2025 Q.1

(a) $\rho_t + \rho_a = -\mu\rho$

Conservation of mass — const percapita death rate μ.

$\rho(0,a) = f(a)$ — initial ($t=0$) age-distribution.

$\rho(t,a) = \int_0^\infty b(t,a)\rho(t,a)\,da$ ← births with percapita birth rate $b(t,a)$

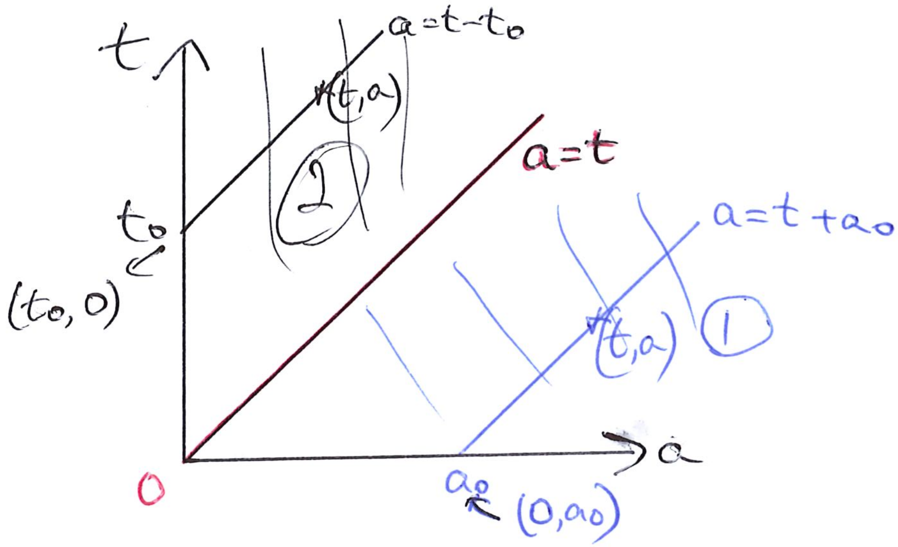
$$\frac{dN}{dt} = \underbrace{N\left(1-\frac{N}{K}\right)}_{\substack{\text{logistic growth}\\ \text{with lin growth}\\ \text{rate 1, carrying}\\ \text{cap } K}} - NP$$

predation

(b) Method of characteristics:

$\frac{d\rho}{dt} = -\mu\rho$ on char. $\frac{da}{dt} = 1 \Rightarrow a = t + \text{const}.$

$$\left[\frac{d\rho}{dt} = \rho_t + \rho_a \underbrace{\frac{da}{dt}}_{1 \text{ from eqn}} = -\mu\rho\right]$$



Two regions: 1 $0<t<a$

$$\frac{d\rho}{dt} = -\mu\rho$$

$$\int_{t=0}^{t} dt : \int_{0}^{t} \frac{d\rho}{\rho} = -\mu\int_{0}^{t} dt$$

$$\Rightarrow \ln\left[\rho(t,a)-\rho(0,a_0)\right] = -\mu t$$

ie $$\frac{\rho(t,a)}{\rho(0,\underbrace{a-t}_{a_0=a-t})} = e^{-\mu t}$$

$$\therefore \rho(t,a) = f(a-t)e^{-\mu t}$$ using $\rho(0,a-t) = f(a-t)$

2 : $0 < a < t$ $\quad \frac{d\rho}{dt} = -\mu\rho$

$$\int_{t_0}^{t} dt : \quad \int_{t_0}^{t} \frac{d\rho}{\rho} = -\mu \int_{t_0}^{t} dt$$

ie $\ln\left(\rho(t,a) - \rho(t_0, 0)\right) = -\mu \underbrace{(t - t_0)}_{a}$

ie $\rho(t,a) = \rho(t_0, 0) e^{-\mu a}$

which becomes, using the birth cond^n:

$$\rho(t,a) = B(t-a) e^{-\mu a}$$

since $t_0 = t - a$
& $\rho(t-a, 0) = B(t-a)$.

