Quantum Field Theory in Curved Space-Time

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Email corrections and queries to the address above.

QFT in Curved Space-Time

Number of lectures: 16 TT2026

No formal assessment; homework completion requirement.
Weight: One unit
Areas: PT, Astro.
Prequels/pre-requisites: Quantum Field Theory (MT), General Relativity I (MT).

Overview

This course builds on both the first courses in quantum field theory and general relativity. The second course in GR and a course on differential geometry will be helpful, but are not essential.

Learning Outcomes

Students will be able to formulate classical and quantum field theories in curved space-time including an understanding of global features.

Syllabus

Non-interacting fields in curved space-time: Lagrangians, coupling to gravity, global hyperbolicity, asymptotic structure, conformal properties. Black hole thermodynamics. Canonical formulation. Quantization, choice of vacuum. Quantum fields in an expanding universe and de Sitter space. Casimir effect. The Unruh effect. Hawking radiation. Holographic principle.

Reading List

For the global structure of space-time there are many texts including:
Hawking & Ellis, The large scale structure of Space-time, 1971 CUP.
Wald, General Relativity
For a unique perspective see:
Penrose & Rindler, Spinors & Space-time, Vols 1 & 2, CUP, 1984 & 1986.

Much of the QFT in curved space-time is covered in:
Mukhanov and Winitzki, Introduction to quantum effects in gravity, 2007, CUP.

The following are also recommended:
R Wald, QFT in Curved Space-time and Black Hole Thermodynamics, Univ Chicago Press, 1994, ISBN 0226-87027-8.
Birrell & Davis, Quantum field theory in curved space-time, CUP.
Ford, Quantum Field theory in Curved space-time, arxiv:9707062.
Gibbons/Hawking/Townsend, Black Holes lecture notes, arxiv:9707012.
Jacobson, Introduction to quantum fields in curved space-time and the Hawking effect, arxiv:0308048.

Contents

1 Introduction

The goal of this course is to study free, i.e., non-interacting, classical and quantum fields in curved space-time. This is a first essential step towards interacting quantum field theory on a curved background, and beyond to quantum gravity. Already, there are two main areas of application

- Black hole thermodynamics: Hawking radiation provides the temperature in Bekenstein's analogies between properties of black holes and thermodynamics, with the area playing role of entropy.

- In cosmology, the cosmic microwave background spectrum is thought to have a quantum origin explained by QFT in curved space-time. These fluctuations are also thought to have caused the creation of galaxies.

More recently these ideas have played a role in AdS/CFT which relates conformal QFTs to quantum gravity on anti de-Sitter spaces and this has limits that can be probed with QFT in curved space-time.

Quantization is a global problem, in which the global structure of space-time plays a crucial role. Thus the first half of the course will be devoted to improving our understanding of classical field theory in curved space-time and global features. Furthermore, Fermions play a basic role in physics, and require the use of spinors —- there is a brief introduction to spinors in curved space-time, although this has been relegated to an appendix including a couple of independent applications such as the positive mass theorem and the geometry of congruences.

1.1 Conventions

Planck units $\hbar = c = G = k = 1 \rightsquigarrow$

- Mass $\sim 10^{-5} g \sim 10^{19}$ GeV.

- distance $\sim 10^{-33}$ cm

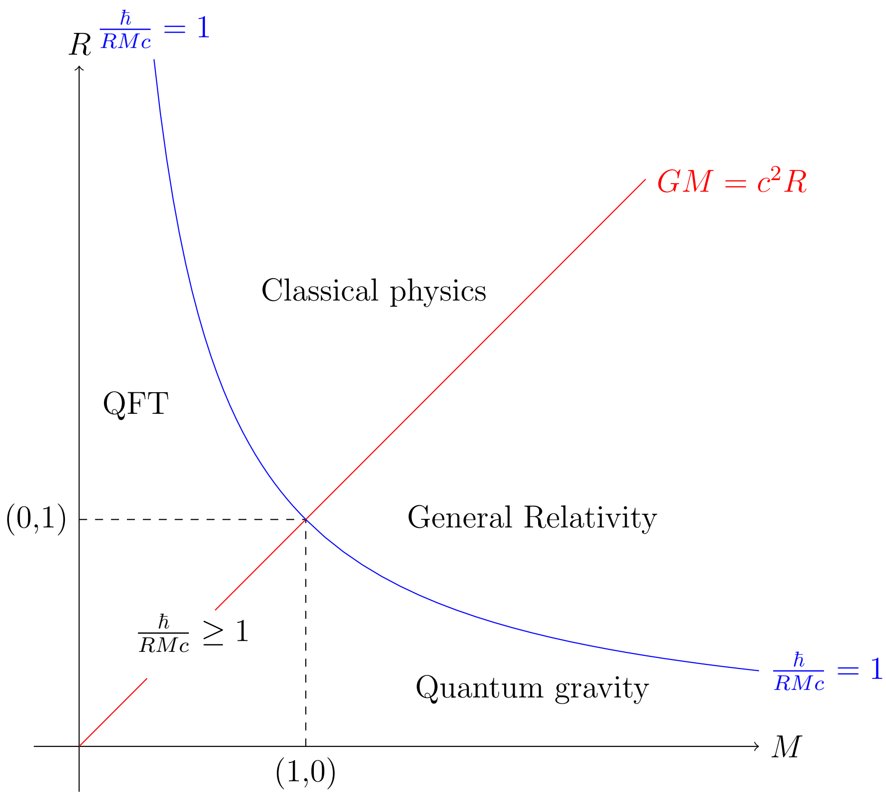
- time $\sim 10^{-44}$ sec

- temperature $\sim 10^{32}$ °K.

A nuclear mass $\sim 10^{-18}$, a Planck mass is almost visible. The cosmological constant is of the order of 3×10^{-122} in these units. For a body of mass M and size R, having

$$\begin{aligned} \frac{GM}{c^2R} &> 1 \qquad \text{must use general relativity (GR)} \\ \frac{\hbar}{MRc} &> 1 \qquad \text{must use quantum field theory (QFT)}\,. \end{aligned}$$

This leads to a (M,R)-plane diagram of validity of theories.



Here M represents the masses and R the size of the bodies involved in the phenomena under discussion both in Planck units, but with factors of $\hbar, G$ and c left in.

Let (M, g_{ab}) be a space-time where for the most part, we will take M to be a 4-dimensional manifold, with local coordinates x^a, $a, b = 0, \ldots, 3$ with metric g_{ab}. Indices are as usual raised and lowered by g_{ab} and its inverse g^{ab}.

We take Penrose conventions:
The metric has signature $(1,3)$ The Ricci identity is

$$[\nabla_a, \nabla_b]V^d = R_{abc}{}^d V^c \,. \tag{1}$$

These conventions are best for spinors but a positive definite sphere has negative curvature but a space-like sphere then has an appropriately positive curvature.[Another very common alternative is to have metric signature $(3,1)$ and a minus sign in the above Ricci identity which conforms better with Riemmannian differential geometry, but less well with QFT and spinors.]

We then have for the Ricci curvature, scalar curvature and Einstein tensors respectively

$$R_{ab} = R_{acb}{}^c \,, \qquad R = R_a^a \,, \qquad G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} \,. \tag{2}$$

The Einstein field equations are

$$G_{ab} + \lambda g_{ab} = -8\pi G T_{ab} \tag{3}$$

where λ is the cosmological constant, G Newton's constant and T_{ab} the stress-energy tensor.

Abstract indices: As an aside, part of Penrose conventions is to use the abstract index notation. Indices are not understood to take on numeric values in general. They simply signify the type of tensor that the object is, having the same downstairs and upstairs indices as would be required if it were to be written out in a coordinate frame. To express a vector in some coordinate or frame basis, we underline the index to refer it to a basis. This avoids the ambiguity in the meaning of

$$\nabla_3 V^2$$

which could be $\partial_3 V^2$ or $\partial_3 V^2 + \Gamma^2_{3a} V^a$ because ∇_3 doesnt know whether to treat V^2 as a scalar or a component of a vector. So ∇ acting on an object with a numerical or concrete underlined index never uses the connection, whereas on an abstractly indexed quantity it does.

2 Causal structure and global hyperbolicity

The wave equation $\Box\phi = 0$ is hyperbolic and, in the massless case, propagates data along null geodesics, see for example the flat space solutions

$f(k_a x^a) = f(t-z)$ where $k_a = (1,0,0,1)$ is a null vector and f an arbitrary wave profile (more generally with some back-reaction, information propagates along causal curves).

Unless otherwise stated, we will take space-time to be both space-time and time orientable and oriented, i.e., we can consistently pick a future directed component of the lightcone at each point, and a non-vanishing four-form.

Time orientability isn't quite enough to rule out almost timelike curves which could be quite bad so we assume

Definition 2.1 *A spacetime (M, g) is* strongly causal *if for all $p \in M$ there exists an open neighbourhood U of p such that no causal (i.e., timelike or null) curve intersects U more than once.*

We expect to be able to solve an initial value problem (IVP) in which we pose initial data $(\phi, \dot{\phi})$ on some space-like[1] 3-surface $\Sigma \subset M$ and let the equation evolve ϕ off the surface. More properly, we will require

Definition 2.2 *a hypersurface Σ is* achronal *if no pair of points in Σ can be connected by a timelike curve.*

We will say that the initial value problem for solutions on some region U with the given data on Σ is *well posed* if there exists a unique solution on U with given data[2] on Σ.

The fact that solutions propagate along null or timelike curves (i.e., causal curves) suggests that the data on Σ can only influence the region

$$J^+(\Sigma) = \{p \in M | \exists \text{ future directed causal curve from } \Sigma \text{ to } p\}\,. \qquad (4)$$

This is the future of the set Σ and can be defined for any type of set. $J^+(\Sigma)$ is also said to be the domain of influence of Σ. We can similarly define the past of Σ,

$$J^-(\Sigma) = \{p \in M | \exists \text{ future directed causal curve from } p \text{ to } \Sigma\}\,. \qquad (5)$$

and one uses $I^\pm$ replacing causal by strictly timelike. These sets are the interiors of the $J^\pm$.

[1] Characteristic initial value problems can also be considered on null hypersurfaces, although the nature of the initial data changes there.

[2] The solution is also usally required to depend continuously on the data, although this is straightforward for linear equations.

Definition 2.3 *The future domain of dependence $D^+(\Sigma)$ of Σ is*

$$D^+(\Sigma) = \{p \in M | \textit{every past inextendible causal curve from } p \textit{ intersects } \Sigma\} . \tag{6}$$

Replacing past by future, we similarly define $D^-(\Sigma)$ and the full domain of dependence by $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.

This is the region on which the initial value problem for wave equations can be proved to be well-posed by PDE techniques. If p is a point lying on a causal curve that cannot be extended in the past through Σ, then one can envisage waves coming in along that curve that are not determined by data on Σ and so would violate the uniqueness assumption.

Definition 2.4 *A spacelike hypersurface Σ is a Cauchy surface for M if $D(\Sigma) = M$. A space-time is said to be globally hyperbolic if it admits a Cauchy surface.*

We have

Theorem 1 (Geroch 1970) *If (M, g_{ab}) is globally hyperbolic, with Cauchy hypersurface Σ then M is diffeomorphic to $\Sigma \times \mathbb{R}$ with the second factor determined by a smooth time coordinate t such that each Σ_t is a Cauchy surface.*

We quote that the the IVP for linear wave equations of the form $\Box\phi + V(x)\phi = f(x)$ are well posed with data given by $(\phi, \dot\phi)$ in Sobolev spaces and other function spaces on a Cauchy hypersurface in globally hyperbolic M. The proofs usually proceed by energy estimates.

3 Conformal infinity and Penrose diagrams

To obtain a good grip on the global structure, one needs to understand asymptotics. A neat way to do that is via *conformal compactification*, which involves adding a conformal boundary to space-time the corresponds to infinity in the physical space-time.

A key feature of the diagrams that we will draw is that they represent the causal structure directly by drawing light rays at 45 degrees. They will give an intuition for the asymptotics by bringing infinity into the finite part of the diagram so that we can see which light rays go where. Such diagrams are known as *Penrose diagrams* (or Penrose-Carter diagrams in Cambridge).

3.1 The homogeneous cases

A first example is $\mathbb{C} \to \mathbb{CP}^1 = S^2$ by stereographic projection and this extends in Euclidean signature to $\mathbb{R}^n \to S^n$. Here coordinates near infinity are mapped to those near the origin via the inversion

$$x^a \to \tilde{x}^a = \frac{x^a}{x^2}\,, \qquad x^2 := x^a x_a\,, \tag{7}$$

under which

$$ds^2 = dx^a dx_a = \frac{d\tilde{x}^a d\tilde{x}_a}{(\tilde{x}^2)^2}\,. \tag{8}$$

Such a transformation that preserves the metric up to a rescaling $g \to \Omega^2 g$ is said to be a conformal motion. Here the rescaling $\Omega = \tilde{x}^2$ returns the RHS to manifest flatness.

The same formulae hold in Lorentz signature, but now the light cone $x^2 = 0$ of the origin $x^a = 0$ is sent to infinity, being interchanged with the light cone $\tilde{x}^2 = 0$ of the point i at infinity given by $\tilde{x}^a = 0$, not just the points. Notice that the scale factor $\Omega = \tilde{x}^2$ vanishes on this light cone at infinity to first order.

To be more systematic, we introduce the full conformal group of (conformally) flat space-time but this can only act on a compactification as it interchanges finite with infinite points. We will denote points at infinity by i and hypersurfaces at infinity by $\mathscr{I}$, pronounced *scri* for script I.

For a flat metric of signature (p, q) the full conformal group is $SO(p+1, q+1)/\mathbb{Z}_2$, and so in four dimensions with Lorentz signature we have the 15 parameter group $SO(2, 4)$. This acts on $\mathbb{R}^6$ with coordinates

$$X^I = (s, w, x^a) = (t, x, y, z, s, w)\,, \qquad a =, 0, \ldots, 3\,, \quad I = 0, \ldots, 5,$$

by orthogonal transformations preserving the quadratic form

$$X^2 := \eta_{IJ} X^I X^J = s^2 - w^2 + x^a x_a\,. \tag{9}$$

Define first the projective space

$$\mathbb{RP}^5 = \mathbb{R}^6 / \{X^I \sim \lambda X^I, \lambda \in \mathbb{R} - \{0\}\}\,. \tag{10}$$

Then we can define conformally compactified Minkowski space to be

$$\mathbb{M} = \{[X^I] \in \mathbb{RP}^5 | X^2 = 0\} \subset \mathbb{RP}^5\,. \tag{11}$$

Lemma 3.1 $\mathbb{M} = S^1 \times S^3/\mathbb{Z}_2$.

This follows by rewriting $X^2 = 0$ and rescaling so that

$$s^2 + t^2 = w^2 + x^2 + y^2 + z^2 = 1\,.$$

Thus (s, t) lie on S^1 and (w, x, y, z) on S^3 with the Cartesian product metric, although note that $X^I \sim -X^I$, hence the $\mathbb{Z}_2$.□

The Einstein cylinder: We can take the universal cover by unwrapping the 'time' S^1 by setting $(s, t) = (\cos\tau, \sin\tau)$. This then gives the *Einstein cylinder* metric

$$ds^2_{EC} = \mathrm{d}\tau^2 - \mathrm{d}s^2_{S^3} \tag{12}$$

where the unit round sphere 3-metric can be given in spherical polars by

$$ds^2_{S^3} = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \qquad (\psi, \theta, \phi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi] \tag{13}$$

where $(w, x, y, z) = (\cos\psi, \sin\psi\sin\theta\cos\phi, \sin\psi\sin\theta\sin\phi, \sin\psi\cos\theta)$, although there will be coordinate singularities at $\theta, \psi = 0, \pi$. We either draw this as a cylinder, or in the (τ, ψ)-strip $\mathbb{R} \times [0, \pi]$.

To obtain the key maximally symmetric examples, we choose a non zero constant vector $K^I \in \mathbb{R}^6$ and define

$$ds^2_{K^2} = \left.\frac{\eta_{IJ} dX^I dX^J}{(K \cdot X)^2}\right|_{X^2=0}\,, \tag{14}$$

where $K \cdot X := K_I X^I\,, X^2 = X_I X^I$ etc.; under $SO(2, 4)$, K^I is distinguished only by its norm K^2 so there are only the 3 cases $K^2 = -1, 0, 1$. With maximal symmetry there is only the scalar curvature (the Weyl tensor and trace-free Ricci tensor must vanish) and its sign is that of K^2. By dividing by a quadratic function, the metric is invariant under constant rescalings of X^I. However, on $X^2 = 0$ the form $X_I dX^I = dX^2/2$ vanishes so it is easy to see that under $X^I \to f(X)X^I$, ds^2_I is invariant for any $f(X)$. Thus we can scale X so that $K \cdot X = 1$.

From the inversion example we see that the set where $K \cdot X = 0$ will correspond to points at infinity. These sets will be denoted $\mathscr{I}$, or $\mathscr{I}^+$ and $\mathscr{I}^-$ if respectively to the future or past of the finite part of space-time where $K \cdot X$ can be scaled to be 1. The isometry group of $ds^2_{K^2}$ is the subgroup of $SO(2, 4)$ that preserves K^I.

There are three cases:

$K^2 = 0$ *Flat space*. We can take $K = (0,0,0,0,1,-1)$, so that $K\cdot X = s + w = 1$. It is then immediate that $X^2 = 0$ gives $s - w = -x^a x_a$ and

$$ds_0^2 = dx^a dx_a \tag{15}$$

i.e., flat space as desired. Thus (14) defines a conformally flat metric.

Had we chosen to rescale so that $s - w = 1$ instead (but with the same K^I), we would have obtain the inverted metric given by the right hand side of (8).

To rewrite this in terms of Einstein cylinder coordinates we must divide (12) by $s + w = \cos\tau + \cos\psi = 2\cos(\frac{\psi+\tau}{2})\cos(\frac{\tau-\psi}{2})$

$$\mathrm{d}s_0^2 = \frac{\mathrm{d}s_{EC}^2}{4\cos^2(\frac{\psi+\tau}{2})\cos^2(\frac{\tau-\psi}{2})} \tag{16}$$

Future infinity $\mathscr{I}^+$ is null defined by $\tau + \psi = \pi$; it is the past lightcone of i^+ with $(\tau,\psi) = (\pi, 0)$ or future lightcone of i^0 with $(\tau,\psi) = (0,\pi)$. Past infinity $\mathscr{I}^-$ is $\tau - \psi = -\pi$ and is the past light cone of i^0 and future lightcone of i^- given by $(\tau,\psi) = (-\pi, 0)$. In $\mathbb{M}$, the three points i^0, i^+ and i^- are identified, and $\mathscr{I}^+$ is identified with $\mathscr{I}^-$.

$K^2 = 1$ *De Sitter space*; Einstein vacuum with cosmological constant $+1$ and isometry group $SO(1,4)$.

Put $K = (0,0,0,0,1,0)$ so $s = 1$ and $X^2 = 0$ gives the hyperboloid

$$1 + t^2 = w^2 + x^2 + y^2 + z^2 \tag{17}$$

This clearly has topology $\mathbb{R}\times S^3$, with 3-spheres of radius $r = \sqrt{1+t^2}$ at time t. Introducing $(t,r) = (\tan\tau, \sec\tau)$ we can rewrite the metric as

$$ds_1^2 = \frac{1}{\cos^2\tau}\left(d\tau^2 - ds^2_{S^3}\right) \tag{18}$$

This is therefore the region $\tau \in [-\pi/2, \pi/2]$ in the Einstein cylinder with future/past infinities $\mathscr{I}^\pm$ both of topology S^3 given by $\tau = \pm\pi/2$.

Alternative coordinates $(t,r) = (\sinh T, \cosh T)$ yield

$$ds_1^2 = dT^2 - \cosh^2 T ds^2_{S^3} \tag{19}$$

emphasizing the hyperbola shape with exponential expansion as appropriate for inflationary cosmology. Here T is proper time for observers fixed in S^3.

$K^2 = -1$ *Anti de-Sitter space*; Einstein vacuum with cosmological constant -1, symmetry group $SO(2,3)$.

Put now $K = (0, \dots, 0, 1)$ then $w = 1$ and we obtain instead the hyperboloid

$$s^2 + t^2 = 1 + x^2 + y^2 + z^2, \tag{20}$$

As before for the Einstein cylinder, unwrap the time S^1 setting

$$(s,t) = \sec\psi(\cos\tau, \sin\tau), \qquad \tan^2\psi = x^2 + y^2 + z^2\,. \tag{21}$$

and this gives

$$\mathrm{d}s^2_{-1} = \sec^2\psi \left(\mathrm{d}\tau^2 - \mathrm{d}s^2_{S^3} \right) \qquad \psi \in [0, \pi/2] \tag{22}$$

Thus we obtain the region $\psi \in [0, \pi/2]$ inside the Einstein cylinder.

Anti-de Sitter is important in the AdS/CFT correspondence.

A number of remarks are in order:

1. Infinity $\mathscr{I}$ is a null hypersurface for flat space, space-like for de Sitter, and time-like for AdS. We will see that the correlation with the sign of the cosmological constant is not a coincidence.

2. These last two representations as hyperboloids are in fact double covers of $\mathbb{M} - \{K \cdot X = 0\}$. In the $K^2 = 1, 0$ cases, $\mathscr{I}^-$ and $\mathscr{I}^+$ are identified in $\mathbb{M}$. This in particular shows that light cones of points of $\mathscr{I}^-$ refocus at the corresponding points of $\mathscr{I}^+$. We unwrap these spaces in order to avoid closed timelike curves.

3. The light cone of a point $X_0^I \in \mathbb{M}$ is the intersection of $X_0 \cdot X = 0$ with $\mathbb{M}$.

4. In the $K^2 = \pm 1$ cases we can still use the coordinates scaled so that $s + w = 1$ as we did for flat space with $K^2 = 0$ to obtain *Poincaré patch* coordinates

$$\mathrm{d}s^2_1 = \frac{\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}{t^2}\,, \qquad \mathrm{d}s^2_{-1} = \frac{\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}{z^2} \tag{23}$$

These have infinity at respectively $t = 0$ or $z = 0$ and taking $t > 0$ or $z > 0$, the patches miss out half the part of the space-times covering

$\mathbb{M}$ (which in turn is double covered by the hyperboloids and so on). Sometimes one puts $t = \exp -T$ to obtain

$$ds_1^2 = dT^2 - \mathrm{e}^{2T}(dx^2 + dy^2 + dz^2)\,. \tag{24}$$

The T is now the proper time of an observer at the origin in three space and emphasizes the exponential expansion seen by that observer; the coordinates cover the region in de Sitter space that can eventually be observed by this observer.

