Problem Sheet 1

1. Section A. By taking the scalar product with an arbitrary constant vector $\mathbf{k}$, use Stokes' Theorem to show that

$$\left[\int_C \mathbf{f} \wedge \mathbf{t}\, \mathrm{d}s\right]_i = \int_S [n_i(\nabla \cdot \mathbf{f}) - n_j \partial_i f_j]\, \mathrm{d}S,$$

for any vector function $\mathbf{f}$, where $\mathbf{t}$ is the unit tangent to a curve C (which bounds the surface S), the vector $\mathbf{n}$ is the unit normal to S and we use the summation convention.

Let $\boldsymbol{\nu} = \mathbf{t} \wedge \mathbf{n}$ and choose $\mathbf{f} = \gamma \mathbf{n}$, defined off the surface by the extension that is independent of position in the normal direction $\mathbf{n}$. Show that

$$\int_C \gamma \boldsymbol{\nu}\, \mathrm{d}s = \int_S [\nabla\gamma - \gamma(\nabla \cdot \mathbf{n})\mathbf{n}]\, \mathrm{d}S.$$

[You may need the standard identities

$$(\mathbf{f} \wedge \mathbf{k}) \cdot \mathbf{t} = -\mathbf{k} \cdot (\mathbf{f} \wedge \mathbf{t}),$$

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

]

2. Section B. Recall the Laplace–Young equation governing the shape $z = h(x)$ of a meniscus near a vertical, planar wall

$$\ell_c^2 \frac{h_{xx}}{(1 + h_x^2)^{3/2}} = h.$$

Integrating the full Laplace–Young equation once (without making an assumption of shallow slopes), show that the rise height of the meniscus on the wall, h_0, is given in terms of the contact angle of the liquid θ by

$$h_0 = \pm \ell_c \left[2(1 - \sin\theta)\right]^{1/2}$$

and discuss when each of the $\pm$ branches of the result are appropriate.

Show that the total area displaced by the meniscus is given by

$$A = \int_0^\infty h\, \mathrm{d}x = \ell_c^2 \cos\theta.$$

Determine the weight of liquid displaced (per unit depth length in the y direction). How is this related to the vertical force exerted by surface tension at the contact line?

3. Section B. With r, ϕ, z denoting cylindrical polar coordinates, fluid occupies the domain

$$\Big\{(r,\phi,z) \;\Big|\; 0<r<\infty, \quad \phi\in[-\alpha,\alpha], \quad z\in[0,h(r,\phi)]\Big\}$$

where $h(r,\phi)>0$ denotes the height of the free surface above the (x,y) plane.

The boundary at $z=0$ is rigid as are the boundaries at $\phi=\pm\alpha$, while the boundary at $z=h(r,\phi)$ is a static free surface, with contact lines at $z=h(r,\pm\alpha)$.

Sketch the domain and show that the linearised Laplace–Young equation enforces the equation

$$h=\ell_c^2\,\nabla^2 h,$$

for the region

$$\Big\{(r,\phi) \;\Big|\; 0<r<\infty, \quad \phi\in[-\alpha,\alpha]\Big\},$$

where the constant ℓ_c is to be determined.

Show that the boundary conditions are

$$\frac{1}{r}h_\phi(\alpha)=\cot\theta, \qquad \frac{1}{r}h_\phi(-\alpha)=-\cot\theta,$$

where θ is the contact angle between the air-fluid interface and the rigid boundary at $\phi=\pm\alpha$.

Explain why one must assume that $|\pi/2-\theta|\ll 1$ for self consistency.

When $\alpha=\pi/4$ show that the unique solution is

$$h=\ell_c\cot\theta\left[e^{-r\sin(\pi/4-\phi)/\ell_c}+e^{-r\sin(\phi+\pi/4)/\ell_c}\right].$$

4. Section B.

Consider a spherical bubble of constant radius a rising in fluid of viscosity μ, density ρ, at speed U in the laboratory reference frame under the influence of gravity, with $\boldsymbol{g} = -g\boldsymbol{e}_z$. The bubble is sufficiently small that the Reynolds number is negligible and its gas is of negligible viscosity and density.

With z the coordinate in the bubble-fixed frame with origin at the centre of the bubble, the surface tension at the bubble-water interface is given by

$$\gamma = \gamma_0 + \gamma' z = \gamma_0 + \gamma' a \cos\theta,$$

where θ is the polar angle (so that $z = a$ on the bubble corresponds to $\theta = 0$ and $z = -a$ corresponds to $\theta = \pi$).

The resulting fluid flow, relative to the bubble-fixed frame, is given via the streamfunction

$$\psi = \left[-\frac{U}{2}r^2 + Cr - \frac{\gamma' a^4}{6\mu}\frac{1}{r}\right]\sin^2\theta, \qquad C = \frac{1}{2}Ua + \frac{\gamma' a^2}{6\mu},$$

where r is the distance from the bubble centre, as derived in the lecture notes.

You are given that the force exerted on the bubble by the fluid is given by

$$F_i = \int_B T_{ij} n_j \mathrm{d}S,$$

where B is the bubble surface, $\boldsymbol{T}$ is the Cauchy stress tensor and $\boldsymbol{n}$ is the outward normal for the bubble.

(i) Without calculation, explain why the force on the bubble only has a non-trivial component in the $\boldsymbol{e}_z$ direction.

(ii) Determine the force on the bubble due to buoyancy. In the remaining parts, consider only forces additional to buoyancy, using the reduced pressure $q = p + \rho g z$.

(iii) Show the flow due to the term scaling with r^2 in the stream function is given by $\boldsymbol{u} = -U\boldsymbol{e}_z$. Show that this contribution to the streamfunction induces the buoyancy force.

(iv) Show that the term scaling with $1/r$ in the streamfunction does not generate a force (*Hint: explicit calculation is not expected.*)

(v) Determine the force on the bubble due to the term scaling with r in the streamfunction. Hence, find value of γ' for which the bubble is stationary, that is $U = 0$.

[Hint. You are given that the flow associated with a sphere of radius a moving at speed $V\boldsymbol{e}_z$ has streamfunction

$$\frac{3}{4}Va\left(r + \frac{\alpha}{r}\right)\sin^2\theta,$$

where α is constant, with the flow exerting a force $-6\pi\mu V a\boldsymbol{e}_z$ on the sphere.]

5. Section C. Consider a static blob of fluid lying underneath a horizontal plate, $z = 0$, and occupying the region

$$\Big\{(x,z)\Big| -x_0 \le x \le x_0, \quad h(x) \le z \le 0\Big\}.$$

The liquid has a contact angle $\theta \ll 1$. Assuming that $|h_x| \ll 1$, show that h satisfies

$$\ell_c^2 h_{xxx} + h_x = 0.$$

where ℓ_c is a constant, to be determined. Show that

$$h(x) = \theta\ell_c\Big[\cot(x_0/\ell_c) - \frac{\cos(x/\ell_c)}{\sin(x_0/\ell_c)}\Big],$$

and confirm that the assumption $|h_x| \ll 1$ is justified under certain circumstances, which you should state. Show that the cross sectional area of the drop, A, is given by

$$\frac{A}{2} \;=\; \theta\ell_c^2\left(1 - \frac{x_0}{\ell_c}\cot(x_0/\ell_c)\right). \tag{1}$$

Is there a maximum cross sectional area of fluid which can hang beneath the plate in this way? Does this accord with your physical intuition? If not, why not?

Comments and corrections to Eamonn Gaffney