Revision

sources

Maxwell eqs: $\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$ $\qquad \underline{\nabla} \cdot \underline{B} = 0$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \qquad \underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}$$

Continuity eq: $\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0$

Lorentz force: $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

Linear PDEs:
- solve all of them (in some setups some are trivial)
- principle of superposition

Electrostatics and magnetostatics in $\mathbb{R}^3$

Scalar and vector potentials

$\underline{E} = -\underline{\nabla}\phi$, $\underline{B} = \underline{\nabla} \times \underline{A}$, satisfy Poisson eq (in Lorenz gauge)

$\underline{\nabla} \cdot \underline{E} = -\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$ similarly for $\underline{A}$

$$\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int dV' \, \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} \, , \quad \underline{A} = \frac{\mu_0}{4\pi} \int dV' \, \frac{\underline{J}(\underline{r}')}{|\underline{r} - \underline{r}'|}$$

$$\underline{E} = \frac{1}{4\pi\varepsilon_0} \int dV' \, \rho(\underline{r}') \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|^3} \, , \quad \underline{B} = \frac{\mu_0}{4\pi} \int dV' \, \frac{\underline{J}(\underline{r}') \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \qquad (*)$$

electric field of p.c.'s added up

Biot-Savart law

Common solution method: integral form of ME's + symm.

$$\int_{\Sigma=\partial R} d\underline{S} \cdot \underline{E} = \frac{Q_{\text{inside of } R}}{\varepsilon_0}, \quad \int_{C=\partial\Sigma} d\underline{r} \cdot \underline{B} = \mu_0 I_{\text{through } \Sigma}$$

Multipole expansion, Dipole field

Macroscopic media and boundary conditions

$$\phi(\underline{r}) = \phi_{\text{free}}(\underline{r}) + \phi_{\text{dipoles}}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int dV' \Big(\rho_{\text{free}}(\underline{r}') \underbrace{- \underline{\nabla}' \cdot \underline{P}(\underline{r}')}_{\rho_{\text{bound}}(\underline{r}')} \Big) \frac{1}{|\underline{r}-\underline{r}'|}$$

electric polarisation (or dipole) density

$$\underline{\nabla} \cdot \underline{E} = \frac{1}{\varepsilon_0} (\rho_{\text{free}} + \rho_{\text{bound}})$$

$$\Rightarrow \underline{\nabla} \underbrace{(\varepsilon_0 \underline{E} + \underline{P})}_{= \varepsilon \underline{E}} = \rho_{\text{free}}$$

Similar logic for $\underline{B}$: $\nabla \times \underline{B} = \mu_0 \left(\underline{J}_{free} + \underline{J}_M \right)$

$\nabla \times \underline{M}$ magnetisation density

$$\Rightarrow \nabla \times \underbrace{\left(\frac{\underline{B}}{\mu_0} - \underline{M} \right)}_{\underline{B}/\mu} = \underline{J}_{free}$$

MEs in medium

$\nabla \cdot (\varepsilon \underline{E}) = \rho_{free}$ $\qquad$ $\nabla \cdot \underline{B} = 0$

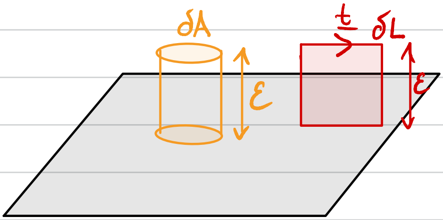
$\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0}$ $\qquad$ $\nabla \times \left(\frac{\underline{B}}{\mu} \right) - \frac{\partial (\varepsilon \underline{E})}{\partial t} = \underline{J}_{free}$

⇝ reorganise (source free)

$\nabla \times \underline{B} - \underbrace{\varepsilon \mu}_{\frac{1}{v^2}} \frac{\partial \underline{E}}{\partial t} = \underline{0}$

$\frac{1}{v^2}$ speed in medium

Matching across bdys

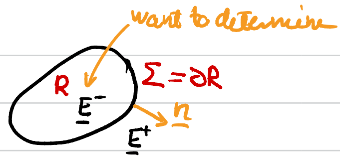


| : curl eq $\qquad$ | : div eq

$\underline{n} \cdot \left(\varepsilon^+ \underline{E}^+ - \varepsilon^- \underline{E}^- \right) = \sigma_{free}$, $\quad \underline{t} \cdot \left(\underline{E}^+ - \underline{E}^- \right) = 0$