$$B(t) \overset{\text{defn.}}{=} \rho(t,0) = \int_0^\infty b(t,a)\rho(t,a)\,da$$

$$= \underbrace{\int_0^t da}_{2} + \underbrace{\int_t^\infty da}_{1}$$

$$= \int_0^t b(t,a) \underbrace{B(t-a) e^{-\mu a}}_{\substack{\text{sol}^n \text{ for} \\ \rho(t,a) \text{ in} \\ 2}} da + \int_t^\infty b(t,a) \underbrace{f(a-t) e^{-\mu t}}_{\substack{\text{sol}^n \text{ for} \\ \rho(t,a) \\ \text{in } 1}} da$$

(4.)

$$\frac{dP(t)}{dt} = \frac{d}{dt}\int_0^\infty \rho(t,a)\,da$$

$$= \int_0^\infty \frac{\partial \rho}{\partial t}\,da$$

$$= \int_0^\infty -\left[\frac{\partial \rho}{\partial a} + \mu\rho\right] da \quad \text{using } \rho_t + \rho_a = -\mu\rho$$

$$= -\left[\rho(t,\infty) - \rho(t,0)\right] - \mu P$$

Assume no predator lives forever $\therefore$ $\rho(t,\infty) = 0$

$$\therefore \frac{dP}{dt} = \underset{\rho(t,0)}{B(t)} - \mu P$$

$$\text{ie } \frac{dP(t)}{dt} + \mu P(t) = B(t) \qquad (*)$$

(c). $\frac{dP}{dt} + \mu P(t) = \int_0^\infty b(t,a) B(t-a) e^{-\mu a} da$

because in $\lim_{t\to\infty}$ $B(t) = \int_0^\infty b(t,a) B(t-a) e^{-\mu a} da$

$$+ \underbrace{\int_\infty^\infty b(t,a) f(t-a) e^{-\mu t} da}_{0}$$

Now : $B(t-a) = \frac{d}{dz}(P(t-a)) + \mu P(t-a)$

where $z = t-a$

$\left[\frac{d}{dz}P(z) = \frac{d}{da}P(z) \cdot \frac{dz}{da} = -\frac{d}{da}P(z)\right]$.

(replace "t" by "t−a" in (*))

$\therefore \int_0^\infty b(t,a) B(t-a) e^{-\mu a} da$ (*)(*)

becomes $\int_0^\infty b(t,a) \left[-\frac{d}{da}(P(t-a)) + \mu P(t-a)\right] e^{-\mu a} da$

since $\frac{d}{dz} P(t-a) = -\frac{d}{da} P(t-a)$

Note that $-\frac{d}{da} P(t-a) e^{-\mu a} + \mu P(t-a) e^{-\mu a}$

$$= -\frac{d}{da}\left[P(t-a) e^{-\mu a}\right]$$

$\therefore$ (*)(*) is $-\int_0^\infty \underbrace{b(t,a)}_{u} \underbrace{\frac{d}{da}\left[P(t-a) e^{-\mu a}\right]}_{dv} da$

$$= -b(t,a) P(t-a) e^{-\mu a}\Big]_0^\infty + \int_0^\infty P(t-a) e^{-\mu a} \frac{db}{da} da$$

$B(0)=0 \Rightarrow b(t,0)=0$, $b(t,\infty) P(t-\infty) e^{-\mu\infty} = 0$

$$\therefore \frac{dP(t)}{dt} + \mu P(t) = \int_0^\infty P(t-a) e^{-\mu a} \frac{db}{da}\, da$$