5. It is clear that Minkowski space and de Sitter are globally hyperbolic, but that AdS is not. For AdS, we need to present, not just data on an initial $t = const.$ hypersurface, but also data, or at least boundary conditions on the time-like infinity. Otherwise, one can imagine incoming radiation from infinity.

3.2 Conformal infinity in conformally curved spaces

Singularities are characterized by incomplete geodesics that cannot be extended beyond some finite time, perhaps because we have removed some region where the curvature is infinite. A space-time is nonsingular if it is *geodesically complete*, that is each geodesic can be extended to infinite affine parameter. Typical examples are isolated systems in which, perhaps some particles, fields or gravitational radiation come in from infinity, and interact without forming a black hole, and then escape again to infinity. In order to understand what is happening at large distances, we can introduce a concept of conformal infinity.

In curved space, we do not have a group of conformal symmetries, but we can nevertheless perform conformal rescalings $g \to \Omega^2 g$ and this leads to the following definition of conformal compactification in curved space.

Definition 3.1 *A conformal compactification of a space-time* (M, g) *is a manifold* $\tilde{M}$ *with boundary* $\mathscr{I} = \partial\tilde{M}$ *and metric* $\tilde{g}$ *such that*

1. $\tilde{g}$ *is smooth on* $\tilde{M}$
2. M *is diffeomorphic to the interior of* $\tilde{M}$,
3. *On* M *we have* $\tilde{g} = \Omega^2 g$ *with* Ω *smooth on* $\tilde{M}$, $\Omega \neq 0$ *on* M,

4. $\Omega = 0$*, and* $d\Omega \neq 0$ *on* $\mathscr{I} = \partial\tilde{M}$*.*

We can also specify the level of differentiability if desired.

We have already seen examples with Minkowski space, de Sitter space, AdS and so on. If M is globally hyperbolic we can see that $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ where future infinity $\mathscr{I}^+$ is to the future of a Cauchy hypersurface and past infinity $\mathscr{I}^-$ to the past.

We have the theorem

Proposition 3.1 *Let* (M, g) *have conformal compactification* $(\tilde{M}, \tilde{g})$*, and suppose that the space-time asymptotically satisfies the Einstein equations with conformally invariant matter (so that the trace of the stress-energy tensor vanishes) with cosmological constant* λ*. Then* $\mathscr{I}$ *is space-like when* $\lambda > 0$*, time-like for* $\lambda < 0$ *and null when* $\lambda = 0$*.*

We furthermore have that if the trace-free Ricci tensor falls off fast enough at $\mathscr{I}$*, then* $\mathscr{I}$ *is umbilic, i.e., the trace-free part of the extrinsic curvature vanishes. [The extrinsic curvature is* $k_{ab} := \nabla_{(a}N_{b)}$ *where* N_a *is the unit normal (and* N^a *is continued off the surface by* $N^a\nabla_a N_b = 0$*).] In the null case this implies that* $\mathscr{I}$ *is shear-free.*

Proof: Define the *Schouten tensor*[3]

$$P_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{12}Rg_{ab}\,. \tag{26}$$

This is constructed so that under a conformal rescaling we have

$$P_{ab} = \tilde{P}_{ab} + \Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega - \Omega^{-2}\tilde{g}_{ab}\tilde{\nabla}_c\Omega\tilde{\nabla}^c\Omega \tag{27}$$

With the vanishing of the trace of the energy momentum tensor, we have in the physical metric $P_a^a = -R/6 = -\lambda/6$. So

$$-\frac{\lambda}{6} = P_a^a = \Omega^2\tilde{g}^{ab}P_{ab} = \Omega^2\tilde{P}_a^a + \Omega\tilde{\Box}\Omega - 4\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\,. \tag{28}$$

[3]This is defined in d-dimensions by

$$P_{ab} = -\frac{1}{d-2}\left(R_{ab} - \frac{1}{2(d-1)}R\right)\,, \tag{25}$$

with the same conformal transformation law. It plays a key role in conformal geometry. Note the sign flip relative to Riemannian definitions.

On $\mathscr{I}$, $\Omega = 0$ and so we have, defining the normal to $\mathscr{I}$ by $N_a = \tilde{\nabla}_a \Omega$

$$\tilde{g}^{ab} N_a N_b = \frac{\lambda}{24} \tag{29}$$

hence the first part of the proposition follows.

To obtain the second part we use the trace-free part of (27) to see that, multiplying through by Ω we have

$$(\tilde{\nabla}_a N_b - \frac{1}{4} \tilde{g}_{ab} \tilde{g}^{cd} \nabla_c N_d)|_{\Omega=0} = 0 \tag{30}$$

The extrinsic curvature is defined to be the projection of the covariant derivative of the normal into the surface and so this shows that its trace-free part vanishes. □

In the null case this implies that $\mathscr{I}$ has the intrinsic geometry of a light cone in Minkowski space in the sense that it is *shear free* in the sense of the Sachs equation.

3.3 Asymptotics and peeling

If the space-time is nonsingular, hence complete, we expect all light rays to make it to infinity both in the past and future, and if so, we say that the space-time is *asymptotically simple*. Such space-times can be thought of as perturbations of Minkowski space, de Sitter space or anti-de Sitter space. We have theorems now that tell us that such solutions can be constructed from generic but small data in some Sobolev norms. In the case of positive cosmological constant the stability of small perturbations of de Sitter was proved by Friedrich in the 1980s and for vanishing cosmological constant in the 1990s by Christodoulou and Klainerman and followers. However, in recent work, anti-de Sitter space has been shown to be unstable in this sense. For a start it is not globally hyperbolic, and one must impose some boundary conditions at $\mathscr{I}$ to obtain a well-posed initial value problem, and then these are usually chosen to be reflective so that waves can bounce back and forth leading to instabilities.

If the unphysical metric is smooth enough, we can also deduce that the Weyl tensor vanishes on $\mathscr{I}$. For zero cosmological constant it is possible to show that $\mathscr{I}$ has topology $S^2 \times \mathbb{R}$ and indeed this is typically also the case in black hole space-times with $\lambda = 0$. If so we can find, perhaps after a further rescaling, Bondi coordinates $(u, \zeta, \bar{\zeta})$ near $\mathscr{I}$ so that the unphysical metric is given by

$$\widetilde{ds}^2 = dud\Omega - \frac{d\zeta d\bar{\zeta}}{(1+|\zeta|^2)^2} + O(\Omega)\,, \tag{31}$$

where the second term is simply the sphere metric.

It is reasonable to expect conformally invariant and massless fields to continue smoothly to $\mathscr{I}$. Thus, if ϕ is a solution to the conformally invariant wave equation, $\tilde{\phi} = \phi/\Omega$ should be smooth on $\mathscr{I}$ in $(\tilde{M}, \tilde{g})$ at least if it is in the domain of dependence of M. When $\lambda > 0$, we can deduce that a linear massless field will evolve past $\mathscr{I}$ as if it wasn't there and so $\phi_{A_1\ldots A_n}/\Omega = \phi^0_{A_1\ldots A_n}$ will be smooth and generically non-vanishing near $\mathscr{I}$ in the unphysical space-time, giving sharp asymptotic falloff of the physical field $\phi_{A_1\ldots A_n} = \Omega\phi^0_{A_1\ldots A_n}$. It is instructive to compare this falloff to that in terms of the affine parameter r along an outward going null geodesic. In the case when $\mathscr{I}$ is null, we find

$$\Omega \sim 1/r. \tag{32}$$

However, in the de Sitter case, it is easily seen that $\Omega \sim \exp(-t)$ when t is proper time along a time-like geodesic going out to $\mathscr{I}$ since $dt = f\tau/\tau$ where τ is the Einstein cylinder coordinate. When $\lambda = 0$, the situation is as before for the wave equation but more subtle for higher spin as different components of the spinor scale differently according to whether they are aligned with $\mathscr{I}$ or transverse. Taking o^A aligned along the null geodesic, we find that if ϕ_r is r contractions of ι and $n - r$ with o^A, then we have

$$\phi_r \sim \frac{1}{r^{n-r+1}}\,. \tag{33}$$

We can construct a spin-two field $\tilde{\psi}_{ABCD}$ on $\tilde{M}$ from the Weyl spinor Ψ_{ABCD} by defining

$$\tilde{\psi}_{ABCD} = \frac{\Psi_{ABCD}}{\Omega}\,. \tag{34}$$

The asymptotics above apply to this field, i.e., ψ_{ABCD} should be finite on $\mathscr{I}$. However, under the rescaling the Weyl tensor itself does not rescale. Thus we learn that the Weyl tensor itself should vanish on $\mathscr{I}$. The argument is more delicate when $\mathscr{I}$ is null, but follows when it has topology $S^2 \times \mathbb{R}$.

4 Black holes

More generally, we do not expect space-times to be complete and we expect singularities to form.

4.1 The Chandrasekhar limit

For a star whose nuclear fuel has burnt out, the pressure p is related to the density ρ by $P = \alpha\rho^\gamma$ for some constants α, γ.

$$\begin{aligned} \text{Gravitational potential energy} &\sim \frac{M^2}{R} \\ \text{Pressure energy} &\sim PV \sim PR^3 \sim \alpha \left(\frac{M}{R^3}\right)^\gamma R^3 \\ \text{Total energy} &\sim \alpha M^\gamma R^{3(1-\gamma)} - \frac{M}{R}\,. \end{aligned}$$

For $\gamma > 4/3$ a stable minimum exists for all M. For $\gamma < 4/3$ no stable minimum exists. The parameter γ measures the stiffness, and one can ask how stiff can matter become? The value $\gamma = 4/3$ value is in fact singled out by fermionic degeneracy pressure arising from the Pauli exclusion principle and represents a maximum stiffness.

For degenerate atoms/neutrons filling Fermi level p_F, the degeneracy implies that we have $n = \#/\text{vol} \sim p_F^3$ with one particle per cube of order of the wavelength. The density is then $\rho \sim m_n p_F^3$, where m_n is atom/neutron mass, pressure $\sim np_F \sim p_F^4$

$$P \sim m_n^{-4/3}\rho^{4/3}\,,$$

giving $\gamma = 4/3$. This implies

$$E = \frac{M^{4/3}}{R}(\alpha - M^{2/3})\,,$$

and so for $M > M_c = \alpha^{3/2}$ collapse is inevitable. According to the above $M_c \simeq 1/m_n^2 \sim$ one solar mass. This is the Chandrasekhar limit for white dwarfs (electron degeneracy) and Landau limit (neutron degeneracy) for Neutron stars.

These back-of the envelope calculations for the existence of black holes from 1930 are bolstered on the one hand by rigorous mathematical arguments in the form of the Hawking-Penrose singularity theorems from the 1960s, and more recently by ample observational evidence see Nasa website for examples.

The final state of gravitational collapse is understood to settle down to the Kerr or more simply the Schwarzschild solutions in which the star disappears inside a radius $R = 2M$, the Schwarzschild radius from which light can no longer escape.

4.2 Schwarzschild and the standard picture

The Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - ds^2_{S^2} \tag{35}$$

provides the prototype nonrotating black hole exterior. It can be completed with an interior by gluing in a collapsing dust Friedman model

$$dt'^2 - R(t')^2(d\chi^2 + \sin^2\chi ds^2_{S^2}) . \tag{36}$$

We relegate this gluing to the exercises.

After collapse, it can be seen that the metric has issues at $r = 2m$ but these are resolved by use of the respectively retarded and advanced (Eddington-Finkelstein) coordinates u, v

$$du = dt - \frac{dr}{1 - \frac{2m}{r}} , \qquad dv = dt + \frac{dr}{1 - \frac{2m}{r}} \tag{37}$$

so that

$$(u, v) = (t - r_*, t + r_*) , \qquad r_* = r + 2m \log\left(\frac{r - 2m}{2m}\right) . \tag{38}$$

where r_* is the Regge-wheeler tortoise coordinate that places the horizon at $r_* = -\infty$. This allows us to put the metric in the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2 ds^2_{S^2} , \tag{39}$$

and similarly with advanced coordinates, showing that there is no singularity at $r = 2m$. We see in fact that $r = 2m$ is a null hypersurface ruled by outgoing null geodesics, but the fact that $r = 2m$ means that the light rays are not escaping to infinity. It is an *event horizon*. For $r > 2m$, light rays with $\dot{r} > 0$ can and do escape. For $r < 2m$, all causal geodesics have future end point at $r = 0$.

These are the best coordinates for examining $\mathscr{I}^+$. The rescaling can be done with $\Omega = 1/r$ because r is an affine parameter on radial null geodesics. This yields unphysical metric

$$\widetilde{ds}^2 = \Omega^2 \left(1 - \frac{2m}{r}\right) du^2 + 2dud\Omega - ds^2_{S^2} \,, \tag{40}$$

and gives rise to the following picture:

In this picture it is clear that the future event horizon $r = 2m$ satisfies the defining property

Definition 4.1 *The event horizon is the boundary of the past of* $\mathscr{I}^+$.

This follows as the boundary is $u \to \infty$ and this follows as $r \to 2m$.

There is a corresponding time-reversed picture using coordinates (v, r, θ, ϕ). However, now we have the puzzle that we see that $r = 2m$ is the past horizon being the boundary $v = -\infty$ of the future of $\mathscr{I}^-$.

Using both we again have a problem at $r = 2m$

$$ds^2 = \left(1 - \frac{2m}{r}\right) dudv - r^2 ds^2_{S^2} \,, \tag{41}$$

So this puzzle isnt resolved.

To do so, introduce the Kruskal coordinates

$$U = -\exp(-u/4m) \,, \qquad V = \exp(v/4m) \tag{42}$$

which yield

$$ds^2 = \frac{32m^3}{r} dUdV - r^2 ds^2_{S^2} \,, \tag{43}$$

and this now extends to negative values of U and V through $U = 0$ and $V = 0$ which give the two components for the past and future event horizons since

$$UV = \left(1 - \frac{r}{2m}\right) \mathrm{e}^{r/2m} \,. \tag{44}$$

These give new asymptotic regions as $U, V \to -\infty$ and gives the full Kruskal extension with Penrose-Carter diagram:

We can see that the singularity $r = 0$ (which is a genuine curvature singularity) is a black hole to the future of every observer that crosses the future event horizon, or a white hole in the past. Time translation by

$$\partial_t = V\partial_V - U\partial_U \tag{45}$$

in this picture is much like a boost in $1+1$ dimensions.

Similar diagrams can be drawn for Reissner-Nordstrom, Kerr and the Kerr-Newman, see Hawking and Ellis although the latter have the novelty of *Cauchy horizons*, hypersurfaces beyond which neither fields nor space-time itself are determined by Cauchy data essentially as a consequence of *naked singularities*, singularities in the past of observers. However, these cannot be seen from infinity. These black hole solutions are unique subject to various assumptions (like the existence of a stationary Killing vector that looks like a time translation at large distances). They have extensions to versions with cosmological constant.

This final state is tightly constrained as in four dimensions we have powerful uniqueness theorems. Birkhoff's theorem says that any spherically symmetric vacuum solution is static, which then implies that it must be Schwarzschild. For Einstein-Maxwell system this extends to show that the only spherically symmetric solution is Reissner-Nordstrom. But suppose we know only that the metric exterior to a star is static. We further have:

Theorem 2 (Israel) *If (M,g) is an asymptotically-flat, static, vacuum spacetime that is non-singular on and outside an event horizon, then (M,g) is Schwarzschild.*

More remarkably we have

Theorem 3 (Carter-Robinson) *If (M,g) is an asymptotically-flat stationary and axi-symmetric vacuum spacetime that is non-singular on and outside an event horizon, then (M,g) is a member of the two-parameter Kerr family. The parameters are the mass M an the angular momentum J.*

The assumption of axi-symmetry has since been shown to be unnecessary by Hawking and Wald , i.e., for black holes, stationarity $\Rightarrow$ axisymmetry.

4.3 A digression on null congrences and hypersurfaces

An important role is played by null hypersurfaces, i.e., hypersurfaces $u(x) =$ constant such that $\nabla_a u$ is a null vector. It is a standard elementary exercise that $l_a = \nabla_a u$ is tangent to a family of null geodesics.

More generally, a null congruence is a foliation of a region of space-time by null geodesics. It can be defined by a null vector field l^a whose integral curves are the null geodesics through each point. If it is tangent to a congruence of affinely parametrised null geodesics, then

$$\nabla_l l^b := l^a \nabla_a l^b = 0 \tag{46}$$

The geometry can be studied by introducing a 2-dimensional *screen space* consisting of tangent vectors perpendicular to the direction of null geodesic. This space can be spanned by a pair of orthonormal vectors X^a, Y^a, $X \cdot X = Y \cdot Y = 1$ and $X \cdot l = Y \cdot l = 0$. Given l, these are defined up to standard 2d rotations. The general vector in this screen space can be parametrized as

$$V^a = xX^a + yY^a \,. \tag{47}$$

It is conventional to introduce the complex coordinate $\sqrt{2}\zeta = x+iy$ on screen space and associated complex null vectors $m^a, \bar{m}^a$ so that

$$m^a = \frac{1}{\sqrt{2}}(X^a - iY^a)\,, \qquad V^a = \zeta\bar{m}^a + \bar{\zeta}m^a \,. \tag{48}$$

We can choose X^a and Y^a to be parallel propagated along the null geodesics, $\nabla_l X^a = \nabla_l Y^a = \nabla_l m^a = 0$.

We wish to measure how images on the screen are distorted as they are propagated along the light rays of the congruence. If V^a connects nearby geodesics of the congruence, then it is Lie derived along l^a, i.e.,

$$[l, V]^a = \nabla_l V^a - \nabla_V l^a = 0 \,. \tag{49}$$

It is easy to see that if $l \cdot V = 0$ initially, then it remains zero. This gives

$$\nabla_l \zeta = -\rho\zeta - \sigma\bar{\zeta} \,, \tag{50}$$

for some complex parameters ρ, σ. These can be interpreted as follows:

1. The imaginary part of ρ is the twist and generates rotations of the ζ plane. It vanishes iff the congruence is hypersurface forming, $l_{[a}\nabla_b l_{c]} = 0$ which implies that there is a rescaling of l_a so that $l_a = \nabla_a u$ for some function u.

2. The real part of ρ gives the *expansion*, $\nabla_a l^a = -2\rho$ and the area element of the orthogonal transverse plane evolves by

$$A = -im_a dx^a \wedge \bar{m}_b dx^b,$$

satisfies

$$\mathcal{L}_l A = -2\rho A \tag{51}$$

3. The complex scalar σ is the shear in the sense that a circle in the ζ plane evolves into an ellipse.

4. Equation (290) implies the geodesic deviation equation

$$\nabla_l \nabla_l V^a = l^b l^c V^d R_{bdc}{}^a \tag{52}$$

and this combines with (291) to give the *Sachs equations*

$$\nabla_l \rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \tag{53}$$
$$\nabla_l \sigma = (\rho + \bar{\rho})\sigma + \Psi_0 \tag{54}$$

Here $\Psi_0 = C_{abcd} l^a m^b l^c m^d$, $\Phi_{00} = \Phi_{ab} l^a l^b = -\frac{1}{2} R_{ab} l^a l^b$ and is positive when the dominant energy condition is satisfied. An important consequence for horizons and singularity theorems is that the whole RHS of (53) is manifestly positive definite.

5. If a null hypersurface has vanishing shear, then it has the intrinsic geometry of a light cone or null hyperplane in Minkowski space up to scaling (i.e. the metric restricts to a multiple of $d\zeta d\bar{\zeta}$ on $\mathbb{R}^3$ or $S^2 \times \mathbb{R}$ where $l^a \partial_a = \partial_v$ for a third coordinate v).

4.4 Horizons and black hole thermodynamics

For an asymptotically flat space-time, we define

Definition 4.2 *The* event horizon $\mathcal{H}$ *is the boundary of the past* $J^-(\mathscr{I}^+)$ *of* $\mathscr{I}^+$*, that is, it is the boundary of the region from which it is possible to escape to infinity along a causal curve.*

Much is known about event horizons under reasonable assumptions appropriate to isolated systems that settle down:

- $\mathcal{H}$ is a null hypersurface being the boundary of a past set (it clearly cannot be time-like as causal paths could then cross both ways, and if it were space-like there would be regions to its past that could not exit to $\mathscr{I}^+$).
- $\mathcal{H}$ is ruled (or foliated) by complete null geodesics.
- If $\mathscr{I}$ has topology $S^2 \times \mathbb{R}$, as appropriate for the exterior of an isolated system, then so does $\mathcal{H}$, with the $\mathbb{R}$ factor being the null geodesics.
- The cross-sectional area is bounded above.

This is a rather excessively global definition that requires knowledge of the whole space-time. One can also define with just local knowledge:

Definition 4.3 *a* closed trapped surface *is a two-surface of topology S^2 such that the outward pointing null geodesics have nonpositive expansion (i.e., the area will drop or be constant in any outward going null direction or $\rho \geq 0$ where ρ is the spin coefficient in the definition of the Sachs equation).*

Penrose's original singularity theorem deduces the existence of a singularity (in the form of geodesic incompleteness) from the existence of such a closed trapped surface. It is easy to see from the signs in the Sachs equations and following the outward going null geodesic normals off hte surface that a closed trapped surface leads to:

Definition 4.4 *an* apparent horizon *is a null hypersurface of topology $S^2 \times \mathbb{R}$ such that the expansion of the outward going null rays is nonpositive (i.e., the area is non-increasing to the future).*

The first of the Sachs equation for a null geodesic congruence generated by l gives

$$\nabla_l \rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \geq \rho^2 \,. \tag{55}$$

Thus if $\rho \geq 0$ then it cannot decrease. (Recall that if A is the area element, $\mathcal{L}_l A = -2\rho A$.)

However, Penrose's theorem doesnt deduce the location of the singularity! In particular it is not clear that an apparent horizon is hidden inside an event horizon and the following is open:

The cosmic censorship hypothesis: All singularities that arise from evolving from an initial data hypersurface are hidden behind an event horizon and so cannot be seen from infinity.

Generally speaking we assume that a black hole settles down to being stationary or static. Then, the event horizon must settle down to a null hypersurface with finite cross sectional area (otherwise geodesics will be escaping to infinity). Once a black hole horizon settles down, its area is constant. Assuming that the black hole is becoming stationary or static, it follows that it is (under suitable analyticity assumptions) a *Killing horizon*:

Definition 4.5 *A Killing horizon is a null hypersurface on which a Killing vector k_a becomes null, so that the surface is defined by $k_a k^a = 0$ and $k^a \neq 0$. Thus k^a is tangent to the null geodesic generators of the horizon.*

The fact that k^a is Killing means that we must have $\rho = \sigma = 0$.

It follows from the black hole uniqueness theorems that, even if we started from some collapse scenario, the final black hole, if essentially static or stationary, is Kerr Newman or Schwarzschild. These all have a similar structure to Schwarzschild in that they can be continued analytically back to a point where the standard future event horizon intersects a past one at a 2-surface C.

