Topological groups, 2022–2023

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Course overview

Groups like the integers, the circle, and general linear groups (over $\mathbb{R}$ or $\mathbb{C}$) share a number of properties naturally captured by the notion of a topological group. Providing a unified framework for these groups and properties was an important achievement of 20th century mathematics, and in this course we shall develop this framework.

Highlights will include the existence and uniqueness of Haar integrals for locally compact topological groups, the topology of dual groups, and the existence of characters in various topological groups. Throughout, the course will use the tools of analysis to tie together the topology and algebra, getting at superficially more algebraic facts by analytic means.

Course synopsis

[7 lectures] Definition of topological groups. Examples and non-examples, and basic topological properties. Subgroups. Quotient groups. The Open Mapping Theorem.
[4 lectures] Complete regularity of topological groups. Continuous partitions of unity and Fubini's Theorem. Existence and uniqueness of Haar integrals.
[5 lectures] The Peter-Weyl Theorem for compact topological groups. Dual groups of topological groups. Local compactness of the dual of a locally compact topological group.

References

There are other notes on similar topics with a slightly different focus *e.g.* [Fol95, Kör08, Kra17, Meg17] and [Rud90].

General prerequisites

The course is designed to be pretty self-contained. We assume basic familiarity with groups as covered in Prelims Groups and Group Actions (see *e.g.* [Ear14]). We shall also assume familiarity with Prelims Linear Algebra (see *e.g.* [May20]) and Part A: Metric Spaces and Complex Analysis (see *e.g.* [McG19]) for material on metric and normed spaces.

Familiarity with topology is essential, though not much is required content-wise. What we use (and more) is covered in Part A: Topology (see *e.g.* [DL18]), with the exception of Tychonoff's Theorem. This can be informally summarised as saying that a non-empty product of compact spaces is compact, and there is no harm in taking it as a black box for the course. Those interested in more detail may wish to consult Part C: Analytic Topology (see *e.g.* [Kni18]).

The Axiom of Choice is sometimes formulated as saying that an arbitrary product of non-empty sets is non-empty, and in this formulation it may be less surprising that it can be used to prove Tychonoff's Theorem. It turns out that the converse is also true, *i.e.* Tychonoff's Theorem (and the other axioms of set theory) can be used to prove the Axiom of Choice[1].

Finally no familiarity with functional analysis is assumed, though there are clear similarities and parallels for those who do have some. See *e.g.* [Pri17] and [Whi19].

Teaching

A first draft of these notes is on the website, but they will be updated after each lecture with any resulting changes. This document was compiled on 25th May, 2023 at 09:44.

Lectures will be supplemented by some tutorial-style teaching where we can discuss the course and also exercises from the sheets. Once I have a list of the MFoCS students attending I shall be in touch to arrange these.

Contact details and feedback

Contact tom.sanders@maths.ox.ac.uk if you have any questions or feedback.

[1]Those unfamiliar and looking for a reference may wish to consult the notes [Ter10].

Group notation

A group G is **written multiplicatively** if the binary operation of the group is written $G^2 \to G; (x,y) \mapsto xy$ and called **multiplication**; the unique inverse is written x^{-1} and the map $G \to G; x \mapsto x^{-1}$ is called **inversion**; and the identity is written 1_G. Given $S, T \subset G$ we write

$$S^{-1} := \{s^{-1} : s \in S\} \text{ and } ST := \{st : s \in S, t \in T\}.$$

For $n \in \mathbb{N}_0$ we define S^n inductively by

$$S^0 := \{1_G\} \text{ and } S^{n+1} := S^n S; \text{ and } S^{-n} := (S^{-1})^n.$$

It will also be convenient to write $xS := \{x\}S$ and $Sx := S\{x\}$ for $x \in G$, which aligns the the usual notation for left and right cosets when S is a subgroup. ⚠This notation has effect that in general $SS^{-1} \neq S^0$ and $S^2 \neq \{s^2 : s \in S\}$.

⚠As an exception to the above notation, G^n denotes the n-fold Cartesian product $G \times \cdots \times G$ *not* the product defined above; that product is just G.

We write $\langle S \rangle$ for the group generated by S, that is $\bigcap\{H \leqslant G : S \subset H\}$, the intersection of all the subgroups of G containing S.

We say $S \subset G$ is **symmetric** if $S = S^{-1}$. If S and T are symmetric then $S \cap T$ is symmetric, and if S is symmetric then $\langle S \rangle = \bigcup_{n \in \mathbb{N}_0} S^n$ by the Subgroup Test.

We write G^{op} for the **opposite group**, that is the group with the same base set as G but group operation given by $G^2 \to G; (x,y) \mapsto yx$. The identity element and the inverse map on G^{op} are the same as those on G, and the map $G \to G^{\mathrm{op}}; x \mapsto x^{-1}$ is a group isomorphism.