(d.) St. st.: $N^*\left(1 - \frac{N^*}{K}\right) = N P^*$

$$\Rightarrow N^* = 0 \text{ or } N^* = K(1 - P^*)$$

$$\mu P^* = \int_0^\infty \underbrace{\left(\mu + \frac{1}{m}\right)^2 N^*(1 + cP^*)}_{b(t,a)} \frac{d}{da}\left(a e^{-\frac{a}{m}}\right) P^* e^{-\mu a}\, da$$

$$\uparrow$$

$P(t-a) \equiv P(t) \equiv P^*$ at st. st.

$$\text{i.e. } P^* = 0 \text{ or } \mu = \left(\mu + \frac{1}{m}\right)^2 N^*(1 + cP^*) \underbrace{\int_0^\infty \underbrace{\frac{d}{da}\left(a e^{-\frac{a}{m}}\right)}_{dv} \underbrace{e^{-\mu a}}_{u}\, da}_{I}$$

$$I = \left[e^{-\mu a} a e^{-\frac{a}{m}}\right]_0^\infty + \mu \int_0^\infty \underbrace{a}_{u} \underbrace{e^{-\frac{a}{m} - \mu a}}_{dv}\, da$$

$$= \mu \left\{ \left[\frac{a e^{-\frac{a}{m} - \mu a}}{\left(-\frac{1}{m} - \mu\right)}\right]_0^\infty + \int_0^\infty \frac{e^{-\frac{a}{m} - \mu a}}{\frac{1}{m} + \mu}\, da \right\}$$

$$= \frac{-\mu}{\left(\frac{1}{m} + \mu\right)^2} \left[e^{-\frac{a}{m} - \mu a}\right]_0^\infty = \frac{\mu}{\left(\frac{1}{m} + \mu\right)^2}$$

∴ we have

$$\mu = \left(\mu + \frac{1}{m}\right)^2 N^*(1+cP^*) \underbrace{\frac{\mu}{\left(\mu+\frac{1}{m}\right)^2}}_{I}$$

i.e. $1 = N^*(1+cP^*)$

$$\therefore N^* = \frac{1}{1+cP^*}$$

Recap: $N^* = 0$ or $N^* = K(1-P^*)$ eqn for N

$P^* = 0$ or $N^* = \dfrac{1}{1+cP^*}$ " " P.

Hence stst are $(N^*, P^*) = (0,0)$

$(K, 0)$

$(\bar{N}, \bar{P})$

where $\bar{N} = K(1-\bar{P})$

$\bar{N} = \dfrac{1}{1+c\bar{P}}$

i.e. $\dfrac{1}{1+c\bar{P}} = K(1-\bar{P})$

Hence $1 = K(1-\bar{P})(1+c\bar{P})$

ie $-Kc\bar{P}^2 - (K-Kc)\bar{P} - 1 + K = 0$

ie $Kc\bar{P}^2 + K(1-c)\bar{P} + 1 - K = 0$

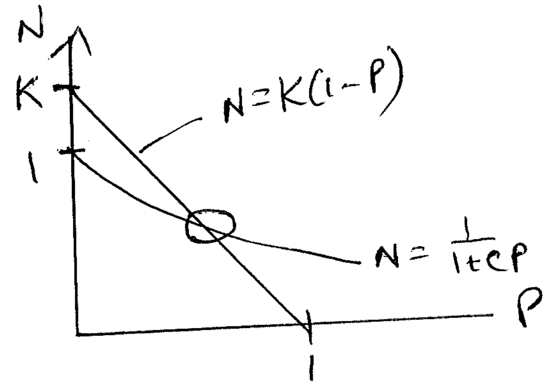
$$\Rightarrow \quad \bar{P} = \frac{c-1}{2c} \pm \frac{K\sqrt{(c-1)^2 - \frac{4(1-K)c}{K}}}{2Kc}$$

$$= \frac{c-1}{2c} \pm \sqrt{\left(\frac{c-1}{2c}\right)^2 - \frac{1-K}{Kc}}$$

$\therefore$ if $K>1$ then we have 2 real roots but only one +ve root $\frac{c-1}{2c} + \sqrt{\left(\frac{c-1}{2c}\right)^2 + \frac{K-1}{Kc}}$

N.B. c can be any +ve value.