Definition 4.6 *Such a Killing Horizon is said to be a bifurcate Killing horizon if there exists cross-section C of topology S^2 on which k^a vanishes—it is bifurcate because then in a neighbourhood of there is a transverse horizon such that $k^a = U\partial_U - V\partial_V$ as for the crossover in Schwarzschild.*

On a Killing horizon we can define

Definition 4.7 *The* surface gravity κ *is defined by*

$$\nabla_a k_b k^b = -2\kappa k_a \,, \qquad or\ k^a \nabla_a k^b = \kappa k^b \,, \tag{56}$$

where in the static case, k_a is understood to be normalized to have $k_a k^a = 1$ at large distances. For Schwarzschild $\kappa = 1/4m$.

Black hole thermodynamics starts with the Bekenstein bound on the entropy S: in a region of radius R and mass-energy E the entropy is constrained by

$$S < \frac{2\pi k R E}{\hbar c} \tag{57}$$

where we have not set the usual fundamental constants $k, \hbar, c$ to unity. It was arrived at by consideration of throwing objects with entropy into black holes and trying to avoid violations of the second law arising from the black hole eating entropy. In this view, the black does have entropy

$$S_{BH} = \frac{kA}{4G} \tag{58}$$

and this is taken to be the maximal entropy state, i.e., the Bekenstein bound is saturated by the black hole entropy.

Classically, one does not think of black holes as having microstates that could give rise to an entropy in view of the black hole uniqueness theorems. These seem to imply that the black hole state is unique, whereas an enropy suggests the existence of many equally likely microstates compatible with given macroscopic observables. The black hole entropy is usually understood as having its origin in quantum gravity.

This chain of reasoning subsequently led to the *Holographic principle*, that the maximum number N of states in a spatial region of radius R satisfies

$$N < \exp S_{BH}(R) \tag{59}$$

This comes from the definition of entropy of a system as $S := -\sum_i p_i \log p_i$ where p_i is the probability of the ith state. If the system is equidistributed, $p_i = 1/N$, where N is the number of states (the dimension of the Hilbert space of the system) we obtain $S = \log N$. This is counter-intuitive without general relativity because one thinks of the number of states in a region as being the exponential of the volume rather than the area. However, gravitational collapse reduces this if there is too much matter (too many particles) and indeed the vast bulk of the entropy is undertood to be gravitational.

The second law of thermodynamics: if the entropy is equated with the area of the event horizon in black hole thermodynamics, the 2nd law states that it can only increase. We have the area theorem

Proposition 4.1 *The area of an event horizon is non-decreasing.*

Proof: This is a simple consequence of the Sachs (or Raychauduri) equations

$$\dot{\rho} = \rho^2 + |\sigma|^2 + \Phi_{00} \,. \tag{60}$$

This shows that in particular $\dot{\rho} \geq \rho^2$ when the dominant energy condition is satisfied. Thus, if $\rho = \rho_0 > 0$ at some affine parameter value $t = 0$ on the generator, it is bounded below by the solution

$$\rho_0(t) = \frac{\rho_0}{1 - \rho_0 t} \,, \tag{61}$$

the solution to $\dot{\rho} = \rho^2$ with the same initial condition. Thus $\rho \to \infty$ in finite time. This introduces a *cusp* after which the null geodesic must then leave the horizon (see picture), contradicting its being a generator of $\mathcal{H}$. $\square$

The first law of black hole thermodynamics: for a variation of a closed system with rotation and charge can be stated as

$$dE = TdS + \Omega dJ + \Phi_H dQ \tag{62}$$

Here E is the total energy, T the temerature, Ω the angular velocity, J the angular momentum, Q the charge and ϕ the electrostatic potential. In the context of black holes, the total energy is the mass, we identify the temperature with the surface gravity by

$$T = \kappa/2\pi \tag{63}$$

and S with the area.

For Reissner Nordstrom, $\Phi = \Phi_H$ is the electric potential at the horizon and Q the total charge, and, in the case of the Kerr solution, Ω is the angular velocity, and J the angular momentum.

There are a number of strategies for proving these formulae. The most basic is to simply establish sufficient relations between the various quantities $(M, A, \Omega_H, J, \phi_H, Q)$ as can be read off from the black hole metric, and then to differentiate it. The simplest example is for Schwarschild where the area is that associated with the Schwarzschild radius $r = 2M$, $A = 16\pi M^2$, upon which differentiation yields

$$dM = \frac{dA}{32\pi M} = \frac{\kappa}{2\pi} dA\,, \tag{64}$$

giving the most basic version.

If we wish to introduce charge, we must consider the Reissner-Nordstrom solution

$$ds^2 = \frac{\Delta(r)}{r^2} dt^2 - \frac{r^2}{\Delta(r)} dr^2 - r^2 ds^2_{S^2}\,, \qquad \Delta(r) = r^2 - 2Mr + Q^2. \tag{65}$$

This satisfies the Einstein equations with electromagnetic potential

$$A = \frac{Q}{r} dt\,. \tag{66}$$

The Killing horizons are where $\Delta = 0$ giving

$$r_\pm = M \pm \sqrt{M^2 - Q^2}\,, \tag{67}$$

assuming $Q < M$. The outer one is the event horizon and a short calculation shows that differentiating the obvious relation $A = \pi r_+^2$ now gives

$$dM = \frac{\sqrt{M^2 - Q^2}}{2\pi r_+^2} dA + \frac{Q}{r_+} dQ\,. \tag{68}$$

The coefficient of dQ is indeed the value of the potential at the horizon. It is a more complicated task to see that the surface gravity does indeed appropriately give the coefficient of dA (see the exercises). Even more nontrivially, this works as stated above for the Kerr-Newman solution where there is also rotation.

The zeroth law of Black hole thermodynamics: In the analogy with thermodynamics, κ plays the role of temperature via $T = \kappa/2\pi$. The zeroth law is that the temperature is constant in equilibrium. It is easy to see that the surface gravity κ is constant up the generators of $\mathcal{H}$, because k^a is Killing. We will see in the problems that for a bifurcate Killing horizon κ is actually constant over the horizon. Hence it is constant everywhere. The next result follows in greater generality but we will not prove it here.

There is also a third law, that the entropy of an object at absolute zero is zero. This fails for black holes for a number of reasons, but a vaguer version, that once cannot approach absolute zero temperature with a finite number of processes does seem reasonable, as $T \to 0$ corresponds to $M \to \infty$.

The glaring omission in all this is of course that the temperature of a black hole classically would seem to have to be zero. This will be seen to be resolved by Hawking radiation.

5 QFT in curved space-time

The main few feature in curved space will be that we will no longer have a clear grasp of the concept of particle. Associated is the lack of uniqueness in defining the vacuum.

5.1 Fields in curved space-time

The main linear fields are Klein Gordon $\phi(x)$, Maxwell $A_a(x)$ and spinor fields (Dirac etc.). When coupling to a metric, we often adopt the minimal coupling prescription, that we take the flat space action, and replace coordinate derivatives by covariant derivatives sufficient to guarantee covariance. However, we could in principle include additional curvature terms if desired. For example, for scalar wave equation (Klein-Gordon) we can have

$$S[\phi] = \tfrac{1}{2} \int_M g^{ab} \partial_a \phi \partial_b \phi - (aR + m^2)\phi^2 \, \mathrm{d}\nu_g \,, \qquad \mathrm{d}\nu_g = \sqrt{-g} d^4 x \,. \tag{69}$$

Here m the mass and a is a number that can be zero, but when non-zero violates minimal coupling. This yields field equations

$$\left(\Box + m^2 + aR\right) \phi = 0 \,. \tag{70}$$

However, the scalar curvature term has some utility because, when[4] and $m = 0$, this equation is conformally invariant under

$$(g_{ab}, \phi) \to \left(\Omega^2 g_{ab}, \frac{\phi}{\Omega} \right) \,, \tag{71}$$

for any $\Omega(x) \neq 0$.

We remark on the differential form version of the kinetic term

$$\int d\phi \wedge {}^* d\phi \tag{72}$$

which leads to the coordinate formula for the wave operator

$$\Box \phi = {}^*(d^* d\phi) = \frac{1}{\sqrt{-g}} \partial_{\underline{a}} (\sqrt{-g} g^{\underline{ab}} \partial_{\underline{b}} \phi) = \nabla_a \nabla^a \phi \,. \tag{73}$$

[4] in d dimensions, when $a = (d-2)/4(d-1)$, although with a different scaling weight.

We determine the energy momentum tensor by

$$T^{ab} = -\frac{\delta S}{\delta g_{ab}} \tag{74}$$

so as to give the source term for the Einstein equations. This yields for $a = 0$

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab}((\partial\phi)^2 - m^2\phi^2) .$$

For an observer with 4-velocity U^a, the field has 4-momentum density $T_{ab}U^b$.

Differential forms come into their own in Maxwell theory. These are equations on a 1-form potential $A = A_a dx^a \in \Omega^1$ defined up to the gauge freedom $A_a \to A_a + \partial_a g(x)$, or $A \to A + dg$, for arbitrary $g(x)$. The field is

$$F = F_{ab} dx^a \wedge dx^b = dA \in \Omega^2 , \qquad F_{ab} = \nabla_{[a} A_{b]} , \tag{75}$$

and the action coupled to gravity is

$$S[A_a] = \frac{1}{4} \int_M F \wedge {}^*F = \frac{1}{2} \int_M F_{ab} F^{ab} d\nu_g , \tag{76}$$

with Bianchi identities $dF = 0$, or $\nabla_{[a} F_{bc]} = 0$ and field equations

$$d^*F = 0 , \qquad \text{or} \qquad \nabla^a F_{ab} = 0 . \tag{77}$$

In order to obtain a deterministic equation, it is normal to impose Lorenz gauge $\nabla^a A_a = 0$ upon which these equations reduce to the wave equation $\Box A_a = 0$ although there is nevertheless still residual gauge freedom under $A \to A + dg$ with $\Box g = 0$. These equations are conformally invariant (see problem sheet).

The stress-energy tensor in this case is

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} . \tag{78}$$

Conformal invariance here is manifested in the fact that $T_a^a = 0$. This is a general property of conformally invariant field theories, and is a consequence of the invariance of the action under $\delta g_{ab} = \omega g_{ab}$.

In both cases we have the positivity of energy manifested in the *dominant energy* condition that for any non-zero timelike or null vector t^a,

$$T_{ab} t^a t^b \geq 0 \tag{79}$$

with equality in the timelike case if only if $F_{ab} = 0$ or $\nabla_a \phi = 0$.

Half-integer spin fields require the introduction of spinors in curved space-times. This is best done in terms of two component spinors which have a variety of applications in GR and is discussed in appendix B.1.

5.2 The field theory phase space

From hereon we will take space-time to be globally hyperbolic with a foliation by Cauchy hypersurfaces Σ_t where $t \in \mathbb{R}$ is a time-like coordinate. The classical phase space V is the vector space of fields satisfying the linear field equations. Such fields are usually taken to have some degree of smoothness and asymptotic fall-off sufficient for the various formulae that we will use to make sense. We will assume that our fields have been so restricted but we will not be too specific unless it is important.

For the wave equation and neutrino equation we have respectively

$$V_\phi = \{\phi(x) \in C^\infty(M) | (\Box + m^2 + aR)\phi = 0\}\,, \tag{80}$$

$$V_\psi = \{\psi_A(x) \in C^\infty_A(M) | \nabla^{AA'}\psi_A = 0\}. \tag{81}$$

We can identify these with initial data on some Σ_t

$$V_\phi = \{(\phi(x), \dot{\phi} = \nabla_n \phi) \in C^\infty(\Sigma_t) \times C^\infty(\Sigma_t)\}\,, \tag{82}$$

$$V_\psi = \{\psi(x) \in C^\infty_A(\Sigma_t)\}\,, \tag{83}$$

where n is the unit normal to Σ_t. For the Dirac equation, being first order, we only need specify the field on Σ_t. For Maxwell, there is the gauge freedom to contend with and so we will leave that until later.[5]

In the case of integral spin, the phase space has a natural symplectic structure or skew form, whereas in half-integral spin we have a symmetric form. This explains the choice of statistics we will impose. These can be obtained in the usual way from a Legendre transformation of the action to obtain a Hamiltonian formalism. We obtain

$$\Omega(\phi_1, \phi_2) := \int_\Sigma {\phi_1}^* d\phi_2 - {\phi_2}^* d\phi_1 = \int_\Sigma (\phi_1 \dot{\phi}_2 - \phi_2 \dot{\phi}_1) d^3\nu_\Sigma\,, \tag{84}$$

$$S(\psi_{1A}, \psi_{2A}) := \int_\Sigma (\psi_{1A}\bar{\psi}_{2A'} + \psi_{2A}\bar{\psi}_{1A'}) d\nu^{AA'}_\Sigma \tag{85}$$

where Σ is a Cauchy hypersurface, $\dot{\phi} = n^a \nabla_a \phi$ where n^a its unit normal and $d\nu_\Sigma$ the volume element of the metric restricted to Σ and

$$d\nu^a_\Sigma = \frac{1}{6}\epsilon^a_{bcd} dx^b \wedge dx^c \wedge dx^d\,, \qquad d\nu_\Sigma = n_a d\nu^a_\Sigma\,. \tag{86}$$

[5] Alternatively there are constraints if we work at the level of the Maxwell tensor.

This is easily seen to be conserved (time-independent) as a consequence of the equations of motion. Such formulae for other theories can be systematically derived as the variation of the boundary term from the first variation of the action evaluated on solutions to the equations.

This skew form Ω is a symplectic form dual to the Poisson bracket on V_ϕ that can be expressed on Σ_t as

$$\{\phi(x), \dot{\phi}(x')\} = \delta^3(x, x')\,, \qquad \{\phi(x), \phi(x')\} = \{\dot{\phi}(x), \dot{\phi}(x')\} = 0\,. \tag{87}$$

where $\delta^3(x, x')$ is the 3-dimensional delta functional normalized against $d^3\nu_\Sigma$ by

$$f(x') = \int f(x)\delta^3(x, x')d^3\nu_\Sigma. \tag{88}$$

[Note that suddenly in these formulae we are regarding the evaluation of $\phi(x)$ at x as a coordinate or function on the phase space V_ϕ.]

The symmetric forms for half-integer spin fields signify the need for Fermi statistics and anticommutators, but we wont develop these further here.

We can then attempt to construct field operators $\hat{\phi}$ satisfying corresponding commutator equations

$$[\hat{\phi}(x), \hat{\dot{\phi}}(x')] = i\hbar\delta^3(x, x') \tag{89}$$

with other commutators zero. There are many ways to achieve this however.

To develop quantum field theory we must relate V_ϕ to a 1-particle Hilbert space $\mathcal{H}$ and defining the Fock space to be

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \odot \mathcal{H}) \oplus \ldots = \oplus_{n=0}^{\infty} \odot^n \mathcal{H}\,. \tag{90}$$

This is done by Fourier transform in flat space-time by decomposing complex fields into plane-wave modes $e^{-ik\cdot x}$

$$\mathbb{C} \otimes V_\phi = V_\phi^+ \oplus V_\phi^-\,. \tag{91}$$

where V_ϕ^+ is made up of the plane waves of positive frequency, i.e., $e^{-ik\cdot x}$ where k_a is the momentum satisfying the *on-shell* condition $k^2 = m^2$ and in particular the time-like component $k_0 = \omega$, the frequency, should be positive for elements of V_ϕ^+ and negative for elements of V_ϕ^-.

On flat space then we discover that Ω yields a positive definite form on V_ϕ^+ and negative definite on V_ϕ^-. This can be seen Fourier transform

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}(\phi_\mathbf{k}^+ \mathrm{e}^{-ik\cdot x} + \phi_\mathbf{k}^- \mathrm{e}^{ik\cdot x}) \tag{92}$$

where $k_a = (\omega, \mathbf{k})$ and $\omega = +\sqrt{\mathbf{k}^2 + m^2} > 0$. We then have

$$\Omega(\bar{\phi}, \phi) = i \int \frac{d^3\mathbf{k}}{2\omega} (|\phi^+_{\mathbf{k}}| - |\phi^-_{\mathbf{k}}|^2) \tag{93}$$

The Hilbert space $\mathcal{H}$ is then that spanned by $\phi^+_{\mathbf{k}} \mathrm{e}^{-ik\cdot x}$ with positive definite norm given above.

We usually formalize this by introducing for each momentum $\mathbf{k}$ raising and lowering operators $a^\dagger_{\mathbf{k}}$ and $a_{\mathbf{k}}$ respectively that create and annihilate Fock space states $|k\rangle = e^{-ik\cdot x} \in \mathcal{H}$. We introduce field operators by

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2} 2\omega} (a_{\mathbf{k}} e^{-ik\cdot x} + a^\dagger_{\mathbf{k}} e^{ik\cdot x}) \, . \tag{94}$$

[Note the dualization here so that $a^\dagger_{\mathbf{k}} = \Omega(\hat{\phi}, \mathrm{e}^{-ik\cdot x})$.] The creation and annihilation operators then satisfy

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}') \, , \tag{95}$$

with the other commutators vanishing and these then imply (87).

5.3 Quantization on a globally static space-time

The Fourier transform will not naturally be available to us in a general curved space-time. However we can expect to be able to perform our splitting when we have a time-like symmetry $\partial/\partial t$ for example with a globally static space-time $\mathbb{R} \times \Sigma$ with coordinates (t, x^i) and metric of the form

$$ds^2 = |g_{00}| dt^2 - h_{ij} dx^i dx^j \, . \tag{96}$$

We can separate variables considering solutions of the form and define V^+_ϕ to be those solutions going like

$$f(x) = \mathrm{e}^{-i\omega t} \chi(\mathbf{x}). \tag{97}$$

and we will have

$$\langle f|f\rangle := i\Omega(\bar{f}, f) = 2\omega \int |\chi|^2 \sqrt{g_{00} h} d^3x \, . \tag{98}$$

Let $\chi_n(\mathbf{x})$ be an orthonormal basis of functions on Σ satisfying the reduced equation which takes the form

$$(\Delta + \omega_n^2)\chi_n = 0\,, \tag{99}$$

for some second order operator Δ on Σ. These might be complex, for example on $\mathbb{R} \times S^2$ with product metric for χ_n we might choose the $Y_{lm}(\theta, \phi)$, but should satisfy the completeness relation

$$\sum_n \bar{\chi}_n(x)\chi_n(x') = \delta^3(x, x')\,, \tag{100}$$

where the delta function is understood be against the measure $\sqrt{g_{00}h}$.

More generally, on a non-compact spatial slicing such as $\mathbb{R}^3$, n, m must be replaced by continuous variables such as $\mathbf{k}, \mathbf{k}'$ for flat spatial slices, for the Fourier transform as above. Then δ_{nm} is replaced by $\delta^3(\mathbf{k} - \mathbf{k}')$ and $\sum_n$ by $\int_{\mathbb{R}^3} d^3\mathbf{k}/\omega_{\mathbf{k}}$ as we will see later. For the more formal development we will stick to the discrete modes but will introduce the continuous alternatives when needed.

Then define an orthonormal basis for the positive frequency solutions to be

$$P_n(x) = \frac{e^{-i\omega_n t}}{\sqrt{2\omega_n}}\chi_n\,, \qquad \langle P_n|P_m\rangle = \delta_{mn}. \tag{101}$$

We can similarly define a basis for the negative frequency solutions by

$$N_n(x) = \frac{e^{i\omega_n t}}{\sqrt{2\omega_n}}\bar{\chi}_n\,, \qquad \langle N_n|N_m\rangle = -\delta_{mn} \tag{102}$$

We take the 1-particle Hilbert space to be $\mathcal{H} = \{P_n\}$.

Introduce harmonic oscillators a_n for each n satisfying the usual

$$[a_n, a_m^\dagger] = \delta_{mn}\,, \qquad [a_n, a_m] = 0 = [a_n^\dagger, a_m^\dagger], \tag{103}$$

with the usual vacuum satisfying $a_n|0\rangle = 0$. The single particle states

$$|n\rangle := a_n^\dagger|0\rangle \tag{104}$$

can be identified now with $P_n \in H$ and N-particle states

$$|n_1, \ldots, n_N\rangle := a_{n_1}^\dagger \ldots a_{n_N}^\dagger|0\rangle \tag{105}$$

can similarly be identified with a basis of the N-particle space, $P_{n_1} \dots P_{n_N}$ of $\odot^N \mathcal{H}$.

Now define the quantum fields $\hat{\phi} : \mathcal{F} \to \mathcal{F}$ by

$$\hat{\phi}(x) = \sum_n (a_n P_n(x) + a_n^\dagger N_n(x)) \,. \tag{106}$$

Note that this is real with our choice of $N_n = \bar{P}_n$.

As a check, define the commutator by

$$\Delta(x, x') := [\hat{\phi}(x), \hat{\phi}(x')] \,. \tag{107}$$

Then the commutation relations imply that

$$i\Delta(x, x') := [\hat{\phi}(x), \hat{\phi}(x')] = \sum_n P_n(x) N_n(x') - P_n(x') N_n(x) \,. \tag{108}$$

Setting $t = t'$ gives zero from the completeness relation, whereas, if we first differentiate with respect to t' before setting $t = t'$ we find

$$[\hat{\phi}(x), \dot{\hat{\phi}}(x')]_{t=t'} = \frac{i}{2} \sum_n \chi_n(\mathbf{x}) \chi_n(\mathbf{x}') + \chi_n(\mathbf{x}') \chi_n(\mathbf{x}) = i\delta^3(\mathbf{x}, \mathbf{x}') \,. \tag{109}$$

where the first equality follows from the normalizations and time derivative, and the second from the completeness relation for the basis χ_n. Thus we have successfully quantized $\phi \to \hat{\phi}$ so that the Poisson bracket turns into the commutator.

It is clear that $\Delta(x, x')$ also satisfies the wave equation in each of x and x' and so, as a function of x, it is the solution the wave equations with initial data $(\phi, \dot{\phi}) = (0, \delta(\mathbf{x}, \mathbf{x}'))$ at $t = t'$. We furthermore now have a unique vacuum $|0\rangle$ and a good particle interpretation.

Although Schwarzschild is static, this isnt sufficient to understand QFT on a standard collapse to black hole background.

5.4 The stress-energy tensor and the Casimir effect

The stress-energy tensor provides one of the most basic observables. In general a killing vector k^μ gives rise to a conserved quantity attached to a Cauchy surface Σ

$$E(k) := \int_\Sigma k^\mu T_{\mu\nu} d\Sigma^\nu \,, \qquad d\Sigma^\nu = \epsilon^{\kappa\lambda\mu\nu} dx_\kappa \wedge dx_\lambda \wedge dx_\mu \,. \tag{110}$$

The simplest examples are provided by translation Killing vectors in flat space-time giving rise to the conserved total 4-momentum

$$P_\mu = \int_{t=const.} T_{\mu\nu} d\Sigma^\nu \,, \tag{111}$$

with time-like component $H = P_0 =$ the Hamiltonian or energy. In quantum field theory this is an operator and we can take its expectation value. However, like many observables in QFT, it diverges in at least two ways.