$\underline{n} \cdot \left(\underline{B}^+ - \underline{B}^- \right) = 0$, $\quad \underline{t} \cdot \left(\frac{\underline{B}^+}{\mu^+} - \frac{\underline{B}^-}{\mu^-} \right) = 0$ $\quad (\cancel{\underline{J}}_{free})$

Solve PS3/Q4 b,c



Electrostatics in finite vol

$$-\underline{n}\cdot\left(\varepsilon^- \underline{E}^-\right)\Big|_\Sigma = \sigma$$

dyn determined in conductor
fixed for insulator

Problem: $\underline{E} = -\underline{\nabla}\phi \Rightarrow$

$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_0}$

$$\boxed{\begin{array}{l} \nabla^2\phi = -\dfrac{\rho(\underline{r})}{\varepsilon_0} \quad \text{in } R \\[2ex] \phi\big|_\Sigma = V(\underline{r}) \quad \text{or} \quad \dfrac{\partial\phi}{\partial n}\Big|_\Sigma = \dfrac{\sigma(\underline{r})}{\varepsilon_0} \end{array}}$$

Solution methods:
- solve for fixed ρ_{free}
- Green's fn
- method of images
- orthonormal fns.

Green's fns

$$\nabla^2 G(\underline{r},\underline{r}') = -4\pi\,\delta(\underline{r}-\underline{r}')$$

unprimed, but
G is symm

True for any Green's fn, need to make choices to make useful:

- $G_D(\underline{r},\underline{r}') = 0 \quad \forall \underline{r} \in R, \underline{r}' \in \Sigma$

$$\phi(\underline{r}) = \frac{1}{4\pi}\left(\int_R dV'\, G(\underline{r},\underline{r}')\frac{\rho(\underline{r}')}{\varepsilon_0} - \int_{\Sigma=\partial R} dS'\, \frac{\partial G(\underline{r},\underline{r}')}{\partial n'}\, \underbrace{\phi(\underline{r}')}_{\text{given}}\right)$$

- $\dfrac{\partial G_N(\underline{r},\underline{r}')}{\partial n'} = -\dfrac{4\pi}{A} \quad \forall \underline{r} \in R, \underline{r}' \in \Sigma$

$$\phi(\underline{r}) = \frac{1}{4\pi}\left(\int_R dV'\, G(\underline{r},\underline{r}')\frac{\rho(\underline{r}')}{\varepsilon_0} + \int_{\Sigma=\partial R} dS'\, G(\underline{r},\underline{r}')\underbrace{\frac{\partial \phi(\underline{r}')}{\partial n'}}_{\text{given}}\right) + \underbrace{\langle \phi \rangle_\Sigma}_{\text{bdy average can be shifted to 0}}$$

Ex: $R = \mathbb{R}^3 \quad G(\underline{r},\underline{r}') = \dfrac{1}{|\underline{r}-\underline{r}'|}$

Method of images (also for Green's fn)

Ex: Conducting plate @ $z=0$, potential of p.c. @ $\underline{r}_0$ in $z\geq 0$

$$\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\underline{r}-\underline{r}_0|} + \frac{q^*}{|\underline{r}-\underbrace{\underline{r}_0^*}_{\text{in LHS}}|}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\underline{r}-\underline{r}_0|} - \frac{1}{|\underline{r}-\underline{r}_0^R|}\right)$$

with $\underline{r}_0^R = (x_0, y_0, -z_0)$

$$G_D(\underline{r}, \underline{r}') = \frac{1}{|\underline{r}-\underline{r}'|} - \frac{1}{|\underline{r}-\underline{r}'^R|}$$

$\propto$ pot of p.c. @ $\underline{r}'$

Other ex's are sphere, spherical shell, wedge, parallel plates

Orthonormal fn's

On $[a,b]$ interval : $f(x) = \sum_n c_n u_n(x)$

any well-behaved fn

$$\int dx\, \overline{u_n(x)}\, u_m(x) = \delta_{nm} \quad \text{orthogonality}$$

$$\sum_n \overline{u_n(x)}\, u_n(x') = \delta(x-x') \quad \text{completeness}$$

$$\Rightarrow c_n = \int dx\, \overline{u}_n(x) f(x)$$

We will do these on other regions as well, want them to play nicely w/ ∇^2:

- $[0,a]$: $u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$ $\qquad \nabla^2 u_n = -\left(\frac{n\pi}{a}\right)^2 u_n$
- $[0,a]\times[0,b]$: $u_{n,m}(x,y) = \sqrt{\frac{4}{ab}} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b} y\right)$

$$\int_0^a dx \int_0^b dy\, \overline{u_{nm}(x,y)}\, u_{n'm'}(x,y) = \delta_{nn'}\delta_{mm'}$$

$$\sum_{n,m=1}^{\infty} \overline{u_{n,m}(x,y)}\, u_{n,m}(x',y') = \delta(x-x')\delta(y-y') =: \delta(\underline{r}-\underline{r}')$$

$$\nabla^2 u_{nm} = -\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] u_{nm}$$

- $\mathbb{R}$: $u_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $f(x) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} dk}_{\text{replaces sum}} \, C(k) e^{ikx}$

$$C(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; e^{-ikx} f(x)$$

(inverse) Fourier transform

Orth : $\frac{1}{2\pi} \int dx \; e^{i(k'-k)x} = \delta(k-k')$

Completeness : $\frac{1}{2\pi} \int dk \; e^{ik(x'-x)} = \delta(x-x')$

$\nabla^2 u_k = -k^2 u_k$

- S^1 circle : $u_m = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ $\qquad \nabla^2_{S^1} u_m = \partial^2_\varphi u_m = -m^2 u_m$

Cylinrical polar : $f = F(r,z) \, u_m(\varphi)$

$$\nabla^2 f = \left[\partial_z^2 F + \frac{1}{r} \partial_r (r \partial_r F) \right] u_m + F \frac{1}{r^2} \underbrace{\nabla^2_{S^1} u_m}_{-m^2 u_m}$$

- S^2 sphere : $u_{\ell m} = Y_{\ell m}(\theta, \varphi) \propto P_{\ell m}(\theta) \, e^{im\varphi}$ $\qquad \nabla^2_{S^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$

Spherical polar : $f = F(r) \, Y_{\ell m}(\theta, \varphi)$

$$\nabla^2 f = \frac{1}{r^2} \partial_r (r^2 \partial_r F) \, Y_{\ell m} + F \frac{1}{r^2} \underbrace{\nabla^2_{S^2} Y_{\ell m}}_{-\ell(\ell+1) Y_{\ell m}}$$

Orth : $\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; \bar{Y}_{\ell m} Y_{\ell' m'} = \delta_{\ell\ell'} \delta_{mm'}$

Completeness: $\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \overline{Y_{\ell m}(\theta,\varphi)}\, Y_{\ell m}(\theta',\varphi') = \frac{1}{\sin\theta}\, \delta(\theta-\theta')\,\delta(\varphi-\varphi')$

Solving electrostatics problems w/ orth fn's

Ex. $R = [0,a]\times[0,b]\times\mathbb{R}$, $\phi(0,y,z) = \phi(a,y,z) = \phi(x,0,z) = 0$

$$\phi(x,b,z) = V_0 + V_k \sin\left(\frac{k\pi}{a}x\right)$$

Basic sol: $\phi = \sin\left(\frac{n\pi}{a}x\right) Y(y)$

$$\nabla^2\phi = \underbrace{\left[-\left(\frac{n\pi}{a}\right)^2 Y(y) + Y''(y)\right]}_{=0 \;\Rightarrow\; Y = \sinh\left(\frac{n\pi}{a}y\right) \;\; n\neq 0, \quad Y = y \;\; n=0} \sin\left(\frac{n\pi}{a}x\right) = 0$$

Principle of sup: $\phi = V_0 \frac{y}{b} + V_k \sin\left(\frac{k\pi}{a}x\right) \sinh\left(\frac{k\pi}{a}y\right) / \sinh\left(\frac{k\pi}{a}b\right)$

Ex. $R = \mathbb{R}^3$, $\rho_{free} = \rho_0 r^{\alpha} Y_{7,3}(\theta,\varphi)$

Look for sol as $\phi = F(r)\, Y_{7,3}(\theta,\varphi)$

$$\nabla^2\phi = \left(F'' + \frac{2}{r}F' - \frac{7\cdot 8}{r^2}F\right) Y_{7,3} = -\rho_0 r^{\alpha} Y_{7,3}$$