If G is Abelian then it is **written additively** if the binary operation of the group is written $G^2 \to G; (x,y) \mapsto x + y$ and called **addition**; inversion is written $G \to G; x \mapsto -x$ and called **negation**; and the identity is written 0_G. ⚠All groups written additively are Abelian, but not all Abelian groups will be written additively.

If G is written additively then the above notation changes in the obvious way so we write $-S$ instead of S^{-1}, $S + T$ instead of ST, nS instead of S^n *etc.*

1 Groups with topologies

A group G that is also a topological space is called a **topologized group**. Without any additional assumptions these are no more than their constituent parts: a group and a topological space. When the group inversion $G \to G$ and the group operation $G^2 \to G$ are both continuous, where G^2 has the product topology, we say G is a **topological group**.

Example 1.1 (Indiscrete groups)**.** For any group G, we write G_I for G endowed with the indiscrete topology. This is a topological group since any map into an indiscrete space is continuous.

Any indiscrete space is compact since the indiscrete topology is finite, so G_I is a compact topological group. G_I is Hausdorff if and only if G is the trivial group.

There are non-compact spaces that retain traces of compactness which we shall find it useful to discuss: A topological space is **locally compact** if every element is contained in a compact neighbourhood; and it is **σ-compact** if it is a countable union of compact sets. ⚠ In the literature sometimes different definitions of local compactness are used – see Remark 1.44 for an example that is relevant to us – but they usually coincide when the space is additionally assumed to be Hausdorff.

Example 1.2 (Discrete groups)**.** For any group G, we write G_D for G endowed with the discrete topology. This is a topological group since the product of two copies of the discrete topology is discrete – so both the topological spaces G and G^2 are discrete – and any map from a discrete space is continuous.

Any discrete space is Hausdorff and locally compact since $\{x\}$ is an open neighbourhood of x which is compact, since it is finite, and disjoint from the open neighbourhood $\{y\}$ if $x \neq y$. Hence G_D is a locally compact Hausdorff topological group. Since the set of singletons in G_D is an open cover of G_D, G_D is compact if and only if it is finite; and it is σ-compact if and only if it is countable.

The reals under addition may be endowed with the discrete or indiscrete topologies to make them into a topological group as above. However, neither of these is the 'usual' topology which is generated by intervals without their endpoints.

Example 1.3 (The real line)**.** The additive group $\mathbb{R}$ endowed with its usual topology is a topological group which we call the **real line**, and which we also denote $\mathbb{R}$. The relevant continuity is just the algebra of limits: in particular, if $x_n \to x_0$ then $-(x_n) = (-1)x_n \to (-1)x_0 = -x_0$; and if additionally $y_n \to y_0$, then $x_n + y_n \to x_0 + y_0$.

The compact sets in the real line $\mathbb{R}$ are exactly the closed and bounded sets (this is the Heine-Borel Theorem for $\mathbb{R}$). We can use this to see that the real line is a non-compact σ-compact locally compact Hausdorff topological group. Local compactness follows since

$[x-1, x+1]$ is a compact neighbourhood of x, and σ-compactness follows since $\mathbb{R} = \bigcup_{n\in\mathbb{N}} [-n, n]$. $\mathbb{R}$ is non-compact since $\{(-x, x) : x \in \mathbb{R}\}$ is an open cover without a finite subcover, and $\mathbb{R}$ is Hausdorff since if $x \neq y$ then putting $\delta := |x-y|/2$ the sets $(x-\delta, x+\delta)$ and $(y-\delta, y+\delta)$ are disjoint open neighbourhoods of x and y respectively.

Example 1.4 (The rationals)**.** The additive group $\mathbb{Q}$ endowed with subspace topology inherited from the real line is a topological group for the same reasons as in Example 1.3.

$\mathbb{Q}$ is Hausdorff, but *not* locally compact (and so certainly not compact) – this is exactly why one constructs the real line! – but *is* σ-compact since $\mathbb{Q}$ is a countable union of singletons each of which is compact since it is finite.

Example 1.5 (Non-zero complex numbers)**.** The non-zero complex numbers, $\mathbb{C}^*$, form a multiplicative group and with the usual topology is a topological group: By the algebra of limits, if $x_n \to x_0$ in $\mathbb{C}^*$ then $x_n^{-1} \to x_0^{-1}$; and if additionally $y_n \to y_0$ then $x_n y_n \to x_0 y_0$.

The compact sets in $\mathbb{C}$ are exactly the closed and bounded sets (this is the Heine-Borel Theorem again, this time for $\mathbb{R}^2$). We can used this as in Example 1.3 to see that $\mathbb{C}^*$ is a topological group that is non-compact, σ-compact, locally compact, and Hausdorff.