Taking the scalar field in a static space-time as in the previous section, we can compute the Hamiltonian operator to be

$$\begin{aligned} \hat{H} &:= \int_{\Sigma_t} \hat{T}_{00} d\Sigma^0 \\ &= \tfrac{1}{2} \int_{\Sigma_t} (\partial_t \hat{\phi})^2 + (\nabla \hat{\phi})^2 \, d\Sigma^0 \\ &= \tfrac{1}{2} \int_{\Sigma_t} ((\partial_t \hat{\phi})^2 - \hat{\phi} \Delta \hat{\phi}) \, d\Sigma^0 \,. \end{aligned} \tag{112}$$

The second equality is an integration by parts. Substituting in (106) we obtain, after using the orthogonality of the χ_n,

$$\hat{H} = \sum_n (a_n a_n^\dagger + a_n^\dagger a_n) \frac{\omega_n}{2} \int_\Sigma |\chi_n|^2 \,, \tag{113}$$

using also the explicit time-dependence of the modes. This gives the vacuum energy

$$E = \langle 0 | \hat{H} | 0 \rangle = \sum_n \frac{\omega_n}{2} \int_\Sigma |\chi_n|^2 d\Sigma^0 \,. \tag{114}$$

The answer is as you would expect for the sum of the zero-point energies of each mode of the field thought of as a harmonic oscillator.

The infrared divergence. In the non-compact case, we dont expect normalizable eigenfunctions. When $\Sigma = \mathbb{R}^3$, eigenfunction are spanned by momentum eigenstates $e^{i\mathbf{k}\cdot\mathbf{x}}$, where

$$\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2} \,, \qquad \sum_n \to \int \frac{d^3\mathbf{k}}{\omega_{\mathbf{k}}} \,, \qquad \delta_{mn} \to \delta^3(\mathbf{k} - \mathbf{k}') \,.$$

The delta functions reflect the fact that these states are not normalizable. Indeed $\int_\Sigma \chi_n^2 \, d\Sigma^0 \to \int_{\mathbb{R}^3} 1 \, d^3\mathbf{x}$, i.e. the volume of $\mathbb{R}^3$. Alternatively this is the momentum delta-function evaluated at the origin $\delta^3(\mathbf{0})$, hence the term infrared, for a low momentum divergence. This is physically reasonable if there is a local energy density, and we can either work on a compact Σ, at which point $\int_\Sigma |\chi_n|^2 d\Sigma^0 = 1$, or consider the coefficient $\sum_n \frac{1}{2}\omega_n$ to be an energy per unit volume.

The ultraviolet divergence. We have $\omega_n \to \infty$ as $n \to \infty$ so the sum never converges. One could perhaps have seen this coming: the standard expressions for $\langle 0|\phi(x)\phi(y)|0\rangle \sim \frac{1}{(x-y)^2}$ in 4d, are all divergent as $x \to y$ as a consequence of the commutation relations (89) and the additional derivatives in the stress-energy tensor do nothing to soften these divergences. These are short-distance or in momentum space ultraviolet divergences.

However, we can regulate these expressions to obtain finite answers. To focus ideas consider the 1-dimensional case case where $\Sigma = S^1$ of length L. The modes of the previous subsection can be taken to be:

$$\chi_n = \sqrt{\frac{2}{L}} \exp\left(\frac{2\pi i n x}{L}\right), \qquad \omega_n = \frac{2\pi |n|}{L}, \qquad n \in \mathbb{Z}. \tag{115}$$

With the two modes at each ω_n we have energy density per unit length

$$\mathcal{E} = \frac{\langle 0|H|0\rangle}{L} = \frac{2\pi}{L^2}\sum_n n\,. \tag{116}$$

The most elegant approach to this is ζ-function regulation. Here we regard the divergence as following from an attempt to evaluate a power series outside its domain of convergence, in this case, that of the ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{117}$$

which is to be evaluated at $s = -1$. This clearly defines $\zeta(s)$ as a holomorphic function for $\Re s > 1$. It requires some hard work to show that it extends as a meromorphic function over the complex plane with a pole at $s = 1$ to find that we can evaluate $\zeta(-1) = -\frac{1}{12}$ to give

$$\sum_{n=1}^{\infty} n \sim -\frac{1}{12}, \qquad \mathcal{E} = -\frac{2\pi}{12L^2}\,.$$

This might seem like too much black magic. A lower tech approach introduces a regulation parameter α by writing

$$\begin{aligned}\mathcal{E}(L,\alpha) &= \frac{2\pi}{L^2}\sum_n n\,\mathrm{e}^{-\frac{n\alpha}{L}} \\ &= -\frac{2\pi}{L}\frac{d}{d\alpha}\sum_n \mathrm{e}^{-\frac{n\alpha}{L}} = -\frac{2\pi}{L}\frac{d}{d\alpha}\frac{1}{1-\mathrm{e}^{\frac{\alpha}{L}}} \\ &= \frac{2\pi}{L^2}\frac{\mathrm{e}^{-\frac{\alpha}{L}}}{(1-\mathrm{e}^{-\frac{\alpha}{L}})^2}\,. \end{aligned} \tag{118}$$

Expanding this in a Laurent series as $\alpha = 0$ gives

$$\mathcal{E}(L,\alpha) = \frac{\pi}{2L^2}\frac{1}{\sinh^2\frac{\alpha}{2L}} = \frac{2\pi}{\alpha^2} - \frac{2\pi}{12L^2} + \frac{1}{L^2}O\left(\frac{\alpha^2}{L^2}\right)\,. \tag{119}$$

Subtracting off the divergent first term, we arrive at the same answer as $\alpha \to 0$ as via ζ-function regularization.

Further remarks:

- Up to a factor, this is the same calculation as that for the Casimir effect in electromagnetism. This predicts a force $F = -dE/dL$ of attraction between two conducting plates separated by a distance L. This is a real effect due to vacuum polarization and is experimentally measured.

  In electromagnetism, the electric field vanishes at the two conducting plates forcing the field to vanish there so that they come in modes $\sin(n\pi x/L)$ much like the calculation above. We therefore expect the same behaviour for quantum field theory in a flat space of non-trivial topology S^1.

- As a regularization procedure, we could have Wick-ordered the stress-energy tensor computing $\langle 0| : T_{\mu\nu} : |0\rangle$ to remove the divergence, but this would have yielded $E(L) = 0$ and removed a genuine physical effect.

- As far as computing the force is concerned, one only needs the derivative of $E(L)$ and so the divergent term in (119) can be safely ignored. However, for the back-reaction on the gravitational field, we would expect the Einstein equations to be sourced by $\langle 0|T_{\mu\nu}|0\rangle$ in some semi-classical approximation at least. Furthermore, the divergent constant might in more complicated circumstances depend on details of the geometry which might also depend on time. These divergences do no arise with ζ-function regularisation.

- Another takewaway is that the renormalised energy is negative, and indeed has to be in order to give the correct sign for the Casimir effect.

5.5 The Bogoliubov transformation

As a first attempt to understand time dependence, consider a space-time M that is flat in past and future regions respectively M^- and M^+ separated by some intermediate region of time dependent disturbance. These will lead to two different definitions of positive and negative frequency and in general we do not expect them to agree.

We can define our quantum field in the two regions as above and in order to study the transition between them, we will impose the condition that the quantum field so constructed is the same in both regions

$$\hat{\phi}(x) = \sum_n P_n^- a_{n-} + N_n^- a_{n-}^\dagger = \sum_n P_n^+ a_{n+} + N_n^+ a_{n+}^\dagger \tag{120}$$

with the $-$ denoting the past region and the $+$ the future.

We wish to deduce from this how to identify the Fock spaces $\mathcal{F}_+$ and $\mathcal{F}_-$ constructed as above in their respective regions M_+ and M_-.

The classical scattering matrix and Bogoliubov coefficients. The $(P_n^\pm, N_n^\pm)$ are independent bases of $\mathbb{C}\otimes V_\phi$ and so are related by linear maps

$$\begin{pmatrix} P_n^+ & N_n^+ \end{pmatrix} = \sum_m \begin{pmatrix} P_m^- & N_m^- \end{pmatrix} \begin{pmatrix} \alpha_{mn} & \beta_{mn} \\ \bar{\beta}_{mn} & \bar{\alpha}_{mn} \end{pmatrix} = \begin{pmatrix} P_m^- & N_m^- \end{pmatrix} S\,, \tag{121}$$

where

$$S = \begin{pmatrix} \alpha_{nm} & \beta_{nm} \\ \bar{\beta}_{nm} & \bar{\alpha}_{nm} \end{pmatrix} \tag{122}$$

is the classical S-matrix and its entries are the *Bogoliubov coefficients* which can be obtained for example as

$$\alpha_{mn} = i\Omega(\bar{P}_m^-, P_n^+)\,, \qquad \beta_{mn} = -i\Omega(\bar{P}_m^-, N_n^+)\,. \tag{123}$$

and similar fomulae related by complex conjugation.

Lemma 5.1 *The* $(P_n^{\pm}, N_n^{\pm})$ *are both orthonormal bases for the pseudo-Hermitian form* $\langle \phi, \phi \rangle := i\Omega(\bar{\phi}, \phi)$ *and so we obtain (pseudo-unitarity) relations*

$$\alpha^\dagger \alpha - \beta^T \bar{\beta} = 1, , \qquad \alpha^\dagger \beta - \beta^T \bar{\alpha} = 0 \,. \tag{124}$$

Now (120) gives transpose relations for the oscillators (summation convention assumed)

$$a_{n-} = \alpha_{nm} a_{m+} + \beta_{nm} a^\dagger_{m+} \,, \qquad a_{n+} = \bar{\alpha}_{nm} a_{m-} - \bar{\beta}_{nm} a^\dagger_{m-} \,. \tag{125}$$

The vacua $|0\rangle_\pm$ in $M^\pm$ willbe defined by

$$a_{n+}|0\rangle_+ = 0 \,, \qquad a_{n-}|0\rangle_- = 0 \,. \tag{126}$$

If $\beta = 0$ these equations are equivalent, and the vacua can be identified at least up to phase. However, when $\beta \neq 0$ they are distinct as follows from computing the expected particle numbers in the two regions. Define particle number operators in the two regions by

$$N^\pm = \sum_n a^\dagger_{n\pm} a_{n\pm} \tag{127}$$

so if we evaluate N^+ in $|0\rangle_-$ we find

$${}_-\langle 0|N^+|0\rangle_- = {}_-\langle 0| \sum_n a^\dagger_{n+} a_{n+} |0\rangle_- = {}_-\langle 0| \sum_{nm} \bar{\beta}_{nm} \beta_{nm} |0\rangle_- = \sum_{mn} |\beta_{mn}|^2 \,. \tag{128}$$

so the changing gravitational field creates particles (in pairs).

The one dimensional example for the simple harmonic oscillator is instructive. Then $|\alpha|^2 - |\beta|^2 = 1$ allows us to write

$$\alpha = \cosh \xi \,, \qquad \beta = \sinh \xi \tag{129}$$

and

$$a_+ = \alpha a_- + \beta a^\dagger_- = \alpha(x + ip) + \beta(x - ip) \tag{130}$$
$$= (\alpha + \beta)x + i(\alpha - \beta)p = e^\xi x + i\mathrm{e}^{-\xi} p \,. \tag{131}$$

so x is scaled by e^{ξ} and p by $\mathrm{e}^{-\xi}$. The new ground state is not the original ground state, but one that is a scaled gaussian, a so-called *squeezed state*, a ground state for an oscillator with a different frequency.

More generally, for free quantum field theory in curved space-time, we imagine that we are dealing with a collection of time dependent harmonic oscillators and the propagation of a vacuum in M^- to M^+ takes the vacuum to an analogue of a multi-harmonic oscillator squeezed state.

We would also like to extend the map S from the classical space V_ϕ to the Fock space $\mathcal{S} : \mathcal{F}_- \to \mathcal{F}_+$. As a first step, we can attempt to explicitly construct the new vacuum from the old in terms of the Bogoliubov coefficients. We have, up to a phase,

$$|0\rangle_+ = \frac{\mathrm{e}^{-F}|0\rangle_-}{|\det\alpha|^2}\,, \qquad F := \tfrac{1}{2}\sum_{nm}(\alpha^{-1}\beta)_{nm}a^\dagger_{n-}a^\dagger_{m-}. \tag{132}$$

Thus we get a picture in which we have an exponential distribution of particle pair creation operators $\frac{1}{2}\sum_{nm}(\alpha^{-1}\beta)_{nm}a^\dagger_{n-}a^\dagger_{m-}$ inserted.

To prove this, we first observe that if $F = \sum_{m,n} M_{mn}a^\dagger_{n-}a^\dagger_{m-}$ we have

$$\mathrm{e}^{-F}a_{n-}\mathrm{e}^{F} = a_{n-} + [a_{n-}, F] = a_{n-} + \sum_m M_{nm}a^\dagger_{m-}\,. \tag{133}$$

The truncation of the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{-F}a\mathrm{e}^{F} = a + \sum_n \frac{1}{n!}[[\ldots[[a,F],F]\ldots,F],F]_n \tag{134}$$

follows because the first commutator gives a combination of $a^\dagger_{n-}$s which commute with F. We therefore see that with $M_{mn} = (\alpha^{-1}\beta)_{mn}$ we obtain a linear combination of the a_{n+} operators (precisely a_{n+} after multiplying through by α).

We check the normalization factor in the one dimensional (or more generally diagonalizable) case

$${}_-\langle 0|\mathrm{e}^{-F^\dagger}\mathrm{e}^{-F}|0\rangle_- = \sum_{r=0}^{\infty}\frac{2r!}{r!^2}\left(\frac{|\beta|^2}{|2\alpha|^2}\right)^r = \frac{1}{(1-|\beta|^2/|\alpha|^2)^2} = |\alpha|^4 \tag{135}$$

The full quantum evolution operator takes the form

$$\mathcal{S} = \exp\left(\sum_{m,n} M_{mn}a^\dagger_{n-}a^\dagger_{m-} - \sum_{m,n}\bar{M}_{mn}a_{n-}a_{m-}\right)\,. \tag{136}$$

This is now unitary as the exponent is skew Hermitian so preserves norms. The coefficients M_{mn} are related non-linearly to the Bogoliubov coefficients.[6] However convergence is an issue, and this follows if $\sum_{mn} |\beta_{mn}|^2 < \infty$.

Fermions: for Fermions, the basic structures are the same, but signs change so that $|\alpha|^2 + |\beta|^2 = 1$. Odd numbers of particles can be produced via spectral drift of eigenvalues across 0.

An important remark is that although here we presented the calculation as a scattering matrix type calculation, we could have simply done this for two different slicings of a space-time with two families of observers.

Complex scalar field

For the real scalar field above we should have used the modes $\sin \mathbf{k} \cdot \mathbf{x}$ as a basis so as to have a real spatial field for both P_i and N_i and $P_i = \bar{N}_i$ conjugating only the $\mathrm{e}^{-i\omega t}$ part. For the complex scalar field we can use a complex basis $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ for the space-like parts. Then P_i and N_i are independent particles. Defining as before the inner product

$$(\phi_1, \phi_2) = \Omega(\bar{\phi}_1, \phi_2) \tag{137}$$

we assume the orthogonality relations

$$(P_i, P_j) = \delta_{ij} = -(N_i, N_j)\,, \qquad (P_i, N_j) = 0\,. \tag{138}$$

We have the interpretation of the positive energy modes P_i as particles with associated annihilation and creation operators $a_i, a_i^\dagger$ and negative energy modes N_i as antiparticles with wave functions $\bar{N}_i$ for which we can introduce creation and annihilation operators $b_n^\dagger, b_n$, i.e., we double the number of oscillators and we introduce the complex field operator

$$\hat{\phi} = \sum_i P_i a_i + N_i b_i^\dagger\,. \tag{139}$$

The Bogoliubov transformation from M^- to M^+ is now

$$\begin{pmatrix} P_i^+ & N_i^+ \end{pmatrix} = \sum_j \begin{pmatrix} P_j^- & N_j^- \end{pmatrix} \begin{pmatrix} \alpha_{ji} & \tilde{\beta}_{ji} \\ \beta_{ji} & \tilde{\alpha}_{ji} \end{pmatrix} = \begin{pmatrix} P^- & N^- \end{pmatrix} S\,, \tag{140}$$

[6] In the scalar case we have for example $\beta = \frac{M}{|M|} \sinh |M|$.

where now

$$S = \begin{pmatrix} \alpha_{ij} & \tilde{\beta}_{ij} \\ \beta_{ij} & \tilde{\alpha}_{ij} \end{pmatrix} , \qquad S^\dagger G S = G \, , \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{141}$$

We have

$$\begin{pmatrix} a_{i+} \\ b_{i+} \end{pmatrix} = \begin{pmatrix} \alpha_{ij} & \tilde{\beta}_{ij} \\ \beta_{ij} & \tilde{\alpha}_{ij} \end{pmatrix} \begin{pmatrix} a_{j-} \\ b^\dagger_{j-} \end{pmatrix} . \tag{142}$$

We can obtain from these the number of particles created of the ith mode of type a and b to be

$$\mathcal{N}_i^a = {}_-\langle 0| a^\dagger_{i+} a_{i+} |0\rangle_- = \sum_j |\beta_{ji}|^2, \qquad \mathcal{N}_i^b = {}_-\langle 0| b^\dagger_{i+} b_{i+} |0\rangle_- = \sum_j |\tilde{\beta}_{ji}|^2 \tag{143}$$

This leads to a relationship between the in and out vacua $|0\rangle_-$ and $|0\rangle_+$ as:

$$|0\rangle_- = \exp\left(\sum_{ij} a^\dagger_{i+} (\alpha^{-1} b^\dagger_{j+} \tilde{\beta})_{ij} \right) |0\rangle_+ / \det(\alpha)^2 \tag{144}$$

In practice for us $\tilde{\beta} = \beta$ as we are complexifying a real field.

6 Cosmological perturbations

6.1 Cosmological models

The Friedmann-Robertson-Walker (FRW) models are models in which we assume a collection of comoving observers for whom the universe is homogeneous (the same for each observer) and isotropic (the same in every direction). It is straightforward to deduce from this that the space-time can divided up into spatial sections of constant curvature with 3-metrics $\mathrm{d}s^2_{E^3_k}$ are for $k = 0, 1, -1$ the flat 3-metric, the round sphere or hyperbolic 3-space respectively. The FRW metrics are

$$\mathrm{d}s^2_{FRW_k} = \mathrm{d}t^2 - a(t)^2\,\mathrm{d}s^2_{E^3_k}\,. \tag{145}$$

They are all conformally flat and therefore can all be represented inside the Einstein cylinder. The value of k is determined by whether the density of the universe is greater than ($k = 1$), or less than ($k = -1$) or exactly equal to ($k = 0$) some critical value. This is currently too close to call.

The $k = +1$ case is the simplest since then

$$\mathrm{d}s^2_{FRW_1} = S(\eta)^2(\,\mathrm{d}\eta^2 - \,\mathrm{d}s^2_{S^3})\,, \qquad \eta(t) = \int^t \frac{dt}{a(t)}\,, \tag{146}$$

and we have abused notation by defining $a(\eta) = a(t(\eta))$. We usually assume perfect fluid energy momentum tensor

$$T_{ab} = \mathrm{diag}(\rho, p, p, p)\,,$$

In this conformal time coordinate, the main Einstein equation is the Friedmann equation is

$$\left(\frac{da}{d\eta}\right)^2 + ka^2 = \frac{8\pi G}{3}\rho a^4 + \frac{\lambda}{3}a^4\,. \tag{147}$$

Another independent equation can be expressed as the conservation equation

$$d\rho = -3(\rho + p)d\log a,. \tag{148}$$

We need an additional equation of state relating $p = f(\rho)$. For dust we have simply $p = 0$ and then the conservation gives simply $M = \rho a^3$ for some constant M (or for radiation $p = \rho/3$ we get $M = \rho a^4$).

The simplest 'dust' $k = 1$ case has, after solving the Friedmann equations

$$t = \frac{C}{2}(\eta - \sin\eta)\,, \qquad a(\eta) = \frac{C}{2}(1 - \cos\eta), \tag{149}$$

giving a cycloid in the (t, a) plane with angular parameter η. Thus we have a big bang $a = 0$ at $\eta = 0$ followed by a big crunch, with $a = 0$ at $\eta = 2\pi$.

In the conformal diagram one sees cosmological horizons very clearly as light rays are at 45° on the Einstein cylinder. We see that in general an observer at later time can see far away regions A and B whose causal pasts do not intersect. This then makes the apparent homogeneity of the universe a surprise. In a diagram that takes into account the size of a which tends to zero at the big bang, it is unclear whether the past of A and B can mix, whereas in the conformal diagram it is completely clear.

This surprising homogeneity is conventionally understood to be resolved by inflation which notes that there is a *surface of last scattering* at some time t_s perhaps 300,000 years or so after the big bang before which we cannot see what was going on and for which we can posit new physics. This surface is where radiation decouples from matter and after this time, we the main information that have from what went before is what we see from the cosmic microwave radiation. They then posit a period of inflation, which although incredibly short in terms of seconds, 10^{-32} or so soon after the big bang, but very long in terms of conformal time η. At the end of this period, the currently observable universe was still compressed into a size of less than one metre. This is modelled by gluing in an exponentially expanding region of de Sitter, which gives the pasts of A and B time to mix and homogenize so as to explain the apparent homogeneity and isotropy of the universe. More explicitly, a region (a Poincaré patch) in de Sitter can be put into the above flat space $k = 0$ Friedmann form with

$$a = \frac{e^{H_\lambda t}}{H_\lambda} = \frac{-1}{H_\lambda \eta}, \qquad \eta = -e^{-H_\lambda t}\,, \qquad \eta < 0\,. \tag{150}$$

It is this exponential relationship that allows the the huge discrpancy between η and proper time t. The with exponential expansion rate is given by the Hubble parameter

$$H_\lambda = \sqrt{\lambda/3} = \frac{1}{L},$$

which is inverse to the de Sitter radius L which gives the radius of cosmological horizons.[7] This is the standard model for the inflationary epoch $\eta \in [\eta_-, \eta_+]$ where $\eta_- \ll \eta_+ < 0$. Other motivations for inflation were the lack of abundance of monopoles and the flatness problem, that $k \sim 0$.