Power Ansatz: $F = c\,\rho_0 r^{\alpha+2}$: $c\left[(\alpha+2)(\alpha+1) + 2(\alpha+2) - 56\right] = -1$

$$c = -\frac{1}{(\alpha+2)(\alpha+3) - 56}$$

Constructing Green's fns using orth. fn's

Ex. $R = \mathbb{R}^3$ using translation symm (Method 1)

$$G(\underline{r}, \underline{r}') = \text{"}\sum_n \frac{\overline{u_n(x)}\, u_n(x')}{\lambda_n}\text{"} = \int \frac{d^3 \underline{k}}{(2\pi)^3} \frac{e^{i\underline{k}\cdot(\underline{r}'-\underline{r})}}{-|\underline{k}|^2} = \frac{1}{|\underline{r}-\underline{r}'|}$$

$\nabla^2 e^{i\underline{k}\cdot\underline{r}} = -|\underline{k}|^2 e^{i\underline{k}\cdot\underline{r}}$

Ex. $R = \mathbb{R}^3$ using rotation symm (Method 1 & 2)

$$\nabla^2 G(\underline{r}, \underline{r}') = -4\pi\, \delta(\underline{r}-\underline{r}') = -4\pi \frac{1}{r^2 \sin\theta} \delta(r-r')\, \delta(\theta-\theta')\, \delta(\varphi-\varphi')$$

inv. Jacobian

$$= -\frac{4\pi}{r^2} \delta(r-r') \sum_{\ell=0}^{\infty} \sum_{m=-\ell} \overline{Y_{\ell m}(\theta,\varphi)}\, Y_{\ell m}(\theta', \varphi')$$

completeness

Inspires writing : $G_D = \sum_{\ell,m} A_{\ell,m}(r, r')\, \overline{Y_{\ell,m}(\theta,\varphi)}\, Y_{\ell,m}(\theta', \varphi')$

Method 2: $y_1(r') = r'^{\ell}$, $y_2 = r'^{-(\ell+1)}$, $P_2 = -\frac{r'^2}{4\pi}$, $W = -\frac{(2\ell+1)}{r'^2}$

reg @ $r=0$ — decay for $r \to \infty$

$$A_{\ell,m} = \frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}}$$

$r_<^{\ell} \leftarrow y_1$, $r_>^{\ell+1} \leftarrow y_2$

$r_< := \min(r, r')$

$r_> := \max(r, r')$

Electrodynamics

$$\underline{\nabla} \cdot \underline{B} = 0 \quad \Rightarrow \quad \underline{B} = \underline{\nabla} \times \underline{A}$$

$$0 = \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underbrace{\left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right)}_{-\underline{\nabla}\phi} \quad \Rightarrow \quad \underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$$

Fix gauge freedom (in vacuum): $0 = \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \underline{\nabla} \cdot \underline{A}$ Lorenz gauge

Other two vacuum MEs: $\Box \phi = -\frac{\rho}{\varepsilon_0}$, $\Box \underline{A} = -\mu_0 \underline{J}$

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

Time dep Green's fn: $\Box \psi = -4\pi f(\underline{r}, t)$

$$\Box G(\underline{r}, t; \underline{r}', t') = -4\pi \delta(\underline{r} - \underline{r}')\delta(t - t')$$

$$\psi(\underline{r}, t) = \int_{\mathbb{R}^3 \times \mathbb{R}} dt' dV' \, G(\underline{r}, t; \underline{r}', t') f(\underline{r}', t')$$

$$G_R(\underline{r},t;\underline{r}',t') = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^3\times\mathbb{R}} d\omega\, d^3k \;\; g(\underline{k},\omega)\; e^{i[\underline{k}\cdot(\underline{r}-\underline{r}')-\omega(t-t')]}$$

$$g(\underline{k},\omega) = \frac{4\pi}{|\underline{k}|^2 - \frac{\omega^2}{c^2}} \quad \text{Method 2}$$

$$= \frac{1}{|\underline{r}-\underline{r}'|}\,\delta\left(t-t'-\frac{|\underline{r}-\underline{r}'|}{c}\right)$$

contour integration

ω contour on UHP

$$\phi(\underline{r},t) = \frac{1}{4\pi\varepsilon_0}\int dV' \,\frac{\rho(\underline{r}',t_r)}{|\underline{r}-\underline{r}'|} \qquad t_r = t - \frac{|\underline{r}-\underline{r}'|}{c}$$

$$\underline{A}(\underline{r},t) = \frac{\mu_0}{4\pi}\int dV' \,\frac{\underline{J}(\underline{r}',t_r)}{|\underline{r}-\underline{r}'|}$$

