region $x > 0$, and is initially in a steady state with dimensionless flux $q = 1$. The influence of a storm over the higher catchment basin is modelled with the boundary condition

$$q(0,t) = \begin{cases} Q & 0 < t < 1, \\ 1 & \text{otherwise}, \end{cases}$$

where $Q > 1$ represents the intensity of the storm.

(i) With the aid of a characteristic diagram, find the water depth $h(x,t)$ for all $x > 0$, $t > 0$, noting that a shock will form at $x = x_s(t)$, which initially propagates at constant speed but later at a slowing speed that asymptotes towards $\dot{x}_s = \frac{3}{2}$. [*You do not need to solve explicitly for $x_s(t)$, but should find the differential equation that it satisfies.*]

(ii) If the banks of the river have height H, what is the value of Q above which you would expect the river to flood?

(c) [8 marks] A certain water company releases sewage into the river during the storm surge. The concentration of sewage $c(x,t)$ is governed by the equation

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(qc) = 0,$$

with initial condition $c(x,0) = 0$, and boundary condition

$$c(x,0) = \begin{cases} 1 & 0 < t < 1, \\ 0 & \text{otherwise}. \end{cases}$$

(i) Show that the equation for the concentration can be rewritten as

$$\frac{\partial c}{\partial t} + h^{1/2}\frac{\partial c}{\partial x} = 0.$$

(ii) Use the method of characteristics to show that the domain over which the sewage is spread is $x_a(t) < x < x_b(t)$, where

$$x_b(t) = \begin{cases} Q^{1/3}t & 0 < t \leqslant 3, \\ 3Q^{1/3}\left(\frac{t-1}{2}\right)^{2/3} & 3 < t \leqslant 1 + 2Q, \\ t - 1 + Q & 1 + 2Q < t, \end{cases}$$

and $x_a(t)$ should be found.

[*You may assume without proof that the speed of the shock $\dot{x}_s$ is always larger than the water speed upstream of it, and that there is no change in c across the shock.*]

3. A layer of sea ice grows downwards from the surface of the ocean, occupying the region $0 < z < H(t)$, where z measures distance vertically downwards. The density difference between ice and water is ignored. The temperature $T(z,t)$ within the ice is governed by the heat equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2}.$$

The temperature at the surface $z = 0$ is assumed equal to the air temperature $T_a(t) = \overline{T}_a - A \cos \omega t$, where $\overline{T}_a$, A and ω are constants. The temperature at $z = H$ is assumed equal to the (constant) melting temperature T_m, and it is assumed that $T_a \leqslant T_m$. There is a constant heat flux F_O supplied to the interface at $z = H$ from the ocean below.

(a) [7 marks] (i) Write down the appropriate Stefan condition relating the time derivative $\mathrm{d}H/\mathrm{d}t$ to the temperature gradient at $z = H$.

(ii) Choosing appropriate scales for the variables, non-dimensionalise the model to arrive at

$$\frac{1}{S}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2}, \quad 0 < z < H(t), \qquad \frac{\mathrm{d}H}{\mathrm{d}t} = \left.\frac{\partial T}{\partial z}\right|_{z=H} - \hat{F}_O,$$

with

$$T = -1 - a \cos t \quad \text{at} \quad z = 0, \qquad T = 0 \quad \text{at} \quad z = H(t),$$

giving definitions of the dimensionless parameters S, a and $\hat{F}_O$.

(b) [10 marks] For the rest of the question, you may assume that $S \gg 1$, so that a quasi-steady approximation to the temperature can be assumed.

(i) Derive the ordinary differential equation satisfied by the ice thickness $H(t)$.

(ii) Assuming that $H = 0$ at $t = 0$, find an approximate expression for the initial growth of the ice thickness, neglecting the small heat flux from the ocean, $\hat{F}_O \ll 1$.

(iii) Explain why this approximation does not hold at large times. By writing $H(t) = \hat{F}_O^{-1} + \hat{F}_O \tilde{H}(t)$, show that the long-term behaviour of H is given approximately by

$$H(t) \approx \frac{1}{\hat{F}_O} + a \hat{F}_O \sin t.$$

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(iv) Use your solution to show that on average over the long term there is no net production of sea ice, and to suggest when during a year the ice is at its thickest and thinnest.

(c) [8 marks] Sea-ice production around Antarctica is influenced by 'katabatic' winds, which blow off the continent and push the growing ice out to sea. This is accounted for by modifying the equation for the evolution of ice thickness $H(x,t)$ to

$$\frac{\partial H}{\partial t} + U \frac{\partial H}{\partial x} = \left.\frac{\partial T}{\partial z}\right|_{z=H} - \hat{F}_O,$$

where x represents distance perpendicular to the coast line, and we now have $H(0,t) = 0$ as well as $H(x,0) = 0$.

(i) What is the physical interpretation of the constant parameter U in this model?

(ii) Find an approximate expression for the ice thickness $H(x,t)$, again neglecting the small heat flux from the ocean, $\hat{F}_O \ll 1$.

(iii) Why does the approximation continue to hold at large times in this case? For the case $a = 0$, show that on average over the long term there *is* now net production of sea ice, and that it increases with increasing wind speed.