The boomerang collaboration more recently has argued that observational data shows both that $k \sim 0$ and furthermore, that the cosmological constant is positive. The positivity of the cosmological constant now generically gives that $a \to \infty$ for finite η whatever tje value of k. To see this, with perfect fluid as above, when a is large, the λa^4 term dominates the RHS of the Friedmann equation, and in conformal time, for large a, the equation approximately gives

$$\frac{da}{d\eta} \simeq \sqrt{\frac{\lambda}{3}} a^2 , \quad a \sim \frac{L}{\eta_{\mathscr{I}} - \eta} , \qquad L = \sqrt{\frac{3}{\lambda}}$$

and $a(\eta)$ has a pole at $\eta = \eta_{\mathscr{I}}$ so that we get a $\mathscr{I}^+$ that looks like that of de Sitter. Thus de Sitter is a good approximation at late times too.

This exponential expansion arises because a positive cosmological constant looks like a stress-energy tensor $T_{ab} = \text{diag}(\lambda, -\lambda, -\lambda, -\lambda)$, so that although the effective energy density is positive, the pressure is negative. At the current age of the universe, the contribution of λ to the energy density is thought to be of the same order as that of the matter including dark matter (visible matter being thought to be 3%, dark matter 30% and cosmological constant about 67% of the critical mass of the universe). Such a ratio of matter to cosmological constant is extremely high at early times, and extremely low at late times, and this sometimes leads to the 'why are we alive now?' question. The later periods are however, very cold and boring, and the early periods rather hot, and too early for structure to form, so there are anthropic arguments here.

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[7] These cosmological horizons can be seen most simply in the static patch of de Sitter given by

$$ds^2 = (1 - r^2/L^2)d\tau^2 - \frac{dr^2}{1 - r^2/L^2} - r^2 d\Omega^2_{S^2} .$$

Both this patch and the Poincaré patch above are the interiors of lightcones of points at infinity on the Einstein cylinder. The $k = 0, -1$ models can similarly be obtained as subsets of the Einstein cylinder, see Hawking and Ellis).

6.2 QFT on a cosmological background

A relatively simple context where we can illustrate and apply the ideas above is in Cosmology. We first give a general discussion of the choice of vacuum and particle interpretation and then go on to one of the successes of QFT in curved space-time: the prediction of the scale invariant density fluctuations from the inflationary universe assumptions.

For simplicity, we work with the FRW metric with flat spatial slices

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x} \cdot d\mathbf{x} = a^2(d\eta^2 - d\mathbf{x} \cdot d\mathbf{x}) , \qquad (151)$$

where t is the proper time of the co-moving observers, a the scale factor, and the conformal time η is obtained by solving $d\eta = dt/a(t)$.

The separable solutions to the wave equation can be written as

$$\phi_{\mathbf{k}} = \frac{v_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}}{a(\eta)} \qquad (152)$$

where $k = |\mathbf{k}|$ and $v_k(\eta)$ are now time-dependent harmonic oscillators

$$\ddot{v}_k + \omega_k^2 v_k = 0 , \qquad \cdot = \frac{d}{d\eta} \qquad (153)$$

with time-dependent frequency $\omega_k(\eta)$ given by

$$\omega_k^2 = k^2 + m^2 a^2 - (1 - 6\xi)\frac{\ddot{a}}{a} . \qquad (154)$$

[In terms of t, the equation has a dissipation term.]

The symplectic form Ω reduces to $W(v_k, v_k') := v_k \dot{v}_k' - v_k' \dot{v}_k$, the Wronskian, on v_k which is conserved. However, now that the space-time is explicitly time-dependent, energy will not be conserved, and this will manifest in the space-time doing work on the field creating particles so there is no fixed vacuum.

With an arbitrary normalized choice of the v_k satisfying $W(v_k, \bar{v}_k) = 2i$ we can define an arbitrary separation into modes by

$$\hat{\phi} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2} a(\eta)} \left(a_{\mathbf{k}} \bar{v}_k(\eta) \mathrm{e}^{i\mathbf{k}\cdot x} + a_{\mathbf{k}}^\dagger v_k(\eta) \mathrm{e}^{-i\mathbf{k}\cdot x} \right) , \qquad (155)$$

with $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}')$ etc., as usual and signs chosen to make the field manifestly real. With this arbitrary choice, we find for the Hamiltonian

$$\hat{H} = \frac{1}{4} \int d^3\mathbf{k} \left(\bar{F}_k a_{\mathbf{k}} a_{-\mathbf{k}} + F_k a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger) E_k \right) \qquad (156)$$

where

$$E_k = |\dot{v}_k|^2 + \omega_k^2 |v_k|^2 \,, \qquad F_k = \dot{v}_k^2 + \omega_k^2 v_k^2 \,. \tag{157}$$

Clearly then with this implicit choice of vacuum, we have

$$E = \langle 0|\hat{H}|0\rangle = \delta^3(0)\frac{1}{4}\int d^3\mathbf{k} E_k \tag{158}$$

where as before, $\delta^3(0)$ is the infrared divergent volume of space whose coefficient is the local energy density.

The choice of vacuum and definition of a particle. In the discussion of Bogoliubov coefficients, we imagined that the vacuum was chosen to be the static one in the past and we then checked to see whether particles were created. At each time there is a natural choice of vacuum.

The instantaneous lowest energy vacuum. Notwithstanding the ultraviolet divergence we can define $|0_{\eta_0}\rangle$ at a fixed time η_0, by minimizing E_k for each mode subject to the normalization $W(v_k, \bar{v}_k) = 2i$. A brief calculation shows that we obtain initial data for (153) of the form

$$(v_k, \dot{v}_k)|_{\eta=\eta_0} = \frac{1}{\sqrt{2\omega_k}(\eta_0)}(1, -i\omega_k(\eta_0)) \tag{159}$$

This is what we would obtain by holding a (or ω_k) constant for an interval and defining the usual positive/negative energy splitting on the interval of time but then letting the interval tend to zero. The particle content measured in this vacuum is that of the instantaneous 'static' observer.

The adiabatic vacuum. When the ω_k are very slowly varying, the adiabatic approximation is to take this to define also the evolution of the vacuum. This arises as the WKB approximation at leading order; more generally, the adiabatic approximation takes the WKB approximation beyond leading order and then there will be particle creation. [8]

- We can now define a Bogoliubov transformation between any two times η_1 and η_2 and in general, as in the previous subsection, find particle creation.

[8] This requires a mechanism for the dissipation of the particles created and otherwise we are working in the full quantum theory in the Heisenberg representation where the state is fixed and the evolution is carried by the operators.

- In a cosmological context the initial conditions are less clear, and the choice of vacuum, and hence of how many particles there are, becomes time-dependent.

- The Bogoliubov coefficients diagonalize for each k but depends on the detailed form of the ω_k as a function of η. The diagonalised 1-dimensional formulae are sufficient for each k.

- The F_k in (156) determine the instantaneous particle creation at each time. For the instantaneous lowest energy vacuum they instantaneously vanish.

- At short length scales, k is large and dominates $\omega_k^2 \sim k^2$. So we only expect these effects to become important for longer wavelengths, i.e., k small and lower frequencies. These then correspond to wavelengths that are of the order of the curvature.

- A novel feature of (154) is that for large length scales, small enough k, m, ω_k^2 can become negative. This, as we will see, leads to the corresponding modes ceasing to oscillate and essentially freezing.

De Sitter space, the Bunch-Davies vacuum and freezing of modes. For simplicity we can take a to be constant in $M_\pm$ although M_- has to start with a big bang by the singularity theorems. Taking $m = \xi = 0$ for simplicity we find

$$\omega_k^2 = k^2 - \frac{2}{\eta^2}\,. \tag{160}$$

This can be solved explicitly in terms of Bessels functions. According to the previous subsection there will be particle creation, between instantaneous lowest energy vacua at different times caused by the expansion of the universe. However, we have some new features here also.

The Bunch-Davies vacuum. This is a unique vacuum for quantum fields on de Sitter that is invariant under the full de Sitter group $SO(1,4)$, regarding it as a homgeneous space. In the flat space Friedmann representation (151), with a given by (150), it can be defined by taking the positive frequency modes to be this that are positive frequency in the normal flat-space way in the far past as $\eta \to -\infty$ where $a \sim -1/\eta \to 0$ so that limit $\omega^2 \simeq k^2$ in that limit. Thus positive frequency modes are those that have η dependence of the form $\mathrm{e}^{-i\omega\eta}$ with $\omega > 0$ as $\eta \to -\infty$. By using the solutions in terms of Bessel's functions, these can be found explicitly for all η, (see §7.2 of Mukhanov & Winitzki) with the positive and negative frequency wave functions given by Hankel functions.

Primordial perturbations from zero-point fluctuations. The Bunch-Davies vacuum is usually taken to be the initial condition for inflation as the expansion of the universe is understood to have red-shifted away any excitations. However, (160) shows that for $m^2 < 2H_\lambda^2$, ω_k^2 will eventually become negative and the corresponding k-mode will stop oscillating.

The physical or proper frequencies and wave-numbers per unit lengths are $\omega_{\text{phys}} = \omega_k/a$ and $k_{\text{phys}} = k/a$, being necessarily scaled by the expansion factor a, and so satisfy,

$$\omega_{\text{phys}}^2 = k_{\text{phys}}^2 - 2H_\lambda^2 \,,$$

for simplicity, taking $m = 0$. In the far past, start with frequency $\omega \gg H$, then the k-mode, has $k_{\text{phys}} = k/a \gg H$. However, as the universe expands, a increases but k is constant, and so $k_{\text{phys}} = k/a \to 0$. Thus eventually ω_{phys}^2 and hence ω_k vanishes and becomes negative. At this point $k_{\text{phys}} = \sqrt{2}H_\lambda$ and the solution is frozen in time. When ω^2 goes negative, the solutions are now either exponentially decreasing or increasing. As $\eta \to 0_-$, the differential equation (153) becomes, in the case of $m/H_\lambda \sim 0$ and ignoring terms of $O(\eta^0)$

$$\ddot{v}_k - \frac{2}{\eta^2} v_k = 0 \tag{161}$$

so that as $\eta \to 0$ we have the approximate solution

$$v_k \sim c_1 \eta^{-1} + c_2 \eta^2 \,. \tag{162}$$

Thus as $\eta \to 0$, we can ignore the second solution and we are left with the dominant modes

$$\phi_{\mathbf{k}} = \frac{v_k \, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}}{a} = c_1 H_\lambda \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} + O(\eta^3) \,. \tag{163}$$

The mode freezes when $k_{\text{phys}} = \sqrt{2}H_\lambda$ which gives $\eta = \sqrt{2}/k$ and the coefficient c_1 is now constant, fixed by the value of v_k at $\eta = \sqrt{2}/k$. This can also be seen from the standard asymptotics of the Bessel functions in the exact expressions for the modes. Thus the modes freeze and become time-independent at late times leaving a spectrum of spatial variations.

This is the mechanism by which inflation is understood to give rise to the observed spectrum of fluctuations in the microwave background.

We can obtain the 'power spectrum as follows. Each $\mathbf{k}$ mode of the field can be thought of as a harmonic oscillator. Noting that the zero point fluctuation of the ordinary harmonic oscillator has $\langle 0|x^2|0\rangle = \hbar/2m\omega$, using $\omega \sim k_{\rm phys}$, we find that for the mode $\phi_{\mathbf{k}} = v_k \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}/a$

$$\langle 0|\phi^\dagger_{\mathbf{k}}\phi_{\mathbf{k}}|0\rangle \sim \left|\frac{v_{\mathbf{k}}}{a}\right|^2 \sim \frac{\hbar}{2V_{\rm phys}k_{\rm phys}} \sim \frac{\hbar H_\Lambda^2}{Vk^3}\,. \tag{164}$$

Here the $V_{\rm phys} = a^3 V$ is the volume of 3-space that arises from normalizing the $\mathbf{k}$-Fourier mode (i.e., we have regulated the infrared divergence by taking the spatial slices to be tori of volume $V \sim L^3$). The first $\sim$ using proper physical values are used for consistent matching to the previous flat space result arising from the zero point fluctuations of the harmonic oscillator and the matching is then done at $\eta = \sqrt{2}/k$. The power spectrum for massive modes can be computed also, but all die off as $\eta \to 0$.

This gives rise to the observationally verified scale invariant *power spectrum* of density perturbations as arising essentially from the zero-point fluctuations from the Bunch-davies vacuum.

7 Thermal states and the Unruh effect

The general principle of relativity states that it should be possible to express the laws of physics as being the same for all observers, even those undergoing acceleration; constant acceleration is equivalent to sitting in a gravitational field. Thus one can ask the question, just in flat space, how does the Minkowski vacuum appear to an accelerating observer? The perhaps surprising answer is that an accelerating observer sees a thermal state. We therefore start with a brief discussion of the treatment of such states.

7.1 Revision of thermal states

Thermal states are a feature of statistical physics where we allow ourselves a probability distribution of physical states. In quantum mechanics such a distribution is given in the form of a

Definition 7.1 *A* density matrix *is an element $\rho \in \mathcal{H} \otimes \mathcal{H}^*$ that is Hermitian, positive definite and of trace* $\mathrm{tr}\rho = 1$.

Such matrices are always diagonalizable and an orthonormal basis $|n\rangle$ can be found so that it can be expressed in the form

$$\rho = \sum_n \rho_n |n\rangle\langle n| \tag{165}$$

where $\rho_n \geq 0$ and $\sum_n \rho_n = 1$. In this context, the expectation value of an observable A is

$$\langle A\rangle_\rho = \mathrm{tr}\rho A = \sum_n \rho_n \langle n|A|n\rangle \,. \tag{166}$$

Such a density matrix represents a pure state when $\rho = |\psi\rangle\langle\psi|$ for some normalized state $|\psi\rangle$, i.e., ρ has rank 1, otherwise states are said to be mixed.

Mixed states arise naturally when part of a quantum system is hidden. Thus we might have a pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ with $\mathcal{H}_L$ hidden. The state $|\psi\rangle$ is said to be entangled if when written as

$$|\psi\rangle = \sum_{r,s} \psi_{r,s} |r\rangle_L |s\rangle_R \tag{167}$$

where $|r\rangle$ is a basis of $\mathcal{H}_L$ and $|s\rangle$ of $\mathcal{H}_R$, the matrix $\psi_{r,s}$ has rank greate than one. The associated density matrix apparent in $\mathcal{H}_R$ is then

$$\rho_R = Tr_L |\psi\rangle\langle\psi| = \sum_r \bar{\psi}_{s_1,r}\psi_{r,s_2} |s_2\rangle\langle s_1| \,. \tag{168}$$

and is a mixed state if $|\psi\rangle$ is entangled.

A thermal state of temperature T is one of the form[9]

$$\rho_\beta = \frac{1}{Z_\beta}\exp(-\beta H) = \frac{1}{Z_\beta}\sum_n \mathrm{e}^{-\beta E_n}|n\rangle\langle n|\,, \qquad \beta = \frac{1}{kT}\,, \tag{169}$$

where H is the Hamiltonian and $|n\rangle$ is a basis of energy eigenstates of energy E_n and

$$Z_\beta = \mathrm{tr}\exp(-\beta H)\,, \tag{170}$$

is the partition function. We will choose units so as to take Boltzmann's constant k to be 1.

We give two ways to characterize thermal states. The first is via the expectation of the particle number and the second by periodicity in imaginary time.

Bose-Einstein distribution. In the context of a harmonic oscillator (i.e., a single mode of a quantum field), we can compute the expectation of the number operator N using $E_n = (n+\frac{1}{2})\omega$ to obtain

$$\mathrm{tr}(\rho_\beta N) = \frac{\sum_n n\mathrm{e}^{-\beta n\omega}}{\sum_n \mathrm{e}^{-\beta n\omega}} = \frac{1}{\omega}\frac{d}{d\beta}\log(1-\mathrm{e}^{-\beta\omega}) = \frac{1}{\mathrm{e}^{\beta\omega}-1}\,. \tag{171}$$

The KMS condition. Named after Kubo, Martin and Schwinger, we define a KMS state to be one for which the time evolution of operators $A \to A_t$ can be continued to complex time in such a way that for a time-independent operator, B, we have

$$\langle A_t B\rangle_{KMS} = \langle B A_{t+i\beta}\rangle_{KMS}\,. \tag{172}$$

To see that our thermal state satisfies this condition, recall that in the Heisenberg representation, an operator A has time dependence $A_t = \mathrm{e}^{iHt}A_0\mathrm{e}^{-iHt}$.

[9] More generally we can incorporate a chemical potential μ associated with the number operator N and conserved quantities, i.e., charge Q for electrostatic potential ϕ, and angular momentum J for angular velocity Ω to give $\exp(-\beta(H-\mu N)+\Omega J+\phi Q)/Z_{\beta,\mu,\ldots}$.

For our thermal density matrix above we compute

$$
\begin{aligned}
\langle A_t B\rangle_\beta &= \frac{1}{Z_\beta}\mathrm{tr}(\mathrm{e}^{-\beta H}A_t B) = \frac{1}{Z_\beta}\mathrm{tr}(\mathrm{e}^{-\beta H+iHt}A_0\mathrm{e}^{-iHt}B) \\
&= \frac{1}{Z_\beta}\mathrm{tr}(A_{t+i\beta}\mathrm{e}^{-\beta H}B) = \frac{1}{Z_\beta}\mathrm{tr}(\mathrm{e}^{-\beta H}BA_{t+i\beta}) \\
&= \langle BA_{t+i\beta}\rangle_\beta\,, \qquad (173)
\end{aligned}
$$

where we have used the cyclic property of the trace.

This is often taken to mean that our system can be analytically continued to Euclidean signature with periodicity in imaginary time.

This applies to the Feynman propagator or its analytic continuation, the Euclidean Green's function or the Wightman function $\langle\phi(x)\phi(y)\rangle$ which is simply the Euclidean Greens function. The thermal Greens function can then be obtained as the Green's function for the relevant operator, i.e., the Laplacian on $\mathbb{R}^3 \times S^1_\beta$ where now S^1 is a circle of length β.

Definition 7.2 *The thermal Green's function is*

$$
G_\beta(x,y) = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta H}\phi(x)\phi(y))}{\mathrm{Tr}(\mathrm{e}^{-\beta H})} \qquad (174)
$$

Proposition 7.1 *With a timelike symmetry, $G_\beta = G_\beta(t-t',\mathbf{x},\mathbf{x}')$ and a thermal propagator satisfies the KMS condition that it be periodic in imaginary time with period $i\beta$.*

Proof: This follows directly as above for the KMS condition. □

More generally, the propagator can be obtained by path-integral methods both in Lorentz and Euclidean signature.

In flat space, such a thermal propagator can therefore be constructed by images in imaginary time of period β. Thus we can identify the thermal greens function on Minkowski space for the massless wave equation as

$$
G_\beta(x,x') = \sum_{n\in\mathbb{Z}} \frac{1}{4\pi^2((t-t'+in\beta+i\epsilon)^2 - \mathbf{x}\cdot\mathbf{x})} \qquad (175)
$$

We can also arrive at the Bose-Einstein distribution from the Green's function by computing the probability $P_{E_1\to E_2}$ that a detector at $\mathbf{x}=0$ detects a transition of the field from $E_1 \to E_2 > E_1$ in unit time. The energy eigenstates along the detector worldline $x(\tau)$ take the form $|E\rangle = \int_{-\infty}^{\infty} d\tau \mathrm{e}^{-iE\tau}\phi(x(\tau))|0\rangle$. The transition probability $\langle E_1|E_2\rangle$ is a double integral from $t=-\infty$ to $t=\infty$ and if G_β depends only on $t-t'$, this will diverge. However, the rate per unit time is given by

$$\begin{aligned} P_{E_1\to E_2} &\propto \int_{-\infty}^{\infty} dt\, \mathrm{e}^{-i(E_2-E_1)t}\, G_\beta(x(t),x(0)) \\ &\propto \int_{-\infty}^{\infty} dt \sum_{n\in\mathbb{Z}} \frac{-\mathrm{e}^{-i(E_2-E_1)\tau}}{4\pi^2(t+in\beta+i\epsilon)^2} \end{aligned} \tag{176}$$

The integral can be analytically continued so as to be evaluated by residues along the negative imaginary axis where it has double poles at $t=-n\beta i$. Thus we obtain the sum of residues

$$P_{E_1\to E_2} \propto (E_2-E_1)\sum_{n=0}^{\infty} \mathrm{e}^{-(E_2-E_1)n\beta} = \frac{(E_2-E_1)}{\mathrm{e}^{(E_2-E_1)\beta}-1}\,, \tag{177}$$

as expected for a detector immersed in a thermal bath of temperature $1/\beta$.

7.2 Accelerating observers and the Rindler wedge

Here we will perform the calculation for massless modes in $1+1$ dimensions where we can use the conformal invariance of the wave equation to perform complete explicit calculations that yield some intuition. The analysis can be performed explicitly also for massive modes that then extend to $3+1$, but only at the expense of Bessel function identities. This is laid out in the form of an optional and unmarked problem sheet in an appendix.

An observer $\mathcal{O}_a$ with constant acceleration a along the x-axis has worldline

$$(t,x) = \frac{1}{a}(\sinh a\tau, \cosh a\tau)$$

where τ is the proper time of the observer. We will be asking the question as to how the accelerating observer sees the Minkowski vacuum. Using the clock and radar method, the observer will set up coordinates (τ,ξ) related to inertial coordinates (t,x) by

$$(t,x) = \frac{\mathrm{e}^{a\xi}}{a}(\sinh a\tau, \cosh a\tau)\,, \qquad dt^2-dx^2 = \mathrm{e}^{2a\xi}(d\tau^2-d\xi^2)\,. \tag{178}$$

The (τ, ξ) coordinates cover only the *Rindler wedge:* $R := \{x > |t|\}$. This is the only portion of space-time that the accelerating observer can measure and see.

In terms of the inertial lightcone coordinates $(u, v) := (t-x, t+x)$ versus those of the accelerating observer $(U, V) := (\tau - \xi, \tau + \xi)$ we can write

$$(u, v) = \frac{1}{a}(-e^{-aU}, \mathrm{e}^{aV}), \qquad ds^2 = dudv = \mathrm{e}^{a(V-U)}dUdV\,. \tag{179}$$

The Rindler wedge is therefore the whole plane in (U, V)-coordinates but is the quadrant with $-u, v > 0$ in (u, v)-coordinates. This also makes it clear that light rays travel along $\tau \pm \xi =$ constant, and that when $\mathcal{O}_a$ measures distances using their clock by the radar method, they will be given by[10] ξ.