[Note that $K>1$



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$K<1 \Rightarrow$ we need $c>1$ for +ve roots

& to be real $\left(\frac{c-1}{2c}\right)^2 \geq \frac{1-K}{Kc}$.

Check: if they are real, are they +ve?

$\frac{c-1}{2c} + \sqrt{\left(\frac{c-1}{2c}\right)^2 - \frac{1-K}{Kc}} > 0$ since $c>1$

Note if $c<1$ then this root is <0 since $\sqrt{} < \frac{c-1}{2c}$.

$\frac{c-1}{2c} - \sqrt{\left(\frac{c-1}{2c}\right)^2 - \frac{1-K}{Kc}} > 0$ since $\sqrt{} < \frac{c-1}{2c}$.

$$\left(\frac{c-1}{2c}\right)^2 \geqslant \frac{1-K}{Kc}$$

$$\Rightarrow (c-1)^2 \geqslant \frac{4c^2(1-K)}{Kc}$$

$$= 4c\left(\frac{1-K}{K}\right)$$

ie $c^2 - 2c - 4c\left(\frac{1-K}{K}\right) + 1 > 0$

ie $c^2 - 2\left(\frac{2-K}{K}\right)c + 1 > 0$

Now consider

$$f(c) = c^2 - 2\left(\frac{2-K}{K}\right)c + 1$$

$$f'(c) = 2c - 2\left(\frac{2-K}{K}\right)$$

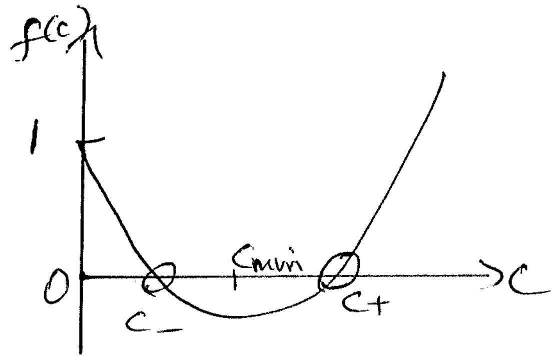
$$= 0$$

$$\Rightarrow c = \frac{2-K}{K}$$

This is a min. $f(c_{min}) = \left(\frac{2-K}{K}\right)^2 - 2\left(\frac{2-K}{K}\right)^2 + 1$

$$= 1 - \left(\frac{2-K}{K}\right)^2$$

$\therefore$ require $\left(\frac{2-K}{K}\right)^2 > 1$



$$c_\pm = \frac{\frac{2-K}{K} \pm \sqrt{4\left(\frac{2-K}{K}\right)^2 - 4}}{2}$$

$$= \frac{2-K}{K} \pm \sqrt{\left(\frac{2-K}{K}\right)^2 - 1}$$

require $c_\pm$ to be real & +ve

ie $\left(\frac{2-K}{K}\right)^2 > 1$

$\Rightarrow (2-K)^2 > K^2$

$\Rightarrow 4 - 4K + K^2 > K^2$

$\Rightarrow 1 - K > 0$ ✓ since $\underline{K<1}$

$c_\pm$ are +ve since $2-K > 0$ ✓

Also need $c > 1$:

Now $c_+ > \frac{2-K}{K} = \frac{2}{K} - 1 > 1$ since $\underline{K<1}$.

$\therefore$ The conditions $c>1$ & $f(c)>0$ certainly hold for $c > c_+ = \frac{2-K}{K}$

$f(c)>0$ also for $0 < c < c_-$

Now is $c_- < 1$ or >1?

Note that for $f(c)=0$, pdt of roots $=1$

Since $c_+ > 1$, then $c_- < 1$.

$\therefore$ $c>1$ & $f(c)>0$ only holds for

$$\underline{\underline{c > \frac{2-K}{K}}}$$