For moving p.c. this evaluates to:

$$\phi(\underline{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{R(t_r)}\,\frac{1}{1-\frac{\underline{R}(t_r)\cdot\underline{v}(t_r)}{R(t_r)\,c}}\,, \qquad \underline{R}(t_r) = \underline{r}-\underline{r}_0(t_r),\; t_r = t - \frac{R(t_r)}{c}$$

$$\underline{A}(\underline{r},t) = \frac{\underline{v}(t_r)}{c}\,\phi(\underline{r},t)$$

For $r\to\infty$: $$\underline{E} = \frac{q}{4\pi\varepsilon_0}\,\frac{\underline{n}\times((\underline{n}-\underline{\beta})\times\underline{\beta}')}{cR(1-\underline{n}\cdot\underline{\beta})^3} + O\left(\frac{1}{r^2}\right), \quad \underline{n} = \frac{\underline{R}}{R}, \quad \underline{\beta} = \frac{\underline{v}}{c}$$

$$\underline{B} = \frac{\underline{n}\times\underline{E}}{c} + O\left(\frac{1}{r^2}\right)$$

E&M waves

In medium source free MEs :

$\underline{\nabla} \cdot \underline{E} = 0 \quad , \quad \underline{\nabla} \cdot \underline{B} = 0$

$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad , \quad \underline{\nabla} \times \frac{\underline{B}}{\mu} - \frac{\partial \varepsilon \underline{E}}{\partial t} = \underline{0}$

$\hookrightarrow \underline{\nabla} \times \underline{B} - \frac{1}{v^2} \frac{\partial \underline{E}}{\partial t} = \underline{0}$

$\Rightarrow \underline{\nabla} \times (\text{Faraday}) = \underline{\nabla} (\underbrace{\underline{\nabla} \cdot \underline{E}}_{0}) - \nabla^2 \underline{E} + \frac{\partial \underline{\nabla} \times \underline{B}}{\partial t} = \left[-\nabla^2 + \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] \underline{E}$

Similarly for $\underline{B}$

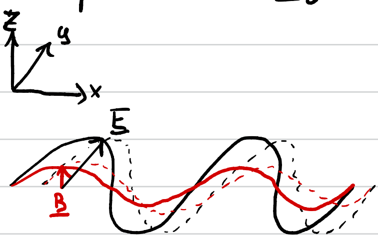
Basic sol : monochromatic plane wave (saw already in Gfn)

$\underline{E}_{\mathbb{C}} = \underline{E}_0 \, e^{i(\underline{k} \cdot \underline{r} - \omega t)}$ $\quad |\underline{k}| = \frac{\omega}{v} \; , \; \underline{E}_0 \cdot \underline{k} = 0 \; , \; \underline{B} = \frac{\underline{e} \times \underline{E}}{v} \qquad \underline{e} = \frac{\underline{k}}{|\underline{k}|}$

$\uparrow$ cplx

Simple case : $\underline{E}_0 = E \, \underline{e}_2$ real , $\underline{k} = k \, \underline{e}_1$, $\underline{E} = E_0 \, \underline{e}_2 \cos(k(x - vt))$

$\underline{B} = \frac{E_0}{v} \underline{e}_3 \cos(k(x - vt))$



check Wiki for animation

Energy

Energy in fields : $\mathcal{E} = \frac{\varepsilon_0}{2}\left(|\underline{E}|^2 + c^2|\underline{B}|^2\right)$

Poynting vec. energy current : $\underline{P} = \frac{1}{\mu_0}\underline{E}\times\underline{B}$

Energy Cons : $\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla}\cdot\underline{P} = \underbrace{-\underline{E}\cdot\underline{J}}_{\text{work density done on E\&M fields}}$

Average energy current in plane wave (PS4/Q7)

$$\underline{E}(\underline{r},t) = \mathrm{Re}(\underline{E}_c(\underline{r},t)) = \underset{\text{Re } \underline{E}_0}{\underline{\alpha}}\cos(\underline{k}\cdot\underline{r} - \omega t) - \underset{\text{Im } \underline{E}_0}{\underline{\beta}}\sin(\underline{k}\cdot\underline{r} - \omega t)$$

$$\underline{B} = \frac{\underline{e}\times\underline{E}}{c}, \quad \underline{P} = \frac{1}{\mu_0}\underline{E}\times\underline{B} = \underline{e}\frac{|\underline{E}|^2}{\mu_0 c} = \frac{\underline{e}}{\mu_0 c}\left[|\underline{\alpha}|^2\cos^2(\ldots) + |\underline{\beta}|^2\sin^2(\ldots) - \underline{\alpha}\cdot\underline{\beta}\sin(2\ldots)\right]$$