In this massless 1+1-dimensional context, the wave equation is conformal and can be trivially solved with left-moving and right-moving modes in each coordinate system. For the flat Minkowski background we have respectively the left and right moving modes

$$\phi_\omega(u) = \mathrm{e}^{-i\omega u}\,, \qquad \tilde{\phi}_\omega(v) = \mathrm{e}^{-i\omega v} \tag{180}$$

that are positive frequency for $\omega > 0$. The adapted coordinates introduced above have the nice property that they are conformal, and so the modes as seen/measured by $\mathcal{O}_a$ to be of frequency ω will be respectively

$$\Phi_\Omega(U) = \mathrm{e}^{-i\Omega U}\,, \qquad \tilde{\Phi}_\Omega(V) = \mathrm{e}^{-i\Omega V}\,. \tag{181}$$

Given the decoupling between left and right-moving modes, we first work with the right moving modes and the left moving modes will work identically.

Define inertial Minkowski right moving field operators as

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} [\mathrm{e}^{-i\omega u} a_\omega + \mathrm{e}^{i\omega u} a_\omega^\dagger]\,, \tag{182}$$

and those for the accelerating observer as follows

$$\hat{\Phi} = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} [\mathrm{e}^{-i\Omega U} A_\Omega + \mathrm{e}^{i\Omega U} A_\Omega^\dagger]\,, \tag{183}$$

[10] See also standard diagrams that give the simultaneities for $\mathcal{O}_a$, $\tau =$ constant, in the Rindler wedge as the lines through the origin.

where a_ω and A_Ω are standard raising and lowering operators satisfying

$$[a_\omega, a^\dagger_{\omega'}] = \delta(\omega - \omega')\,, \qquad [A_\Omega, A^\dagger_{\Omega'}] = \delta(\Omega - \Omega')\,, \tag{184}$$

etc., and we have introduced standard normalizations appropriate to one spatial dimension. These operators will be related to each other via Bogoliubov coefficients. Implicit in these definitions are the Minkowski vacuum $|0_M\rangle$, satisfying $a_\omega|0\rangle_M = 0$, and the Rindler vacuum satisfying $A_\Omega|0\rangle_R = 0$.

We now wish to compute the distribution of the number of particles of frequency Ω detected by $\mathcal{O}_a$ in the Minkowski vacuum. To do this we first compute the Bogoliubov coefficients $(\alpha_{\Omega\omega}, \beta_{\Omega\omega})$ that express

$$A_\Omega = \int_0^\infty d\omega(\alpha_{\Omega\omega} a_\omega - \beta_{\Omega\omega} a^\dagger_\omega)\,. \tag{185}$$

These are determined by imposing $\hat{\phi} = \hat{\Phi}$ on the Rindler wedge, multiplying by $e^{\pm i\Omega U}$ and integrating, using

$$\int_{-\infty}^{\infty} \mathrm{e}^{i(\Omega-\Omega')U} dU = 2\pi\delta(\Omega - \Omega')\,, \tag{186}$$

to obtain, using also $U = \frac{1}{a}\log(-au)$,

$$\begin{aligned}
\begin{pmatrix} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{pmatrix} &= \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{\infty} \begin{pmatrix} \mathrm{e}^{-i\omega u} \\ \mathrm{e}^{i\omega u} \end{pmatrix} \mathrm{e}^{i\Omega U} dU \\
&= \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^{0} \begin{pmatrix} \mathrm{e}^{-i\omega u} \\ -\mathrm{e}^{i\omega u} \end{pmatrix} (-au)^{-i\frac{\Omega}{a}-1} du\,, \\
&= \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}} \begin{pmatrix} \mathrm{e}^{\frac{\pi\Omega}{2a}} \\ -\mathrm{e}^{-\frac{\pi\Omega}{2a}} \end{pmatrix} \left(\frac{\omega}{a}\right)^{i\frac{\Omega}{a}} \Gamma(-i\Omega/a)\,.
\end{aligned} \tag{187}$$

Where the final equality is left as a hard exercise.[11] The main takeaway from this calculation is the relation

$$|\alpha_{\Omega\omega}|^2 = \mathrm{e}^{2\pi\Omega/a}|\beta_{\Omega\omega}|^2\,. \tag{188}$$

[11]The integral can be converted to that for the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}dt$, but the analytic continuations required are delicate, see appendix of Mukahnov & Winitzki.

whose independent proof is a question on the problem sheet. This allows us to compute the Expectation value of the number operator N_Ω of the Ω frequency modes detected by $\mathcal{O}_a$ in the Minkowski vacuum:

$$\begin{aligned}\langle N_\Omega\rangle_M &:= \langle 0_M|A_\Omega^\dagger A_\Omega|0_M\rangle \\ &= \langle 0_M|\int_0^\infty d\omega(\bar{\alpha}_{\omega\Omega}a_\omega^\dagger - \bar{\beta}_{\omega\Omega}a_\omega)\int_0^\infty d\omega'(\alpha_{\omega'\Omega}a_{\omega'} - \beta_{\omega'\Omega}a_{\omega'}^\dagger)|0_M\rangle \\ &= \int_0^\infty d\omega|\beta_{\omega\Omega}|^2 \end{aligned} \tag{189}$$

This can be simplified by using the normalization condition for Bogoliubov coefficients from lemma 5.1 which in our context reads

$$\int_0^\infty d\omega(\alpha_{\Omega\omega}\bar{\alpha}_{\Omega'\omega} - \beta_{\Omega\omega}\bar{\beta}_{\Omega'\omega}) = \delta(\Omega-\Omega')\,. \tag{190}$$

Evaluating at $\Omega = \Omega'$ we reinterpret the right hand side $\delta(0) = V$ as the volume of space, regularized as normal by taking a finite box. This allows us to deduce the particle number density

$$\frac{\langle N_\Omega\rangle_M}{V} = \frac{1}{\mathrm{e}^{2\pi\Omega/a}-1}\,, \tag{191}$$

using (188). This is the main result of this section as we now recognize this as the Bose Einstein distribution with *Unruh temperature*

$$T_{\text{Unruh}} = \frac{a}{2\pi}\,. \tag{192}$$

This is the main result of this section: an accelerated observer sees a thermal bath of particles of temperature $a/2\pi$. This is a real temperature that you could cook with (but at great cost to maintain the acceleration).

We conclude that not only does the choice of vacuum, and hence concept of particle become time-dependent, it is also observer dependent, even in flat space-time.

The Rindler vacuum. The Rindler Fock space $\mathcal{F}_R$ is built on the Rindler vacuum $|0\rangle_R$ using the creation and annihilation operators $A_\Omega^\dagger$ and A_Ω. It is based on the Rindler modes (181) measured by $\mathcal{O}_a$ to be modes of a fixed frequency, i.e., in Rindler coordinates (τ,ξ) or (U,V)-plane. This is the state in which $\mathcal{O}_a$ sees no particles.

For states in the Rindler Fock space we expect the renormalized stress-energy tensor in Rindler coordinates to be bounded locally. On right-going modes this will be ${}_R\langle T_{UU}\rangle_R =_R \langle\hat{\Phi}_U^2\rangle_R$. However, the Minkowski stress tensor for these modes is

$$ {}_R\langle\hat{\phi}_u^2\rangle_R = \left(\frac{\partial U}{\partial u}\right)^2 {}_R\langle\hat{\Phi}_U^2\rangle_R = \frac{1}{a^2u^2}{}_R\langle\hat{\Phi}_U^2\rangle_R \tag{193} $$

and so diverges at the Rindler horizon $u = 0$.

As it stands, there can be no identification between the Minkowski Fock space $\mathcal{F}_M$ and the Rindler Fock space $\mathcal{F}_R$ as $\hat{\Phi}(U)$ only determines[12] $\hat{\phi}(u)$ for $u < 0$ and is not defined for $u > 0$.

To determine $\hat{\phi}$ and $\mathcal{F}_M$ from Rindler type data, we can introduce a Fock space $\mathcal{F}_L$ for the left hand Rindler wedge $L := \{x < -|t|\}$ whose null coordinates (U_L, V_L) are related to inertial coordinates (u, v) by

$$ (t,x) = \frac{\mathrm{e}^{-a\xi_L}}{a}(\sinh a\tau_L, -\cosh a\tau_L), \qquad (u,v) = \frac{1}{a}(\mathrm{e}^{aU_L}, -\mathrm{e}^{-aV_L}) $$

with $(U_L, V_L) = (\tau_L - \xi_L, \tau_L + \xi_L)$ as before so that all coordinates increase into the future but now $u > 0$, $v < 0$ so $x < -|t|$.

We can now introduce modes in L and set up a Fock space for L based on a Rindler vancuum $|0\rangle_L$ exactly as before. We can further extend the Bogoliubov coefficients (185) to what will now be an invertible system on both sides. The Minkowski Fock space is now a tensor product

$$ \mathcal{F}_M = \mathcal{F}_L \otimes \mathcal{F}_R\,. \tag{194} $$

However, the Minkowski vacuum $|0\rangle_M$ is entangled in this product, $\neq |0\rangle_L|0\rangle_R$. The Unruh state ρ_U as measured by $\mathcal{O}_a$ will be the density matrix

$$ \rho_U = \mathrm{tr}_{\mathcal{F}_L}|0\rangle_{M\,M}\langle 0|\,. \tag{195} $$

A key point is that time translation ∂_τ for $\mathcal{O}_a$ in R is given by the boost Killing vector on R but in L it is given by minus the boost Killing vector:

$$ B := x\partial_t + t\partial_x = v\partial_v - u\partial_u = \frac{\partial}{\partial\tau_R} = -\frac{\partial}{\partial\tau_L} \tag{196} $$

[12]This was all we needed to obtain the Bogoliubov coefficients we needed, identifying $\hat{\phi}(u) = \hat{\Phi}(U)$ for $u = -\mathrm{e}^{-aU/a} < 0$.

The Minkowski vacuum is however Lorentz and hence boost invariant. Thus we must have an entangled product of the form

$$|0\rangle_M = \sum_n f_n |n\rangle_L |n\rangle_R \,, \tag{197}$$

for some f_n and our calculations show that $f_n = \mathrm{e}^{-\beta E_n}$, where $\beta = 1/T_{\text{Unruh}}$.

Remarks:

- These consideration extend to include, the left moving modes, 3 spatial dimensions and other spins of fields (but with increasing complexity). In particular, going to higher dimensions is no worse than introducing a mass, but we lose the conformal invariance that we have exploited. We naturally obtain Bessel functions for the propagators.

- The KMS condition gives an alternative argument for the thermal state. We can re-express the Wightman function

  $$G_W(x,x') = {}_M\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle_M = \frac{1}{(x^0 - x'^0 - i\epsilon)^2 - (\mathbf{x}-\mathbf{x}')^2}$$

  in Rindler coordinates to obtain

  $$G_F(x,x') = \frac{a^2}{\mathrm{e}^{2a\xi} + \mathrm{e}^{2a\xi'} - \mathrm{e}^{a(\xi+\xi')}\cosh a(\tau-\tau') - d^2 - 2i\epsilon(x^0 - x'^0)}\,,$$

  where $d^2 = (y-y')^2 + (z-z')^2$. In Rindler coordinates, G_F is periodic in complex τ with period $i\beta = \frac{2\pi}{a}$. Thus, for $\mathcal{O}_a$, the Minkowski propagator is a thermal Greens function of temperature T_{Unruh}.

- As before, we can make the previous remark more explicit by restricting to the accelerating worldline $x = x(\tau), x' = x(\tau = 0)$ to obtain

  $$\begin{aligned} G_W(x(\tau), x(0)) &= \frac{a^2}{(\sinh a\tau - i\epsilon)^2 - (\cosh a\tau - 1)^2} \\ &= \frac{-1}{2(1 - \cosh a\tau + i\epsilon \sinh a\tau)} \end{aligned}$$

This can be used to compute the probability $P_{E_1\to E_2}$ that the accelerating detector detects a transition of the field from $E_1 \to E_2$ as

$$
\begin{aligned}
P_{E_1\to E_2} &\propto \int_{-\infty}^{\infty} d\tau\, \mathrm{e}^{-i(E_2-E_1)\tau}\, G_W(x(\tau),x(0)) \\
&\propto \int_{-\infty}^{\infty} d\tau\, \frac{-\mathrm{e}^{-i(E_2-E_1)\tau}}{2(1-\cosh a\tau + i\epsilon \sinh a\tau)}
\end{aligned} \tag{198}
$$

The integral can be analytically continued so as to be evaluated by residues along the negative imaginary axis where it has double poles at $a\tau = 2n\pi i$. We obtain the sum of residues

$$
P_{E_1\to E_2} \propto (E_2-E_1)\sum_{n=0}^{\infty} \mathrm{e}^{-2\pi(E_2-E_1)n/a} = \frac{(E_2-E_1)}{\mathrm{e}^{2\pi(E_2-E_1)/a}-1}\,, \tag{199}
$$

as expected for a detector immersed in a thermal bath of the Unruh temperature.

- In terms of Euclideanization, the analytic continuation of the Rindler wedge metric can be written, with $\rho = \mathrm{e}^{a\xi}$, as

$$
ds^2 = -a^2(d\rho^2 + \rho^2 d\theta^2)\,, \qquad \theta = \frac{i\tau}{a}\,. \tag{200}
$$

This is the flat metric, but has a conical singularity at $\rho = 0$ unless θ is identified with $\theta + 2\pi$. This is essential for regularity at the horizon. This gives the imaginary periodicity $\tau \sim \tau + i\beta$ for $\beta = 1/T_{\mathrm{Unruh}}$.

This is a theme that can be taken much further in curved space-times.

8 Hawking radiation

Recall the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{201}$$

One might be surprised that anything interesting can happen as surely one can use the Schwarzschild timelike Killing vector (at least at large distances) to define positive and negative frequency. This defines the Boulware vacuum. However, it is unphysical, firstly because the standard model of a collapsing black hole is time dependent in the collapse region, and modes coming in from past infinity, pass through this region and exit. Secondly, it cannot give a sensible positive/negative splitting for observers that cross the horizon as the t coordinate is singular on the horizons; the Schwarzschild t is directly analogous the Rindler accelerating observer's time τ and like the Rindler vacuum, the Boulware vacuum will be singular along the horizons.

To make the horizon's explicit introduce $r_* = r + 2m\log(r - 2m)$, Wheeler's tortoise coordinate, and Kruskal coordinates by

$$U = -e^{-\kappa u}\,, \quad V = e^{\kappa v}\,, \quad u = t - r_*\,, \quad v = t + r_*\,, \tag{202}$$

with $\kappa = 1/4m$ the surface gravity. In these coordinates the metric becomes

$$ds^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dr_*^2) - r^2 d\Omega^2 \tag{203}$$

$$= \frac{32m^3 e^{-r/2m}}{r} dU dV - r^2 d\Omega^2 \tag{204}$$

Drawing now the appropriate Penrose diagrams, we can perhaps see a close analogy with the transform between Minkowski and Rindler coordinates[13] and Penrose diagram that already suggests that we might expect to see a temperature $\kappa/2\pi$.

The standard collapse picture. We dont expect to see the full Kruskal extension, an 'eternal Schwarzschild', but something more like a ball of collapsing dust falling in to form a black hole.

[13]although unfortunately the notation lower-case/upper-case is reversed.

Define the $-$ or in-vacuum $|0\rangle_-$ to arise from positive frequency modes at $\mathscr{I}^-$ of the form $e^{-i\omega v}$. We similarly take for the $+$ vacuum at $\mathscr{I}^+$ positive frequency modes of the form $e^{-i\Omega u}$. Half the modes will also fall into the black-hole across the future horizon H^+ where $U = 0$, but we wont need to specify a choice of vacuum there as we will trace over those modes.

Solutions to the wave equation can be separated as

$$\phi = \frac{1}{r} f(u, v) Y_{lm}\,, \qquad \partial_u \partial_v f + V_e f = 0. \tag{205}$$

where the effective potential is given by

$$V_{\text{eff}} = \left(1 - \frac{2m}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{18m^3}{r^3}\right). \tag{206}$$

This potential falls off exponentially in r_* as $r_* \to -\infty$, or $r \to 1$ approaching the horizon and polynomially as $r \to +\infty$. For large l the potential barrier suppresses transmission by e^{-l} in the WKB approximation so our considerations will be dominated by nearly spherical waves and we will restrict our attention to these.

To obtain the state at $\mathscr{I}^+$ it will be sufficient to express the positive frequency modes $e^{-i\Omega u}$ at $\mathscr{I}^+$ back to $\mathscr{I}^-$. We can no longer perform exact calculations, but in the WKB approximation at $l = 0$ we obtain

$$e^{-i\Omega u} \to T_\Omega \theta(-v)(-v)^{i\Omega/\kappa} = T_\omega \exp\left(\frac{i\Omega}{\kappa}\log(-v)\right). \tag{207}$$

Here T_ω is a transmission coefficient or grey-body factor reflecting attenuation by the potential, $\kappa = 1/4m$ is the surface gravity and we have chosen $v = 0$ to be the value of v at which the horizon H^+ forms. We refer to the $v = 0$ hypersurface as H^- by analogy with the past horizon in the Kruskal diagram.

To understand how this formula arises, come back in time from $\mathscr{I}^+$. No data from $\mathscr{I}^+$ can map back to $v > 0$ from the future. Alternatively all data for $v > 0$ at $\mathscr{I}^-$ falls in through the horizon. Fields from $\mathscr{I}^+$ must arrive via reflection at $r = 0$ before the horizon has formed. Near the horizon, it is the U, V Kruskal coordinates that are regular and smooth and regularity of ϕ at $r = 0$ requires $f = 0$ at $r = 0$. However, near $r = 0$ and H^+ immediately before the formation of the horizon, we must use the U coordinate as u diverges there; U should now be an affine parameter on H^- and we assume that it is approximately given by its Kruskal form here also. Thus at $r = 0$

we identify v with $U = -e^{-\kappa u}$ or alternatively u with $-\frac{1}{\kappa}\log -v$. Thus we obtain infinite blue shift near the horizon H^-.

To compute Bogoliubov coefficients replace index i by ω and summation by integration $\int_0^\infty d\omega/\sqrt{2\pi}$. We obtain

$$T_\omega \int_{-\infty}^{0} (-v)^{i\Omega/\kappa} \mathrm{e}^{i\omega v} dv = \begin{cases} \beta_{\Omega,-\omega}\,, & \omega < 0 \\ \alpha_{\Omega,\omega}\,, & \omega > 0\,, \end{cases} \tag{208}$$

where we have absorbed the step function into the limits.

The astute reader will recognize that in this WKB approximation we have landed on the 1 + 1 Rindler calculation of the previous section. Analytic continuation $v \to \pm iv$ depends on $\pm\Omega > 0$ and we obtain

$$\alpha_{\Omega\omega} = \mathrm{e}^{\pi\Omega/\kappa}\beta_{\Omega\omega}\,. \tag{209}$$

Combining this with our condition $\sum|\alpha|^2 - |\beta|^2 = 1$ on the Bogoliubov coefficents we obtain

$$(\mathrm{e}^{2\pi\Omega/\kappa} - 1)\int_0^\infty \frac{d\omega}{\sqrt{2\pi}}|\beta_{\Omega\omega}|^2 = 1\,. \tag{210}$$

The integral is essentially the expected number operator for states of frequency Ω leading to, including normalizations,

$$\langle 0|N_\omega|0\rangle = \frac{|T_\Omega|^2}{\mathrm{e}^{2\pi\Omega/\kappa} - 1} \tag{211}$$

giving the Hawing temperature $T_H = \kappa/2\pi = 1/8\pi m$. The numerator removes the infrared divergence, particles with low frequency do not transmit.

- With this T_H, the first law of thermodynamics $dE = TdS$ with $E = m$ integrates to give $S_{BH} = \pi m^2 = A/4$ as proposed by Bekenstein.
- This temperature is $T_H \sim 6 \times 10^{-8}\frac{m_\odot}{m}K$, much smaller than the $2.7K$ microwave background for a solar mass black hole. The wavelengths are order the Schwarzschild radius.
- A late time observer, for u large, sees exponentially small $U \propto \exp(-u/4m)$ near the horizon and hence fluctuations of very high energy (but redshifted as it escapes to $\mathscr{I}^+$). So the WKB approximation is robust. We can do a 2-point function argument, as a near horizon Fermion two point function $(dUdU')^{1/2}/(U - U')$ becomes thermal in u at $\mathscr{I}^+$.

- As T_H decreases with increasing $M = E$, black holes have negative heat capacity.

- The calculation is quite robust: repeating the calculation for other types of field, or on rotating and charged black holes leads to the same formula for T_H in terms of surface gravity.

- The Hawking temperature refers to the Killing energy for $k = \partial_t$. As $|k| \to 1$ at infinity, this is the energy measured by a static observer at infinity.

- A static observer $\mathcal{O}$ at finite radius r measures a blue-shifted temperature $T_{\mathcal{O}} = T_H/|k(r)|$. As $r \to \infty$ this is the Hawking temperature at ∞, but diverges at the horizon $|k| = 0$ due to the infinite acceleration of the static observer at the horizon; this is the Unruh effect. A freely falling observer sees no divergence as they cross the horizon.

8.1 The Hawking thermal state and friends

The thermal state can be seen mode by mode from the formula (144) for

$$|0\rangle_- \propto \exp\left(\mathrm{e}^{-\pi\Omega/\kappa} a^\dagger_\Omega b^\dagger_\Omega\right)|0\rangle_+ \tag{212}$$

using $\frac{\beta_{\Omega\omega}}{\alpha_{\Omega\omega}} = \mathrm{e}^{-\pi\Omega/\kappa}$.

By analogy with the L/R Fock spaces in the Rindler case we can identify L with the Fock space built from modes on the horizon H^+, and R with those on $\mathscr{I}^+$: data on $\mathscr{I}^-$ determines and is determined by that on $H^+ \cup \mathscr{I}^+$. As before for Rindler, we obtain the state

$$|0\rangle_- \propto \sum_n \mathrm{e}^{-n\pi\Omega/\kappa}|n\rangle_L|n\rangle_R \tag{213}$$

We then obtain tracing over the horizon modes

$$|0\rangle_{\mathscr{I}^+} = \mathrm{Tr}_{H^+}|0\rangle_{--}\langle 0| \propto \sum_n \mathrm{e}^{-n\pi\Omega/\kappa}|n\rangle_{RR}\langle n| \tag{214}$$

which gives the desired thermal state.

The Hawking state was what arose from an essentially Minkowskian vacuum at $\mathscr{I}^-$ in the collapsing scenario, but other states are natural for the Kruskal eternal' Schwarzschild.

The fake horizon H^- can be identified with the true past horizon in the Kruskal diagram. The following vacua can be defined with positive frequency states given as for $\omega > 0$:

Hartle-Hawking vacuum $\|H^2\rangle$	$e^{-i\omega U}$ on $H^-, e^{-i\omega V}$ on $\mathscr{I}^-$
Unruh Vacuum $\|U\rangle$	$e^{-i\omega U}$ on $H^-, e^{-i\omega v}$ on $\mathscr{I}^-$
Boulware vacuum $\|B\rangle$	$e^{-i\omega u}$ on $H^-, e^{-i\omega v}$ on $\mathscr{I}^-$

- The Hartle-Hawking vacuum state represents a black hole in equilibrium with a thermal path. It is a thermal state at $\mathscr{I}^-$ with the same temperature as the outgoing thermal state at $\mathscr{I}^+$.
- Black-hole collapse brings about the Unruh state in region I.
- The Boulware vacuum, as we have mentioned, like the Rindler vacua becomes singular at the horizons.

