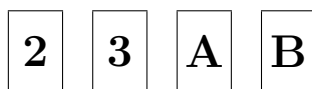


Problem sheet 2

1. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = \sin x$ for all real numbers x .
 - (i) What are $f([0, \pi])$, $f([0, 2\pi])$, $f([0, 3\pi])$?
 - (ii) What are $f^{-1}(\{0\})$, $f^{-1}(\{1\})$, $f^{-1}(\{2\})$?
 - (iii) Let $A = [0, \pi]$ and $B = [2\pi, 3\pi]$. Show that $f(A \cap B) \neq f(A) \cap f(B)$.
 - (iv) Let $A = [0, \pi]$. Find $f^{-1}(f(A))$ and $f(f^{-1}(A))$.
2. In each of the following cases, either find a function $f: \mathbb{R} \rightarrow \mathbb{R}$ with the required properties, or show that no such function exists.
 - (i) $f((0, 1)) = \mathbb{R}$ and $f^{-1}(\mathbb{R}) = (0, 1)$.
 - (ii) $f((0, 1)) = (0, 2)$ and $f((0, 2)) = (0, 1)$.
 - (iii) $f((0, 1)) = (0, 2)$ and $f^{-1}((0, 2)) = (0, 1)$.
 - (iv) $f((0, 1)) = (0, 2)$ and $f^{-1}((0, 2)) = (0, 2)$.
 - (v) $f((0, 1)) = (0, 1)$ and $f([0, 1]) = (0, 2)$.
3. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.
 - (i) How many functions $f: A \rightarrow B$ are there?
 - (ii) How many functions $f: B \rightarrow A$ are there?
 - (iii) How many injective functions $f: A \rightarrow B$ are there?
 - (iv) How many injective functions $f: B \rightarrow A$ are there?
 - (v) How many surjective functions $f: A \rightarrow B$ are there?
 - (vi) How many surjective functions $f: B \rightarrow A$ are there?
4. Which of the following statements about natural numbers are true, which false?
 - (i) 2 is prime or 2 is odd.
 - (ii) 2 is prime or 2 is even.
 - (iii) If 2 is odd then 2 is prime.
 - (iv) If 2 is even then 2 is prime.
 - (v) For all $n \in \mathbb{N}$, if n is a square number then n is not prime.
 - (vi) For all $n \in \mathbb{N}$, n is not prime if and only if n is a square number.
 - (vii) For all even primes $p > 2$, $p^2 = 2026$.
 - (viii) "If 2 is even then 2 is prime, and if 2 is prime then 2 is odd" implies that "if 2 is even, then 2 is odd".
5. Organise the following assertions about a function $f: \mathbb{R} \rightarrow \mathbb{R}$ into pairs such that one member of the pair is true if and only if the other is false. Re-write each statement using symbols ($\in$, $\forall$, $\exists$, etc.)
 - (1) for all real numbers x, y there exists a real number $z > x$ such that $f(z) > y$;
 - (2) for every real number x there exists a real number y such that for all real z either $z \leq y$ or $f(z) \neq x$;
 - (3) there exist real numbers x, y such that for every real number z if $z \leq y$ then $f(z) = x$;
 - (4) there is a real number x such that for every real number y there is a real number $z > y$ for which $f(z) = x$;

- (5) for all real x and y there is a real number z such that $z \leq y$ and $f(z) \neq x$;
 (6) there exist real numbers x and y such that for every real number z either $z \leq x$ or $f(z) \leq y$.

6. Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:



You are permitted to turn just two cards over in order to test the following hypothesis: *every card that has an even number on one side has a vowel on the other*. Which two cards should you turn? Or is it impossible?

7. Let X , Y and Z be sets, and let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be functions. Which of the following statements are true, which false? In each case either carefully prove the statement or give a specific counter-example.
- (i) if f and g are surjective then $g \circ f$ is surjective.
 - (ii) if g is surjective then $g \circ f$ is surjective.
 - (iii) if $g \circ f$ is surjective then f is surjective.
 - (iv) if $g \circ f$ is surjective then g is surjective.
8. Let $n \geq 2$ be an integer, let S be the set of $n \times n$ matrices with real coefficients, and define $f: S \rightarrow \mathbb{R}$ by

$$f(A) = \sum_{i=1}^n a_{ii} \quad \text{for} \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

Prove or disprove each of the following statements:

- (i) $f(A + B) = f(A) + f(B)$ for all $A, B \in S$;
 - (ii) $f(AB) = f(A)f(B)$ for all $A, B \in S$;
 - (iii) $f(AB) = f(BA)$ for all $A, B \in S$;
 - (iv) $f(A) = 0$ if and only if A is not invertible.
9. Let $f: \mathbb{R} \rightarrow \mathbb{R}$. Translate the following statement into English:

$$\forall \varepsilon > 0 \quad \forall a \in \mathbb{R} \quad \exists \delta > 0 \quad \forall x \in \mathbb{R} \quad |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Express the following statement using the symbols $\forall$, $\exists$ and $\Rightarrow$:

for every positive real number ε there exists a positive real number δ such that
 for all real numbers a and x , $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

Do these statements say the same thing? Can you prove whether or not they hold for the case $f(x) = x^2$?

10. Suppose A , B , C , D are finite sets with $|A| \leq |C|$ and $|B| \leq |D|$. Let X be the set of functions from A to B , and let Y be the set of functions from C to D . Is $|X| \leq |Y|$? Prove it, or find a counter-example.