$(\underline{e}, \underline{E}, \underline{B})$ right-handed orthogonal

$$\overline{\underline{P}} = \frac{1}{T}\int_0^T dt\, \underline{P} = \frac{\underline{e}}{\mu_0 c}\,\frac{|\underline{\alpha}|^2 + |\underline{\beta}|^2}{2}$$

$T = \frac{2\pi}{\omega}$

$\overline{\sin^2(\,)} = \frac{1}{2}$

Wave reflection problems

- $\underline{E} \propto \underline{e}_z$ in lectures
- $\underline{E}$ in x-z plane exercise (results in the notes)
- PS4/Q7
- 2022/Q3 :

(c) [3 marks] Consider a *plane conducting mirror* at $\{x = 0\} \subset \mathbb{R}^3$, where the electric field $\mathbf{E}$ is zero for $x > 0$. Using Maxwell's equations, argue that the components of $\mathbf{E}$ tangent to the mirror are continuous across the mirror, and hence zero.

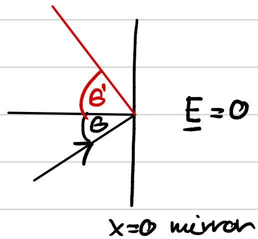
(d) [12 marks] In the set-up of part (c), an incident electromagnetic plane wave of the form given in part (b) arrives from $x < 0$, linearly polarized in the y-axis direction.

(i) Write down the total electric field for $x < 0$, including a *reflected* plane wave of the form given in part (b), where you may assume this has the same angular frequency as the incident wave.

(ii) Show that the directions of the incident wave, reflected wave and normal to the mirror are necessarily coplanar, and that the angle of incidence is equal to the angle of reflection.

(iii) Show that the polarization direction of the reflected wave is the same as that of the incident wave. Hence determine the electric field, where you should express your answer entirely in terms of quantities defined for the incident wave, and you should simplify the expression as much as possible.

[*In this question you may use without proof the identities* $\nabla \wedge (\nabla \wedge \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}$, *and* $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$.]



$$\Rightarrow \underline{E}\cdot\underline{t} = 0$$

$$\underline{E} = E_0\, \underline{e}_2 \cos(\underline{k}\cdot\underline{r} - \omega t) + \underline{C}\cos(\underline{k}'\cdot\underline{r} - \omega t)$$

$$\underline{k}\cdot\underline{e}_2 = 0 \Rightarrow \underline{e} = (\cos\theta, 0, \sin\theta)$$

BCs: $\underline{E}\cdot\underline{e}_3\big|_{x=0} = 0 \Rightarrow \underline{C}\cdot\underline{e}_3 = 0$

$$\underline{E}\cdot\underline{e}_2\big|_{x=0} = 0 \Rightarrow 0 = E_0 \cos\left(\frac{\omega}{c}(\sin\theta\, z - ct)\right)$$

$$+ \underline{C}\cdot\underline{e}_2 \cos\left(\frac{\omega}{c}(e_2' y + e_3' z - ct)\right)$$

$$\Rightarrow e_2' = 0,\ e_3' = \sin\theta,\ e_1' = -\cos\theta,\ \theta' = \theta$$

lin indep

$$\Rightarrow \underline{C} = -E_0\, \underline{e}_2$$

$\underline{C}\cdot\underline{e}' = 0$

$\underline{C}\cdot\underline{e}_3 = 0$

$$\underline{E} = E_0\, \underline{e}_2 \left[\cos\left(\frac{\omega}{c}(\cos\theta\, x + \sin\theta\, z - ct)\right) - \cos\left(\frac{\omega}{c}(-\cos\theta\, x + \sin\theta\, z - ct)\right)\right]$$

$$= 2E_0\, \underline{e}_2 \sin\left(\frac{\omega}{c}\cos\theta\, x\right) \sin\left(\frac{\omega}{c}(\sin\theta\, z - ct)\right)$$