8.2 Thermal Green's functions and Wick rotation

The Hartle-Hawking Greens function is an example of a Thermal Green's function

$$G_\beta^{H^2} := \langle H^2|\hat{\phi}(x)\hat{\phi}(y)|H^2\rangle \tag{215}$$

A key statement is that this extends to complex time in Schwarzschild and is periodic in imaginary time with $\beta = 1/T_H = 8\pi M$. This periodicity is seen in the geometry of Euclidean Schwarzschild. Sending $t \to i\tau$, the Kruskal coordinates give

$$ds^2 = \frac{32M^3 e^{r/2M}}{r} dU d\bar{U} + r^2 d\Omega^2 \tag{216}$$

where

$$U = \rho e^{i\tau/4M}, \qquad \rho = |1 - r/2M|^{1/2} e^{r/2M} \tag{217}$$

This metric has a conical singularity at $r = 2M$, $\rho = 0$ unless τ has period $1/8\pi M$. This gives an alternative 'justification' of the Hawking temperature analogous to the Euclideanization argument for the Unruh temperature given earlier; it is remarkable that this argument extends so directly to the case of a curved background.

To summarize, after Wick rotation, thermal states to extend as follows:

1. inertial states in flat $1+1$-space extend to $\mathbb{R} \times S^1$.

2. accelerating observers on flat $1+1$ space extend to $\mathbb{R}^2$

3. ignoring the S^2 factor, Euclideanized Schwarschild is a curved surface of revolution, interpolating the above two pictures.

8.3 Euclidean path integrals

In the path-integral approach to the quantization one expresses the amplitude to go from a field configuration ϕ_1, at a time t_1, to a field configuration ϕ_2, at time t_2, as

$$\langle\phi_1, t_1|\phi_2, t_2\rangle = \int [d\phi] \exp iI[\phi] \tag{218}$$

where the functional integration is over all fields ϕ on $[t_1, t_2]$ taking the value ϕ_1 at t_1 and ϕ_2 at t_2. We also have

$$\langle\phi_1, t_1|\phi_2, t_2\rangle = \langle\phi_1, t_1|\mathrm{e}^{-iH(t_2-t_1)}|\phi_2, t_2\rangle\,, \tag{219}$$

where H is the Hamiltonian. If we now set $t_2 - t_1 = i\beta$ and set $\phi_1 = \phi_2$ and sum over such ϕ_1, we obtain the partition function for the canonical ensemble at temperature $T = 1/\beta$

$$Z := \mathrm{tre}^{-\beta H} = \int [d\phi]\mathrm{e}^{iI[\phi]} \tag{220}$$

where now the functional integral is now over all fields that are periodic in imaginary time with period β.

To make sense of this for black holes, we can analytically continue to Euclidean signature and take the semiclassical approximation to obtain a formula for the partition function

$$Z_{sc} = \mathrm{e}^{-I_{bh}} \tag{221}$$

where I_{bh} is the action of our Wick-rotated black-hole. This can be computed giving a derivation of the thermodynamic potential and the first law of black hole thermodynamics [Gibbons & Hawking 1977, PRD, vol 15, n10, p2752].

8.4 Black hole evaporation

It is usually understood that if we take the back-reaction into account, Hawking radiation will lead to black-hole evaporation. The negative energy modes tunnelling into the black hole will lower the mass. This can be estimated using Stefan's law for the evaporation of a black body

$$\frac{dE}{dt} \propto -AT^4 \tag{222}$$

where A is the area. With $E \sim m$, $A \sim m^2$, $T \sim 1/m$ we obtain $dm/dt \propto -1/m^2$ and this leads to the black-hole evaporating in finite time of the order of $10^{71}(M/M_\odot)^3$ seconds. It is instructive to draw the corresponding Penrose diagram.

8.5 Holographic principal and information paradox.

The entropy can be systematically defined in quantum mechanics as *entanglement entropy* between two subsystems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Then the entanglement entropy of a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be defined in terms of the density matrix over $\mathcal{H}_A$

$$\rho_{A,B} = \mathrm{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi| \tag{223}$$

by

$$S_{\mathrm{ent}} := \mathrm{tr}_{\mathcal{H}_A}(\rho_{A,B} \log \rho_{A,B}) \,. \tag{224}$$

For $|\psi\rangle = \sum_i p_i |i\rangle_B |i\rangle_A$ this gives the formula $S := -\sum_i p_i \log p_i$ alluded to earlier. If the system is maximally entangled, $p_i = 1/N$, so equi-distributed, where N is the number of states (the dimension of the Hilbert space of the system) we obtain $S = \log N$.

The Bekenstein bound then gave the *Holographic principle*, that the maximum number N of states in a spatial region of radius R satisfies

$$N < \exp S_{BH}(R) \tag{225}$$

where now $S_{BH}(R) \sim R^2$ is given by the area. This was so that the total entropy including the black hole would only increase $\Delta M = 1/\lambda$ for absorbing a photon which has entropy ~ 1 from its spin.

This is counter-intuitive without general relativity because one thinks of the number of states in a region as being the exponential of the volume rather than the area. However, gravitational collapse reduces this if there is too much matter (too many particles) and indeed the vast bulk of the entropy is undertood to be gravitational. Thus the Bekenstein entropy encodes the large amount of information that can fall into a black hole.

However, Hawking radiation implies that the Black hole can in principle radiate away. According to the above, this is uncontroversial as the information remains encoded in entanglement. This is the case until we get to a small black hole of Planck size. If it evaporates completely then a pure state has evolved into a mixed state with loss of unitarity. Alternatively the remaining Planckian nugget has enough states to be as entangled with the rest of the universe as the macroscopic black hole was (and we have left the problem of what happens to the Planckian nugget to some as yet undiscovered theory of quantum gravity). Even then, it is problematic that such a Planck size nugget could have so many possible states.

- The evolution of a pure state into a mixed state seems to require a modification of quantum mechanics.
- Hawking: the black hole evaporates and there is enough entanglement in the radiated fields to reconstruct the original configuration (and modifies evolution of ρ).
- Penrose: takes a realist view over collaps of the wave function in measurement. Collapse also entails evolution from pure to mixed states.

8.6 Holography and AdS/CFT

The biggest modern movement in the subject is AdS/CFT. This is a proposal by Maldacena, made concrete by Witten, whereby we have the 'duality'

$$\text{Maximal susy Yang-mills on } \mathbb{M}^4 = \text{Type IIB strings on AdS}_5 \times S^5 \quad (226)$$

where $\mathbb{M}^4 = \partial\ \text{AdS}_5$. This has a large N strong coupling limit on the left hand side where the right hand side is IIB supergravity on $\text{AdS}_5 \times S^5$.

There are a great many generalizations and the correspondence is expressed incorporated in the ansatze

$$\left\langle \exp\left(\int_{\mathbb{M}} \phi_0 \mathcal{O}\right)\right\rangle_{\text{CFT}} = \int [d\phi] \exp(-iI[\phi])\,. \quad (227)$$

Here $\mathcal{O}$ is an operator in the CFT (i.e., super Yang-Mills) and ϕ some corresponding field in IIB supergravity (or string theory). This has by now been checked for a great many examples.

A Further geometry: differential forms

Differential forms often simplify formulae both computationally and conceptually. A p-form $\alpha \in \Omega^p$ is a totally skew covariant tensor. We usually suppress the p skew downstairs indices by introducing formal objects dx^a so that

$$\alpha = \alpha_{a_1 a_2 \dots a_p} dx^{a_1} \wedge \dots dx^{a_p} = \alpha_{[a_1 a_2 \dots a_p]} dx^{a_1} \wedge \dots dx^{a_p} \in \Omega^p. \tag{228}$$

The $\wedge$ symbol signifies that the tensor is skew symmetrized, so that

$$dx^{a_1} \wedge \dots \wedge dx^{a_p} = dx^{[a_1} \wedge \dots \wedge dx^{a_p]} := \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma dx^{a_{\sigma(1)}} \wedge \dots \wedge dx^{a_{\sigma(p)}}$$

In concrete indices these are just the infinitesimal coordinate variations $dx^{\underline{a}}$. There are two key operations with differential forms, the wedge product

$$\alpha \wedge \beta := \alpha_{[a_1 \dots a_p} \beta_{a_{p+1} \dots a_{p+q}]} dx^{a_1} \wedge \dots \wedge dx^{a_{p+q}} \in \Omega^{p+q}, \tag{229}$$

where α is a p-form and β a q-form. This product is graded commutative

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \tag{230}$$

We also have the exterior derivative defined by

$$d\alpha := dx^a \wedge \nabla_a \alpha. \tag{231}$$

Key features are:

Lemma A.1 *The exterior derivative does not depend on the choice of torsion-free covariant derivative. We have $d^2\alpha = 0$ for all α asa consequence of the commutation of partial derivatives (or symmetry of a torsion-free connection).*

Thus it is metric independent and can be defined just using the coordinate derivative in any coordinate system. The fact that $d^2 = 0$ allows us to define cohomology groups

$$H^p(M) = \{\alpha \in \Omega^p | d\alpha = 0\} / \{\alpha = d\beta\}, \tag{232}$$

because the *exact* forms, those that can be expressed as $d\beta$, are a subset of the closed forms, those that satisfy $d\alpha = 0$. These encode the topology of M

because $d\alpha = 0$ implies that locally there exists a β with $\alpha = d\beta$ (Poincaré lemma). As an example, consider $d\theta$ on the circle. Although clearly closed, $\theta \in \mathbb{R}/2\pi$ is not a single valued function on the circle, so it is not globally exact.

The exterior derivative satisfies the graded Leibnitz rule

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta \,. \tag{233}$$

We also have the interior product with a vector V^a that takes a p-form α to the $p-1$-form

$$(V \lrcorner \alpha)_{a_2 a_3 \dots a_p} = p V^{a_1} \alpha_{a_1 \dots a_p} \,. \tag{234}$$

This also satisfies a graded leibnitz property,

$$V \lrcorner (\alpha \wedge \beta) = (V \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (V \lrcorner \beta) \,. \tag{235}$$

It plays a role in the Cartan formula for the Lie derivative of a form

$$\mathcal{L}_V \alpha = V \lrcorner d\alpha + d(V \lrcorner \alpha) \,. \tag{236}$$

When we have a metric, we can define Hodge duality: in d dimensions a p-form α is dualized to a $d-p$ form ${}^*\alpha$ by

$$({}^*\alpha)_{a_{p+1} \dots a_d} := \frac{1}{p!} \varepsilon_{a_1 \dots a_d} \alpha^{a_1 \dots a_p} \tag{237}$$

where $\varepsilon_{a_1 \dots a_d} = \varepsilon_{[a_1 \dots a_d]}$ and $\varepsilon_{0\,1 \dots d-1} = \sqrt{-g}$ is the metric volume form.

A key application is to integration. Being a covariant tensor, a p-form naturally 'pulls back' under a map, and restricts to provide a p-form on a submanifold. On a p-dimensional submanifold, it can naturally be integrated subject to the choice of an orientation on the surface.

Definition A.1 *A p-surface Σ^p is said to be orientable if it is possible to choose a non-vanishing p-form. Such a choice provides an* orientation *on Σ^p.*

The key point is that under a change of coordinates on the p-surface Σ^p, a p-form transforms with the determinant of the Jacobian of the coordinate transformation, whereas the change of variables formula for integration requires the modulus of the determinant which can introduce additional signs, and so we must restrict the coordinate transformations to those that preserve the sign of the chosen form making sure that the sign in question is positive.[14] The standard example of a non-orientable manifold is $\mathbb{RP}^{2n} = S^{2n}/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts by the antipodal map which reverses the sign of the volume form.

The main theorem concerning integration on manifolds is Stoke's theorem:

Theorem 4 (Stokes) *Let Σ be a p-surface with boundary S with compatible orientations (i.e., the orientation on S is obtained from that on Σ by use of an outward pointing normal vector), and let α be a $p-1$-form on Σ, then*

$$\int_\Sigma d\alpha = \int_S \alpha \,. \tag{238}$$

Another application is the Cartan formulation of connections and curvature.

A.1 Connections and curvature

We first choose an orthonormal frame of one-forms $e^{\underline{a}} := e^{\underline{a}}_a dx^a$ satisfying

$$g_{ab} = \eta_{\underline{ab}} e^{\underline{a}}_a e^{\underline{b}}_b \,, \tag{239}$$

where $\eta_{\underline{ab}} = \text{diag}(1,-1,-1,-1)$ is the flat Lorentz metric. The $e^{\underline{a}}_a$ and its inverse $e^a_{\underline{a}}$ can be used to freely convert abastract indices into concrete indices and back again. The connection acting on this frame can be obtained from the Cartan structural equation

$$de^{\underline{a}} = \Gamma^{\underline{a}}{}_{\underline{b}} \wedge e^{\underline{b}} \tag{240}$$

where $\Gamma_{\underline{ab}} = \Gamma_{[\underline{ab}]} = dx^c \Gamma_{c\underline{ab}}$ are the connection 1-forms. These are as many equations as unknowns being 4 2-forms and are nondegenerate, so admit a unique solution for $\Gamma^{\underline{a}}{}_{\underline{b}}$. We can then define the full connection to be

$$\nabla_a e^{\underline{b}}_c = \Gamma_a{}^{\underline{b}}{}_{\underline{c}} e^{\underline{c}}_c \tag{241}$$

[14]The issue is seen in one dimension: under the transformation $y = -x$,

$$\int_a^b f(x)dx = \int_{-a}^{-b} -f(-y)dy = \int_{-b}^{-a} f(-y)dy \,,$$

so that there is no sign change if we are to integrate from the lower limit to the upper in each case.

so that for a general 1-form $A_a = e_a^{\underline{a}} A_{\underline{a}}$ we have

$$\nabla_a A_b = (\nabla_a A_{\underline{b}} - \Gamma_a{}^{\underline{c}}{}_{\underline{b}} A_{\underline{c}}) e_b^{\underline{b}} , \tag{242}$$

where according to the abstract index convention the first term is the ordinary derivative of the components of A_a and doesnt involve the connection. The skew symmetry of $\Gamma_a{}^{\underline{c}}{}_{\underline{b}}$ on its concrete indices then can be seen to be equivalent to the requirement that it preserves the metric $\nabla_a g_{bc} = 0$.

The connection 1-forms determine the curvature 2-form by

$$R_{\underline{a}}{}^{\underline{b}} := dx^c \wedge dx^d R_{cd\underline{a}}{}^{\underline{b}} = d\Gamma^{\underline{a}}{}_{\underline{b}} - \Gamma^{\underline{a}}{}_{\underline{c}} \wedge \Gamma^{\underline{c}}{}_{\underline{b}} . \tag{243}$$

which satisfy Bianchi identities

$$R^{\underline{a}}{}_{\underline{b}} \wedge e^{\underline{b}} = 0 , \qquad dR^{\underline{a}}{}_{\underline{b}} + \Gamma^{\underline{a}}{}_{\underline{c}} \wedge R^{\underline{c}}{}_{\underline{b}} - \Gamma^{\underline{c}}{}_{\underline{b}} R^{\underline{a}}{}_{\underline{c}} = 0 . \tag{244}$$

These essentially follow from $d^2 = 0$.

B Spinors in curved space-time

B.1 Two component spinors

To discuss the Dirac and Rarita-Schwinger equations in curved space-time we need to introduce spinors. In flat space we introduce 2-component spinors via the identification of $\mathbb{R}^4$ with Hermitian 2×2 matrices:

$$dx^{AA'} := \sigma_a^{AA'} dx^a := \frac{1}{\sqrt{2}} \begin{pmatrix} dt + dz & dx + idy \\ dx - idy & dt - dz \end{pmatrix} , \qquad A = 0, 1, A' = 0', 1' . \tag{245}$$

The matrices $\sigma_a^{AA'}$ are sometimes known as Van de Waerden symbols. The determinant is a multiple of the metric. This can be expressed by introducing

$$\varepsilon_{AB} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \varepsilon_{A'B'} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{246}$$

so that

$$ds^2 = \eta_{ab} dx^a dx^b = \varepsilon_{AB} \varepsilon_{A'B'} dx^{AA'} dx^{BB'} . \tag{247}$$

We use the ε_{AB} and its inverse ε^{AB} to raise and lower indices via

$$\psi^A \varepsilon_{AB} = \psi_B , \qquad \psi^A = \varepsilon^{AB} \psi_B , \tag{248}$$

and similarly for the primed version; beware signs, particularly when differentiating with respect to spinors.

Let $\mathbb{S}$ denote the two-dimensional complex vector space of *spinors* ψ^A and $\mathbb{S}'$ primed spinors $\psi^{A'}$. Although the above has been written out in a concrete basis, it can be undertood to express the abstract isomorphism

$$T = \mathbb{S} \otimes \mathbb{S}' \,, \tag{249}$$

where T here is the tangent space. We will often use this to replace vector indices by pairs of spinor indices all thought of as abstract indices[15] so we can write for abstract indices only

$$V^a = V^{AA'} \,. \tag{250}$$

The above establishes the Lorentz invariant identification of $(\mathbb{R}^4, \eta_{ab})$ with $(\mathbb{S}\otimes\mathbb{S}', \varepsilon_{AB}, \varepsilon_{A'B'})$ underpinned by the spinor isomorphism between the space and time orientation preserving Lorentz group $SO_+(1,3)$ and $SL(2,\mathbb{C})/\mathbb{Z}_2$ given by

$$L^b_a \sigma_b^{AA'} = L^A_B \bar{L}^{A'}_{B'} \sigma_a^{BB'} \,, \quad L^a_b \in SO_+(1,3) \,, \quad L^A_B \in SL(2,\mathbb{C}) \,. \tag{251}$$

Since primed spinors transform with the complex conjugate $SL(2,\mathbb{C})$ there is a complex conjugation map

$$\overline{\mathbb{S}} = \mathbb{S}' \,, \qquad \psi^A \to \bar{\psi}^{A'} \,. \tag{252}$$

For infinitesimal Lorentz transformations $l_{ab} = l_{[ab]}$, this is given in spinors by

$$l^{ab} \sigma_a^{AA'} \sigma_b^{BB'} = l^{AA'BB'} = \varepsilon^{A'B'} l^{AB} + \varepsilon^{AB} \bar{l}^{A'B'} \tag{253}$$

where

$$l^{AB} = l^{(AB)} = \tfrac{1}{2} l^{AA'BB'} \varepsilon_{A'B'} \tag{254}$$

so that on the Lie algebra level $so(1,3) = sl(2,\mathbb{C}) \oplus \overline{sl(2,\mathbb{C})}$.

To prove this note first that $\psi^{AB} - \psi^{BA} = \varepsilon^{AB} \psi_C{}^C$ as skew matrices in 2d are necessarily multiples of ε^{AB}. We can use the skew symmetry of l_{ab} to write

$$l^{AA'BB'} = \tfrac{1}{2} l^{AA'BB'} - \tfrac{1}{2} l^{BA'AB'} + \tfrac{1}{2} l^{AA'BB'} - \tfrac{1}{2} l^{AB'BA'} . \tag{255}$$

[15]this doesnt work well in dimensions greater than 6.

where the two terms with minus signs are equal and opposite by skew symmetry of l^{ab}. The first pair of terms therefore reduces to $\varepsilon^{AB}\bar{l}^{A'B'}$ and the second its conjugate.

These are reduced (chiral) spinors. They are related to Dirac spinors by $\psi^\alpha = (\psi^A, \phi^{A'})$. The Clifford matrices are represented in terms of Van de Waerden symbols by

$$\gamma^\alpha_{c\beta} = \sqrt{2}\begin{pmatrix} 0 & \sigma_{cB'}{}^A \\ \sigma_{cB}{}^{A'} & 0 \end{pmatrix}, \qquad \gamma_a\gamma_b + \gamma_b\gamma_a = -2I\eta_{ab} \tag{256}$$

suppressing the the Dirac spinor indices. The Dirac equation $\gamma^a\partial_a\psi = m\psi$ in this notation becomes

$$\partial_{AA'}\psi^A = m\phi_{A'}\,, \qquad \partial_{AA'}\phi^{A'} = m\psi_A\,, \tag{257}$$

where we have introduced the notation $\partial_{AA'} = \sigma^a_{AA'}\partial_a$.

This can be extended to curved space by introducing an orthonormal tetrad $e^{\underline{a}}_a := (e^0_a, e^1_a, e^2_a, e^3_a)$ such that

$$g_{ab} = \eta_{\underline{ab}}e^{\underline{a}}_a e^{\underline{b}}_b\,. \tag{258}$$

We can then use $\sigma^{AA'}_{\underline{a}}$ to introduce spinors with respect to the orthonormal frame.

To extend the Dirac equation to curved space, we must introduce covariant differentiation for spinors. In an orthonormal frame we introduce the Ricci rotation coefficiencts via

$$\nabla_b e^{\underline{a}}_c = \Gamma_b{}^{\underline{a}}{}_{\underline{c}}e^{\underline{c}}_c\,, \tag{259}$$

where ∇_b is the covariant derivative, and the abstract index notation is now being used to indicate that the derivative uses the space-time connection on abstract but not concrete indices. Since $\nabla_a g_{bc} = 0$ and $\eta_{\underline{ab}}$ are constant,

$$\Gamma_{b\underline{ac}} = \Gamma_{b[\underline{ac}]} \tag{260}$$

and so converting to spinors using the Van de Waerden symbols we can define the spin connection by

$$\Gamma_{b\underline{AB}} = \tfrac{1}{2}\Gamma_{b\underline{AA'}\underline{B}}{}^{\underline{A'}}\,, \qquad \Gamma_{b\underline{A'}\underline{B'}} = \tfrac{1}{2}\Gamma_{b\underline{A'}\underline{AB'}}{}^{\underline{A}}\,. \tag{261}$$

These define the covariant derivative of the spin frame $\varepsilon_A{}^{\underline{A}}$ that corresponds to our choice of orthonormal frame by

$$\nabla_a \varepsilon_A{}^{\underline{A}} = \Gamma_{a\underline{B}}{}^{\underline{A}} \varepsilon_A{}^{\underline{B}} \tag{262}$$

and this together with the complex conjugate determines the covariant derivatives on all spinors by the relations

$$\varepsilon_B{}^{\underline{B}} \nabla_a \alpha^B = \nabla_a \alpha^{\underline{B}} - \Gamma_{a\underline{C}}{}^{\underline{B}} \alpha^{\underline{C}} . \tag{263}$$

where the first term on the right, according to the abstract index convention denotes the ordinary derivative of the components of $\alpha^{\underline{B}}$ whereas on the left, $\nabla_a \alpha^B$ is necessarily a covariant derivative.

