B1.1 Logic

Sheet 1 — MT26

Section A

1. Prove carefully that for any formula[1] ϕ, the number of left parentheses occuring in ϕ is equal to the number of right parentheses occuring in ϕ.

2. Show that for any formulas $\phi, \psi \in \mathrm{Form}(\mathcal{L}_{\mathrm{prop}})$, the following formula is logically valid:

$$\alpha := ((\phi \vee \psi) \leftrightarrow ((\phi \to \psi) \to \psi)).$$

[1] On this problem sheet, "formula" always means: formula of $\mathcal{L}_{\mathrm{prop}}$.

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Section B

3. Which of the following strings are formulas of $\mathcal{L}_{\text{prop}}$? Give reasons.

   (i) $\neg(p_3 \to p_1)$

   (ii) $p_1 \to p_2 \to p_3$

   (iii) $((\neg p_5 \wedge \neg p_6) = \neg p_{11})$

   (iv) $(p \leftrightarrow \neg p)$

   (v) $((p_1 \vee \neg p_1) \to (\neg p_2))$

4. (a) Show that no proper initial segment[2] of a formula is a formula.

   (b) Prove the Unique Readability Theorem.

5. By drawing up truth tables, show that the following formulas are logically valid, where $\phi, \psi, \chi \in \text{Form}(\mathcal{L}_{\text{prop}})$ are arbitrary formulas.

   (a) $\alpha := ((p_0 \vee (p_1 \wedge p_2)) \leftrightarrow ((p_0 \vee p_1) \wedge (p_0 \vee p_2)))$.

   (b) $\beta := ((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$.

6. Let ϕ be a formula, and let $\psi_0, \ldots, \psi_n$ be formulas. Let ϕ' be the result of replacing every instance of p_i in ϕ with ψ_i, for $i = 0, \ldots, n$.

   (a) Show that if ϕ is logically valid, then so is ϕ'.

   (b) Show that the converse fails, by giving an example where ϕ' is logically valid but ϕ is not.

   (c) Show that the converse does hold in the case of a formula in one variable substituted into itself. Namely, suppose p_0 is the only propositional variable occurring in ϕ, let $\psi_0 = \phi$, and suppose that ϕ' is logically valid. Show that ϕ is logically valid.

7. Prove that the following logical equivalences hold for any $n \geq 1$ and any formulas $\psi, \phi_1, \ldots, \phi_n \in \text{Form}(\mathcal{L}_{\text{prop}})$.

   (a) $\neg \bigwedge_{i=1}^{n} \phi_i \vDash\dashv \bigvee_{i=1}^{n} \neg\phi_i$.

   (b) $(\psi \vee \bigwedge_{i=1}^{n} \phi_i) \vDash\dashv \bigwedge_{i=1}^{n} (\psi \vee \phi_i)$.

   (c) $(\psi \wedge \bigvee_{i=1}^{n} \phi_i) \vDash\dashv \bigvee_{i=1}^{n} (\psi \wedge \phi_i)$.

[2] A *proper initial segment* of a string s is the string consisting of the first k characters of s for some $k < \text{len}(s)$.

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Section C

8. In this question, we consider an alternative notation for propositional formulas, known as "Polish notation", which achieves unique readability without parentheses. (We will not use this notation in the course.)

   We recursively define a Polish formula as follows:

   - A propositional variable is a Polish formula.
   - If ϕ is a Polish formula then so is $\neg\phi$.
   - If ϕ and ψ are Polish formulas then so is $\star\phi\psi$ for any $\star \in \{\rightarrow, \wedge, \vee, \leftrightarrow\}$.
   - Nothing else is a Polish formula.

   Prove the unique readability theorem for Polish formulas.

9. This question is set in Smullyan City. Each denizen of Smullyan City is either a Knight or a Knave. Every statement made by a Knight is true, while every statement made by a Knave is false. They are all well-trained in formal logic.

   (a) Lost on the way to the train station, you encounter a pair of denizens of the city.

   The first exclaims: "The station lies North, or I'm a knight!".

   The second then also exclaims: "The station lies North, or I'm a knight!". Gesturing to the first denizen, the second then adds in a whisper: "My colleague here, however, is certainly a knave."

   Express in formulas of $\mathcal{L}_{\text{prop}}$ the information this gives you, and determine whether you should indeed head North.

   (b) You pass a tree heavy with red berries which look delicious but possibly poisonous. You ask a nearby denizen about them, but it quickly becomes clear that although they understand English perfectly and know the berries well, they are willing to speak only in Smullyan. Having totally failed to prepare for the trip, you learnt only two words of the language: "bal" and "da". One of them means "yes" and the other "no" – but you forget which is which! The denizen (who remember is either a knight or a knave) looks impatient, but would probably answer a single question before leaving. Is there a question you could ask to determine whether the berries are safe to eat?

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