Revision Lecture (04/05/2026)

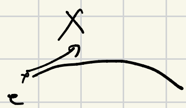
1/ G Lie group Regular value thm $SL(n,\mathbb{R})$ $O(n)$ $U(n)$

$\mathbb{R}^n$ $GL(n,\mathbb{R})$ $SO(n)$ $SU(n)$

Lie group hom

2/ $\mathfrak{g} = \mathrm{Lie}(G) = \{X \mid (L_g)_* X = X\}$ $\mathfrak{g} \xrightarrow{\sim} T_e G$, $X \mapsto X_e$

X



$\alpha_X : \mathbb{R} \to G$

Lie group hom

Integral curve of X through e

3/ $[-,-]$ on $\mathfrak{g}$ $[X,Y]$ $[-,-]$ of left-invt v. fields is left-invt

$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ $C_g(x) = g x g^{-1}$ Commutator of matrices

$\quad = (C_g)_e$ $\mathrm{Ad}(g)$ Lie alg hom adjoint rep

$\mathrm{ad} = d\mathrm{Ad}_e : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ $\mathrm{ad}(X)(Y) = [X,Y]$

4/ $\exp : T_e G = \mathfrak{g} \longrightarrow G$ $d_e \exp = \mathrm{Id}$

$X_e \mapsto \exp(X_e) = \alpha_e(1)$ exp local diffeo

G connected abelian Lie group $\Leftrightarrow$ $G \simeq T^k \times \mathbb{R}^{n-k}$

5/ G Lie subgroup: $H \overset{i}{\hookrightarrow} G$ Lie subg

$\uparrow$ Lie group & i Lie gr. hom

Embedded if homeo on its image.

<u>Thm</u>: $H \subset G$ $\quad H$ emb Lie subg of $G \Leftrightarrow H$ closed.

6/ <u>Thm</u>: G Lie group $\quad \left\{ \begin{matrix} \text{Lie subalg} \\ \mathfrak{h} \subset \mathfrak{g} \end{matrix} \right\} \xrightarrow[\sim]{1:1} \left\{ \begin{matrix} \text{connected Lie} \\ \text{subgroups } H \subset G \end{matrix} \right\}$

$$\mathrm{Lie}(H) \longleftarrow H$$

7/ G connected $\quad \exists\, \tilde{G}$ simply connected $\pi: \tilde{G} \to G$ covering

"univ cover"

<u>Ex</u>: $G = SO(3) \qquad \tilde{G} = SU(2)$

$$SO(3) = SU(2)/\{\pm I\}$$

Without proof: <u>Thm</u> (Lie 3^{rd} thm)

$$\underset{\text{f.d.}/\mathbb{R}}{\{\text{Lie alg}\}}/\sim \quad \underset{\sim}{\overset{1:1}{\longleftrightarrow}} \quad \left\{ \begin{matrix} \text{Simply connected} \\ \text{connected Lie groups} \end{matrix} \right\}/\sim$$

—

$\rho: G \to GL(V)$ rep $\quad$ irreducible, reducible, completely irreducible

8/ Schur's lem: V, W irrep of $G \qquad A: V \to W$ G-inv.

Then a) $A = 0$ or A isom

b) $/\mathbb{C} \qquad A = \lambda I \quad \lambda \in \mathbb{C}$

G compact Lie group $\exists$ Haar measure $\forall$ rep $\exists$ G-invt inner product

$\Rightarrow$ Every rep completely reducible.

$\checkmark$ Character $\chi_V(g) = \mathrm{tr}\, \rho(g)$ $\quad \chi_V : G \to \mathbb{C}, \mathbb{R}$

G compact

$$V, W \text{ rep} \quad \langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$$

$$\langle \chi_V, \chi_W \rangle \overset{..}{=} \int_G \overline{\chi}_V \chi_W$$

Cor: V, W irred $V \not\cong W \Rightarrow \langle \chi_V, \chi_W \rangle = 0$.

Thm: G compact V, W rep: $\chi_V = \chi_W \Rightarrow V \simeq W$

Thm (Peter-Weyl) $\quad L^2(G) \simeq \underset{V \text{ irred rep}}{\widehat{\bigoplus}} \underset{\text{left}}{V} \otimes \underset{\text{right}}{V^\vee}$

$$\chi_V \longleftarrow \mathrm{id}_V$$

$$\rho_{ij} \longleftarrow v_i \otimes v_j^*$$

$$\rho_{ij}(g) = \langle \rho(g) v_i, v_j \rangle$$

10/ G compact connected Lie group T max torus

$W = N(T)/T$ finite group

Thm: T max torus. Every $g \in G$ $\in$ conj of T

. Every max tori conj

11/

$Ad|_T \quad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_a \mathfrak{g}^a$ ⤳ 2-dim irred of T

$e^{2i\pi\theta_a}$

Roots: $\pm \theta_a \in \mathfrak{t}^*$

$\ker \theta_a \subset \mathfrak{t}$ root hyp W gen by reflections in root hyperplanes.

12/ $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{t} \otimes \mathbb{C} \oplus \left(\bigoplus_a \mathfrak{g}_{\theta_a} \oplus \mathfrak{g}_{-\theta_a} \right)$

13/ $\qquad e^{2i\pi\theta_a} \qquad e^{-2i\pi\theta_a}$

$R(G)$ rep ring

$\cong$

$\chi(R(G))$ char ring $R(U(1)) = \mathbb{Z}[t^{\pm}]$ U_n $\chi_{U_n}(t) = t^n$

G conn compact $R(G) \longrightarrow R(T)^W$

Restriction

14/ <u>Thm</u> (Weyl $\int$ formula) $f \in \mathcal{E}(G)$ class function

$$\int_G f = \frac{1}{|W|} \int_{t \in T} \underbrace{\det\left(\left(Ad(t^{-1}) - I\right)\big|_{t^\perp}\right)}_{\displaystyle \prod_a \left(e^{-2i\pi\theta_a} - 1\right)\left(e^{2i\pi\theta_a} - 1\right)} f(t)$$

G Lie group $\mathfrak{g}$

Killing form on $\mathfrak{g}$: $(X,Y) = tr(ad(X)\,ad(Y))$ $\forall X, Y \in \mathfrak{g}$

Ad-inv't sym bilinear

G compact: $(X,X) \leq 0$ $\forall X$

$(X,X) = 0 \iff X \in Z(\mathfrak{g})$

- $\mathfrak{g}$ simple if no non-triv ideal & $\mathfrak{g}$ non-abelian

  semisimple if $\oplus$ simple

- G simple (connected) G non-ab & $H \subset G$ conn normal Lie subgroup $\Rightarrow H = \{e\}$ or $H = G$

  $\iff \mathfrak{g}$ simple

G semisimple if $\mathfrak{g} = Lie(G)$ semisimple

15/ <u>Thm</u>: $\mathfrak{g}$ semisimple $\iff (-,-)$ non-deg

Killing

Classification of compact simple simply connected Lie groups.

$Sp(n)$ $\quad n \geq 1$

$SU(n)$ $\quad n \geq 3$

$Spin(n)$ $\quad n \geq 7$

G_2, F_4, E_6, E_7, E_8

16/ G_2