Once we are happy using fully abstract indices, we can incorporate the isomorphism $TM = \mathbb{S} \otimes \mathbb{S}'$ given by the abstract $\sigma_a^{AA'}$ into equations writing for example

$$\nabla_a \alpha^B = \nabla_{AA'} \alpha^B . \tag{264}$$

The curvature on spinors is given by spinorial Ricci identities

$$[\nabla_{AA'}, \nabla_{BB'}] \alpha^C = \left(\varepsilon_{A'B'} \left(\Psi_{ABD}{}^C - \frac{R}{12} \varepsilon_{D(A} \varepsilon_{B)}{}^C \right) + \varepsilon_{AB} \Phi_{A'B'D}{}^C \right) \alpha^D \tag{265}$$

Here R is the scalar curvature,

$$\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} , \qquad \Phi_{ab} = -\frac{1}{2} \left(R_{ab} - \frac{R}{4} g_{ab} \right) \tag{266}$$

the trace-free Ricci curvature, and $\Psi_{ABCD} = \Psi_{(ABCD)}$ is the spinorial version of the Weyl curvature

$$C_{abcd} = \varepsilon_{A'B'} \varepsilon_{C'D'} \Psi_{ABCD} + \varepsilon_{AB} \varepsilon_{CD} \bar{\Psi}_{A'B'C'D'} . \tag{267}$$

It is also called the conformal curvature because $C_{abc}{}^d$ is invariant under $g_{ab} \to \Omega^2 g_{ab}$. It can be written in terms of the regular curvature as

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} - 4 P_{[a}^{[c} \delta_{b]}^{d]} \tag{268}$$

where P_{ab} is the Schouten tensor

$$P_{ab} = -\frac{1}{2} R_{ab} + \frac{1}{12} R g_{ab} , \tag{269}$$

which we will see later because of its good conformal variations properties.

In this notation we have the Bianchi identities

$$\nabla^D_{A'}\Psi_{ABCD} = \nabla^{B'}_{(A}\Phi_{BC)A'B'} \qquad \nabla^a\Phi_{ab} + \nabla_b R/8 = 0 \tag{270}$$

The Massless field equations: We can now write down massless field equations for arbitrary half-integral helicity s on space-time. These are equations on a symmetric spinor field $\phi_{A_1A_2\dots A_{2s}} = \phi_{(A_1A_2\dots A_{2s})}(x)$

$$\nabla^{A_1}_{A'}\phi_{A_1\dots A_{2s}} = 0\,. \tag{271}$$

For $s < 0$ we have the complex conjugate equation on primed spinors (and at $s = 0$ the scalar wave equation). We have key examples:

1. $s = 1/2$, the Weyl neutrino equation (chiral massless Dirac).

2. $s = 1$ we obtain the spinor form of the Maxwell Field equations

   $$F_{ab} = \varepsilon_{A'B'}\phi_{AB} + \varepsilon_{AB}\bar{\phi}_{A'B'}\,. \tag{272}$$

   The Maxwell equations $\nabla^a F_{ab} = 0$, and $\nabla_{[a}F_{bc]} = 0$ become

   $$\nabla^{AA'}\phi_{AB} = 0. \tag{273}$$

   To see this it is helpful to note that under *Hodge duality* we have

   $$\tfrac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd} = i\varepsilon_{A'B'}\phi_{AB} - i\varepsilon_{AB}\bar{\phi}_{A'B'}\,. \tag{274}$$

   which follows from the expression

   $$\varepsilon_{abcd} = i\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - i\varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'}\,. \tag{275}$$

   Thus ϕ_{AB} defines a self-dual two form ($+i$ eigenvalue under Hodge duality) and $\bar{\phi}_{A'B'}$ anti-self-dual.

3. For $s = 2$ we obtain the vacuum Bianchi identity on the Weyl Spinor

   $$\nabla^{AA'}\Psi_{ABCD} = 0\,, \tag{276}$$

   thus describing gravity.

There are two key results for the general massless field equations

Proposition B.1 *The massless field equations are conformally invariant under $g_{ab} \to \Omega^2 g_{ab}$ with $\phi_{A_1 \dots A_{2s}} \to \Omega^{-1}\phi_{A_1\dots A_{2s}}$.*

Proof: by direct calculation using the conformal variation formulae under $g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$ that give $\nabla_a \to \hat{\nabla}_a$ such that

$$\hat{\nabla}_{AA'}\xi^{B'\dots}_{B\dots} = \nabla_{AA'}\xi^{B'\dots}_{B\dots} - \Upsilon_{A'B}\xi^{B'\dots}_{A\dots} - \dots + \varepsilon_{A'}{}^{B'}\Upsilon_{AC'}\xi^{C'\dots}_{B\dots} + \dots\,, \qquad (277)$$

where $\Upsilon_a = \nabla_a \log\Omega$ and the ... can include terms with primed exchanged by unprimed indices in the obvious way, with one term for each index on ξ. These formulae can be obtained for example from the Cartan structure equations. If all the indices are downstairs and in addition $\hat{\phi}_{B_1\dots B_n} = \phi_{B_1\dots B_n}/\Omega$ we have

$$\Omega\hat{\nabla}_{AA'}\hat{\phi}_{B_1\dots B_n} = \nabla_{AA'}\phi_{B_1\dots} - \Upsilon_{AA'}\phi_{B_1\dots} - \Upsilon_{A'B_1}\phi_{AB_2\dots} - \dots\,, \qquad (278)$$

and so the resulting expression is symmetric in its unprimed indices and so will vanish on contraction with the skew ε^{AB_1}. □

Note that the law satisfied by the Weyl spinor is that under $g_{ab} \to \Omega^2 g_{ab}$, $\Psi_{ABCD} \to \Psi_{ABCD}$ and not the spin-2 variation given above, so Einstein's field equations are not conformally invariant as could be expected. Nevertheless, the vacuum Bianchi identity allows us to rescale the Weyl spinor into a solution to the conformally invariant spin-2 equation.

Proposition B.2 *The massless field equations* (271) *with $s > 1$ are overdetermined and inconsistent.*

Proof: A symmetric spinor $\phi_{A_1\dots A_{2s}}$ has $2s+1$ components whereas there are $2 \times 2s$ equations, i.e., a surfeit of $2s-1$ equations. These only lead to problems in curved space where taking a further derivative of the equation and using the Ricci-identities (265) we obtain

$$0 = \nabla^{A_1}_{A'}\nabla^{A_2A'}\phi_{A_1\dots A_{2s}} = \Psi^{B_1B_2B_3}{}_{(A_3}\phi_{A_4\dots A_{2s})B_1B_2B_3}\,. \qquad (279)$$

This is vacuous in spin one giving a Bianchi identity amongst between the field equations (corresponding to charge conservation) and it then gives Bianchi identities showing the equations are consistent, but with higher spin in curved space it implies new relations on the fields and the equations rapidly become inconsistent. □

The Weyl tensor itself escapes via an algebraic identity that gives the automatic vanishing of the RHS when $\Psi_{ABCD} = \phi_{ABCD}$. Otherwise, spin-2 fields are inconsistent on curved space.

Spin 3/2 fields are a key ingredient of supergravity theories. They escape the Buchdahl conditions in a Ricci-flat background via the Rarita-Schwinger equation, a potential modulo gauge version appropriate for gauging the supersymmetry. This is best understood as an analogue of a Maxwell potential

$$\rho_A = dx^b \rho_{bA}\,, \qquad \text{modulo gauge freedom} \qquad \delta\rho_A = d\xi_A := dx^b \nabla_b \xi_A\,. \tag{280}$$

The action is

$$S = \int_M i\bar{\rho}_{A'} \wedge dx^{AA'} \wedge d\rho_A\,, \tag{281}$$

which gives the field equations

$$dx^{AA'} \wedge d\rho_A := dx^d \wedge dx^c \wedge dx^b \sigma^{AA'}_{[d} \nabla_c \rho_{b]A} = 0\,. \tag{282}$$

Here $\sigma^{AA'}_b$ are the abstract Van der Waerden symbols. In flat space, this relates to the spin 3/2 field above because a consequence of this equation is that

$$d\rho_A = \phi_{ABC} \varepsilon_{B'C'} dx^{BB'} \wedge dx^{CC'}\,, \tag{283}$$

where ϕ_{ABC} is a spin 3/2 massless field in the sense above. However, in curved space, a pure gauge field gives¡ using the Ricci identity (265) with vanishing Ricci tensor in differential form version gives

$$d^2\xi_A = -dx^{BB'} \wedge dx^{CC'} \varepsilon_{B'C'} \Psi_{ABC}{}^D \xi_D\,,$$

so that ϕ_{ABC} is not a gauge invariant quantity, changing by $\Psi_{ABC}{}^D\xi_D$.

It is nontrivial that the field equation is compatible with the gauge freedom, but we have the identity

$$d\left(dx^{AA'} \wedge d\sigma_A\right) = dx^{AA'} \wedge d^2\sigma_A = iG^{AA'}_b {}^*dx^b \wedge \sigma_A\,. \tag{284}$$

Here ${}^*dx^a = \varepsilon^a_{bcd} dx^b \wedge dx^c \wedge dx^d$, and $d^2 \neq 0$ because it is acting on an abstractly indexed quantity and hence requires a commutator that gives rise to curvature. Since G_{ab} is the Einstein tensor, in vacuum we can take ρ_A to be a 0-form, hence proving the consistency of the gauge freedom with the field equations, or as a 1-form, providing a Bianchi identity that gives the consistency of the overdetermined field equations amongst themselves. These identities play a key role in Witten's positive mass theorem.

B.2 Null congruences via spinors

A null congruence is a foliation of a region of space-time by null geodesics. It can be defined by a null vector field l^a whose integral curves are the null geodesics through each point. If it is tangent to a congruence of affinely parametrised null geodesics, then

$$\nabla_l l^b := l^a \nabla_a l^b = 0 \tag{285}$$

Spinors are particularly natural for describing null congruences because a 4-vector l^a is null iff it can be expressed as $l^a := o^A \bar{o}^{A'}$. This follows because the vanishing determinant of $l^{AA'}$ implies that it has rank 1 and conversely. It is always possible to choose the phase of o^A so that o^A is parallel also

$$o^A \bar{o}^{A'} \nabla_{AA'} o^B = 0\,. \tag{286}$$

It is a standard fact that for spinors α^A, β^A, $\alpha^A \beta_A = 0$ iff they are proportional

$$\alpha^A = f \beta^A \tag{287}$$

for some f (i.e., they are proportional) as spin space is two-dimensional and the inner product skew. We can deduce that there is a pair of complex scalars ρ, σ such that

$$o_B \bar{o}^{A'} \nabla_{AA'} o^B = -\rho o_A\,, \quad o_B o^A \nabla_{AA'} o^B = -\sigma \bar{o}_{A'}\,. \tag{288}$$

These have the following geometric interpretation: parametrize the two-plane orthogonal and transverse to l^a by $\zeta \in \mathbb{C}$ by

$$X^a = \zeta \bar{m}^a + \bar{\zeta} m^a\,, \qquad m^a := o^A \bar{\iota}^{A'} \tag{289}$$

for some choice of ι^A with $o_A \iota^A = 1$, and m^a is a complex null vector defined modulo l^a. We can choose ι^A so that $\nabla_l \iota^A = 0$, and then we will also have $\nabla_l m^a = 0$. If X^a connects nearby geodesics of the congruence, then it is Lie derived along l^a, i.e.,

$$[l, X]^a = \nabla_l X^a - \nabla_X l^a = 0\,. \tag{290}$$

This gives

$$\nabla_l \zeta = -\rho \zeta - \sigma \bar{\zeta}\,. \tag{291}$$

This can be interpreted as follows:

1. The imaginary part of ρ is the twist and generates rotations of the ζ plane. It vanishes iff the congruence is hypersurface forming, $l_{[a}\nabla_b l_{c]} = 0$ which implies that there is a rescaling of o^A so that $o_A o_{A'} = \nabla_{AA'} u$ for some function u.

2. The real part of ρ gives the *expansion*, $\nabla_a l^a = -2\rho$ and the area element of the orthogonal transverse plane evolves by

$$A = -im_a dx^a \wedge \bar{m}_b dx^b,$$

   satisfies

$$\mathcal{L}_l A = -2\rho A \tag{292}$$

3. The complex scalar σ is the shear in the sense that a circle in the ζ plane evolves into an ellipse.

4. Equation (290) implies the geodesic deviation equation

$$\nabla_l \nabla_l X^a = l^b l^c X^d R_{bdc}{}^a \tag{293}$$

   and this combines with (291) to give the *Sachs equations*

$$\nabla_l \rho = \rho\bar{\rho} + \sigma\bar{\sigma} + \Phi_{00} \tag{294}$$

$$\nabla_l \sigma = (\rho + \bar{\rho})\sigma + \sigma\bar{\sigma} + \Psi_0 \tag{295}$$

   Here $\Psi_0 = \Psi_{ABCD} o^A \dots o^D$, $\Phi_{00} = \Phi_{ab} l^a l^b = -\frac{1}{2} R_{ab} l^a l^b$ and is positive when the dominant energy condition is satisfied. An important consequence for horizons and singularity theorems is that the whole RHS of (294) is manifestly positive definite.

5. If a null hypersurface has vanishing shear, then it has the intrinsic geometry of a light cone or null hyperplane in Minkowski space up to scaling (i.e. the metric restricts to a multiple of $d\zeta d\bar{\zeta}$ on $\mathbb{R}^3$ or $S^2 \times \mathbb{R}$ where $l^a \partial_a = \partial_v$ for a third coordinate v).

B.3 Problems on Spinors

1. *[This question and the next are not on the course and are based on the 2-component spinor material in this appendix. They are left in for interest only.]*

Show that in 2-component spinors, the stress-energy tensors for Maxwell theory is

$$T_{ab} = \phi_{AB}\bar{\phi}_{A'B'}\,,$$

and for the scalar massless wave equation ($a = 0$)

$$T_{ab} = \nabla_{AB'}\phi\nabla_{BA'}\phi\,.$$

Hence prove the weak version of the dominant energy condition, that $T_{ab}l^a l^b \geq 0$ when l^a is a future-pointing null vector.

2. (This question is harder.)

Use the spinor Ricci identities to prove the relation

$$d(idx^{AA'}d\chi_A) = -\frac{1}{2}{}^*dx^b G_b^{AA'} \wedge \chi_A\,,$$

where χ_A is an indexed form of degree 0 or 1.

Explain briefly how this relation justifies the consistency of the Rarita-Schwinger equations. [The point is that there are more Rarita-Schwinger equations than components of the field minus the number of gauge degrees of freedom. Show that the identity provides as many Bianchi identities for the field equation as this excess number of equations.]

Deduce the Sen-Witten identity

$$d(idx^{AA'}\bar{\xi}_{A'}d\xi_A) = -\frac{1}{2}G_b^{AA'}\bar{\xi}_{A'}\xi_A{}^*dx^b - idx^{AA'} \wedge d\bar{\xi}_{A'} \wedge d\xi_A\,. \quad (296)$$

This lies at the heart of the Witten positive energy proof. Show that the first term on the RHS is positive by the dominant energy condition when the Einstein equations are satisfied. Harder: show that if ξ_A satisfies the Witten equation on a space-like surface Σ, i.e., it satisfies the 4d massless Dirac equation together with the condition that $n^b\nabla_b\xi_A = 0$ where n^a is the normal to Σ, then the second term on the right is positive definite. [*This material is covered p430 on of Vol 2 of Penrose & Rindler's Spinors and Space-time; to complete the argument, the integral of the left hand side gives a boundary term by Stoke's theorem that gives the 4-momentum component along $\xi^A\bar{\xi}^{A'}$ at space-like infinity when ξ_A is asymptotically constant.*]

Solutions to problems on spinors

1. *[This question and the next are not on the course and are based on the 2-component spinor material in the appendix. They are left in for interest only, and optional, i.e. not required for homework completion.]*

   Show that in 2-component spinors, the stress-energy tensors for Maxwell theory is
   $$T_{ab} = \phi_{AB}\bar{\phi}_{A'B'}\,,$$
   and for the scalar massless wave equation ($a = 0$)
   $$T_{ab} = \nabla_{AB'}\phi\nabla_{BA'}\phi\,.$$
   Hence prove the weak version of the dominant energy condition, that $T_{ab}l^a l^b \geq 0$ when l^a is a future-pointing null vector.

   Solution: We start with the formula
   $$T_{ab} = F_a{}^c F_{cb} + \frac{1}{4}g_{ab}F_{cd}F^{cd} \tag{297}$$
   (the staggering of the indices is important) and insert $F_{ab} = \varepsilon_{AB}\phi_{A'B'} +$ *c.c.*, and hope that the signs and factors of 2 come out right... For the wave equation this is a simple index manipulation. The positivity follows from the fact that when written in spinors $T_{ab}l^a l^b$ factorizes as a complex number times its conjugate.

2. (This question is harder and could be the start of a project.)

   Use the spinor Ricci identities to prove the relation
   $$d(idx^{AA'}d\chi_A) = -\frac{1}{2}{}^*dx^b G_b^{AA'} \wedge \chi_A\,,$$
   where χ_A is an indexed form of degree 0 or 1.

   Explain briefly how this relation justifies the consistency of the Rarita-Schwinger equations. [The point is that there are more Rarita-Schwinger equations than components of the field minus the number of gauge degrees of freedom. Show that the identity provides as many Bianchi identities for the field equation as this excess number of equations.]

   Deduce the Sen-Witten identity
   $$d(idx^{AA'}\bar{\xi}_{A'}d\xi_A) = -\frac{1}{2}G_b^{AA'}\bar{\xi}_{A'}\xi_A{}^*dx^b - idx^{AA'} \wedge d\bar{\xi}_{A'} \wedge d\xi_A\,. \tag{298}$$

This lies at the heart of the Witten positive energy proof. Show that the first term on the RHS is positive by the dominant energy condition when the Einstein equations are satisfied. Harder: show that if ξ_A satisfies the Witten equation on a space-like surface Σ, i.e., it satisfies the 4d massless Dirac equation together with the condition that $n^b\nabla_b\xi_A = 0$ where n^a is the normal to Σ, then the second term on the right is positive definite. [*This material is covered p430 on of Vol 2 of Penrose & Rindler's Spinors and Space-time; to complete the argument, the integral of the left hand side gives a boundary term by Stoke's theorem that gives the 4-momentum component along* $\xi^A\bar{\xi}^{A'}$ *at space-like infinity when* ξ_A *is asymptotically constant.*]

Solution: We have $d(dx^a) = 0$ for a torsion-free connection.[16]

Thus we also have $d(dx^a) = 0$. The Ricci identity will in general give $d^2\chi_A = -R^B{}_A \wedge \chi_B$ for any indexed form χ^A where $R_{AB} = dx^c \wedge dx^d R_{cdAB}$ is the spinor curvature 2-form with R_{cdAB} which is half of the RHS of (265). We are trying to compute

$$d(dx^{AA'} \wedge d\chi_A) = -dx^{AA'} d^2\chi_A = -dx^{AA'} \wedge dx^c \wedge dx^d R_{cdA}{}^B \chi_B \,,$$

and from Hodge duality $dx^a \wedge dx^b \wedge dx^c = \varepsilon^{abcd*}dx_d$ so

$$d(dx^{AA'} \wedge d\chi_A) = -\varepsilon^{abcd} R_{cdA}{}^B \chi_B{}^* dx_d \,.$$

Substituting in the expression for the curvature from (265) and the spinor version of ε^{abcd} from (275) we must perform the contractions in

$$(i\varepsilon^{AC}\varepsilon^{BD}\varepsilon^{A'D'}\varepsilon^{B'C'} - i\varepsilon^{AD}\varepsilon^{BC}\varepsilon^{A'C'}\varepsilon^{B'D'}) \left(\varepsilon_{C'D'}\left(\Psi_{CDA}{}^B - \frac{R}{12}\varepsilon_{A(C}\varepsilon_{D)}{}^B\right) + \varepsilon_{CD}\Phi_{C'D'A}{}^B\right)$$

which I wouldnt do in public, but which yields the relevant multiple of the Einstein tensor $-\frac{1}{2}G_{ad} = \Phi_{ad} + g_{ad}R/8$.

[16] More generally, the Cartan structural equation for the exterior derivative of a basis of forms $e^{\underline{a}}$ reads

$$de^{\underline{a}} - \Gamma_{\underline{b}}{}^{\underline{a}} e^{\underline{b}} = T^{\underline{a}}_{\underline{ab}} e^{\underline{b}} \wedge e^{\underline{c}}$$

where T^a_{bc} is the torsion. Converting to concrete indices we would have cancellation of the connection forms in $d(e^{\underline{a}}_{\underline{b}} e^{\underline{b}})$, because it acts on both the upper and lower index but the Cartan structure equation above would leave us with the torsion if the connection had been chosen to have torsion.

This identity immediately gives, with χ_A replaced by ξ_A, an indexed zero-form, the condition that pure gauge satisfies the Rarita-Schwinger equation.

The Rarita-Schwinger equations are $dx^{AA'} \wedge d\chi_A = 0$ for χ_A an indexed 1-form. This has $2 \times 4 = 8$ components bbut two can be set to zero with a gauge transformation leaving 6. The RS equations are 2×4 equations being an indexed 3-form equation so there are two equations extra that can in an initial value problem be thought of as the constraints obtained by restricting the 3-form to the initial data surface (the d can only then have tangential derivatives). The identity just proved then provides two Bianci identities that evolve the constraints if satisfied initially. [The simple model for this is spin-1 where the field equation $d^*F = 0$ is 4 equs on the 3 components of the potential after gauge fixing is taken into account. However, $d^{2*}F = 0$ is an identity that automatically evolves one of the equations if it is satisfied initially. In EM the obvious constraint that is so evolved is $\nabla \cdot E = 0$.]

For the positivity of the RHS of the Sen-Witten identity, the first term just follows from the Eisntein equations and dominant energy condition so long as one knows that on restriction to a space-like surface Σ, ${}^*dx^a|_\Sigma = n^a d\nu_\Sigma$, where $d\nu_\Sigma$ is the volume form of the restricted metric and n^a the unit time-like normal.

The positivity of the second term uses

$$dx^{AA'} \wedge d\bar{\xi}_{A'} \wedge d\xi_A = \varepsilon^{AA'bcd} \nabla_b \bar{\xi}_{A'} \nabla_c \xi_A {}^* dx_d$$

and expanding out the spinor volume form as before. One needs to use the fact that $n^a \nabla_a \xi_B = 0$ and the Witten equation is then $n^{AA'} \nabla_{BA'} \xi_A = 0$. You also need to know that timelike $n^{AA'}$ defines a unitary inner product on spinors.