BC manifestly true   1D plane wave

Note that we didn't need the magnetic BCs

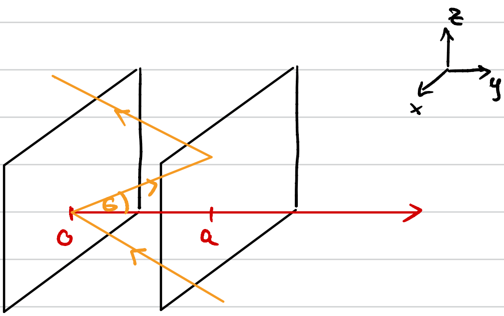
• 2020/Q3

(c) [14 marks] Plane monochromatic electromagnetic waves with frequency ω propagate in the region between two parallel infinite conducting plates. The conductors coincide with the planes $y = 0$ and $y = a$. Choose the direction of propagation vector $\mathbf{k}$ of the waves to lie in the yz-plane making an angle θ with the y-axis.

(i) Derive the boundary conditions that the electric field satisfies on the surfaces $y = 0$ and $y = a$.

(ii) Assuming that the electric field is parallel to the x-axis, find the electric and magnetic fields of the waves in the region between the plates.

(iii) Show that for frequencies ω smaller than a certain value ω_{min}, which you should determine, there are solutions corresponding to fields that are exponentially damped in the z-direction.



Same BC as in previous problem

$$\underline{E} = 2E_0\, \underline{e}_1 \sin\left(\frac{\omega}{c}\cos\theta(y-a)\right) \sin\left(\frac{\omega}{c}(\sin\theta\, z - ct)\right)$$

$x \leftrightarrow y$

$y \to y-a$

$$= -2E_0\, \underline{e}_1 \sin\left(\frac{n\pi}{a}y\right) \sin\left(\frac{\omega}{c}\left(\sqrt{1-\left(\frac{n\pi c}{\omega a}\right)^2}\, z - ct\right)\right)$$

can become imaginary

BC @ $y=0$

$\sin\left(\frac{\omega}{c}\cos\theta\, a\right) = 0$

$\frac{\omega}{c}\cos\theta\, a = n\pi$

$\cos\theta = \frac{n\pi c}{\omega a}$ can formally exceed 1

ω_{min} is when $\frac{\pi c}{\omega_{min} a} = 1 \Rightarrow \omega_{min} = \frac{\pi c}{a}$ for $\omega > \omega_{min}$ we have prop modes (more allowed n's for larger ω)

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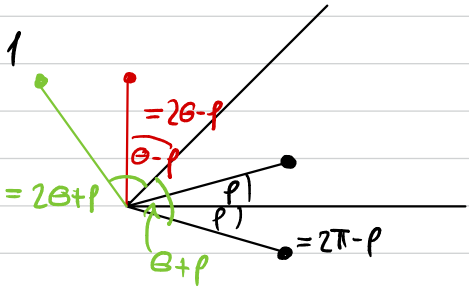


0^{th} $\quad$ 2^{nd}

$\rho,\ 2\theta + \rho,\ \dots$

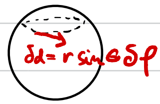
$2\pi - \rho,\ \dots$

$$\Rightarrow \rho_n = \begin{cases} n\theta + \rho & n \text{ even} \\ (n+1)\theta - \rho & n \text{ odd} \end{cases}$$

Required formulas

No need to memorise div, curl, Laplacian, but look at them to see if they make sense. Grad is easy:

$\underline{\nabla}$ is needed, e.g. $\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \underline{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$



$$\underline{\nabla} \, |\underline{r} - \underline{r}'| = \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|}, \quad \underline{\nabla} \times (\underline{\nabla} \psi) = 0, \quad \underline{\nabla} \cdot (\underline{\nabla} \times \underline{f}) = 0$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a}\,\underline{c}) - \underline{c}(\underline{a}\,\underline{b})$$

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b}) = -\underline{b}(\underline{a} \times \underline{c})$$

$$\underline{\nabla} \times (\underline{a} \times \underline{b}) = \underline{a}(\underline{\nabla} \cdot \underline{b}) + (\underline{b} \cdot \underline{\nabla})\underline{a} - \underline{b}(\underline{\nabla} \cdot \underline{a}) - (\underline{a} \cdot \underline{\nabla})\underline{b}$$

$$\underline{\nabla} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{\nabla} \times \underline{a}) - \underline{a} \cdot (\underline{\nabla} \times \underline{b})$$