

Problem sheet 1

1. Given n positive numbers $x_1, x_2, \dots, x_n$ such that $x_1 + x_2 + \dots + x_n \leq \frac{1}{3}$, prove by induction that

$$(1 - x_1)(1 - x_2) \times \dots \times (1 - x_n) \geq \frac{2}{3}.$$

[Hint: for the inductive step, consider $x_1, x_2, \dots, x_{n-1}$, and $(x_n + x_{n+1})$.]

2. Using the recursive definition of addition given in lectures, follow the steps below to show that addition on $\mathbb{N}$ is commutative; that is, $x + y = y + x$ for all $x, y \in \mathbb{N}$.

- (i) First prove the result for $y = 0$ and all x by induction on x .
- (ii) Next prove the result for $y = 1$ and all x .
- (iii) Prove the general result by induction on y .

3. (i) Let A, B , and C be subsets of a set S . Write out a proof that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

- (ii) Let D, E , and F be subsets of a set S . Use (i) and De Morgan's laws to show that

$$D \cup (E \cap F) = (D \cup E) \cap (D \cup F).$$

4. Let A and B be finite sets.

- (i) Assuming for this part that A and B are disjoint, and adopting the recursive definition of cardinality given in lectures, use induction on $|B|$ to show that $A \cup B$ is finite and that

$$|A \cup B| = |A| + |B|.$$

- (ii) Show that for general A and B , $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$. Deduce that

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

5. Which of the following relations on $\mathbb{N}$ are reflexive, which are symmetric, which are transitive?

- (i) the relation $a|b$ (read as ' a divides b ').
- (ii) the relation $a \nmid b$ (does not divide).
- (iii) a, b are related if a, b leave the same remainder after division by 1729.
- (iv) a, b are related if $\text{hcf}(a, b) > 1729$.
[$\text{hcf}(a, b)$ denotes the highest common factor, or greatest common divisor, of a and b . It is the largest natural number that divides both a and b .]

6. How many partitions are there of a set of size 1?

How many partitions are there of a set of size n for $n = 2, 3, 4, 5$?

7. Let $S = \{(m, n) : m, n \in \mathbb{Z}, n \geq 1\}$.

- (i) Show that $\sim$ defined by

$$(m_1, n_1) \sim (m_2, n_2) \quad \text{if and only if} \quad m_1 n_2 = m_2 n_1$$

is an equivalence relation on S .

- (ii) The set of equivalence classes is denoted $S/\sim$. Show that $\oplus$ and $\otimes$ defined by

$$\overline{(m_1, n_1)} \oplus \overline{(m_2, n_2)} = \overline{(m_1 n_2 + m_2 n_1, n_1 n_2)}$$

and

$$\overline{(m_1, n_1)} \otimes \overline{(m_2, n_2)} = \overline{(m_1 m_2, n_1 n_2)}$$

are well-defined binary operations on $S/\sim$.

What set is $S/\sim$ a model for?