Example 1.6 (The positive reals)**.** The set $\mathbb{R}_{>0}$ of positive reals under multiplication with the subspace topology inherited from the usual topology on $\mathbb{C}$ is a topological group for the same reasons as in Example 1.5; it is non-compact, σ-compact, locally compact, and Hausdorff.

Example 1.7 (The circle group)**.** The set $S^1 := \{z \in \mathbb{C}^* : |z| = 1\}$ under multiplication with the subspace topology inherited from the usual topology on $\mathbb{C}$ is a topological group for the same reasons as in Example 1.5, and we shall call it the **circle group**; it is compact and Hausdorff.

We denote the topological groups of Examples 1.3–1.7 without subscripts and SMALL CAPS, and in general only include disambiguating subscripts when the topology may not be clear.

Group actions

Groups often arise with actions, and topological groups are no exception to this. For a left action of a group G on a topological space X, we say it is an **action by continuous functions** if the maps $X \to X; x \mapsto g.x$ are continuous for all $g \in G$. ⚠ In the literature the term 'continuous group action' is reserved for an action of a *topological* group G on a topological space X that is continuous as a map $G \times X \to X$.

Observation 1.8. The maps $X \to X; x \mapsto g.x$ are continuous for all $g \in G$ if and only if they are homeomorphisms since $g^{-1}.(g.x) = x = g.(g^{-1}.x)$ for all $x \in X$ and $g \in G$.

Example 1.9 (Homeomorphisms of topological spaces)**.** For a topological space X and group G of homeomorphisms of X under composition, the map $G \times X \to X; (g, x) \mapsto g(x)$ is an action by continuous functions, and is called the **evaluation action**.

The case of X a metric space, and G a group of isometries is a special case of this. ⚠While isometries are always injective, they need not be surjective. To form a group, however, they must be invertible and so in particular surjective.

Some particular groups of homeomorphisms are studied in the Prelims Analysis II course:

Example 1.10. For $X = [0, 1]$ with the subspace topology from $\mathbb{R}$, let G be the group of all homeomorphisms of X fixing the endpoints. Then – [Qia20, Theorem 1.3.31] – the functions in G are exactly the strictly increasing bijections of X.

Given a left action of a group G on a topological space X, the **topology of pointwise converge on G w.r.t. this action** is the weakest topology on G such that the maps $G \to X; g \mapsto g.x$ are continuous for all $x \in X$. In particular, given a base $\mathcal{B}$ for X, the sets

$$U(x_1, \ldots, x_n; U_1, \ldots, U_n) := \{g \in G : g.x_1 \in U_1, \ldots, g.x_n \in U_n\}$$

with $x_1, \ldots, x_n \in X$ and $U_1, \ldots, U_n \in \mathcal{B}$ form a base for the topology of pointwise convergence w.r.t. the given action.

Proposition 1.11. *Suppose that X is a metric space and G is a group of isometries of X. Then G with the topology of pointwise convergence w.r.t. the evaluation action is a topological group.*

Proof. Write d for the metric on X and $B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\}$ so that the balls $\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ form a base for the topology on X. If $f_0 \in U(x_1, \ldots, x_n; U_1, \ldots, U_n)$ then there is $\epsilon > 0$ such that

$$U(x_1, \ldots, x_n; B_\epsilon(f_0(x_1)), \ldots, B_\epsilon(f_0(x_n))) \subset U(x_1, \ldots, x_n; U_1, \ldots, U_n).$$

For $f \in G$, since f is an isometry, we have

$$d(f^{-1}(f_0(x_i)), f_0^{-1}(f_0(x_i))) = d(f_0(x_i), f(x_i)) = d(f(x_i), f_0(x_i))$$

and so the preimage under inversion of $U(x_1, \ldots, x_n; U_1, \ldots, U_n)$ contains the preimage of $U(x_1, \ldots, x_n; B_\epsilon(f_0(x_1)), \ldots, B_\epsilon(f_0(x_n)))$, which is $U(f_0(x_1), \ldots, f_0(x_n); B_\epsilon(x_1), \ldots, B_\epsilon(x_n))$ which is a neighbourhood of f_0^{-1}. Hence inversion is continuous.

Now suppose $f_0 \circ g_0 \in U(x_1, \ldots, x_n; U_1, \ldots, U_n)$, and $f \in U(g_0(x_1), \ldots, g_0(x_n); B_{\epsilon/2}(f_0 \circ g_0(x_1)), \ldots, B_{\epsilon/2}(f_0 \circ g_0(x_1)))$ and $g \in U(x_1, \ldots, x_n; B_{\epsilon/2}(g_0(x_1))), \ldots, B_{\epsilon/2}(g_0(x_n)))$ then again since f is an isometry

$$\begin{aligned} d(f \circ g(x_i), f_0 \circ g_0(x_i)) &\leqslant d(f \circ g(x_i), f \circ g_0(x_i)) + d(f \circ g_0(x_i), f_0 \circ g_0(x_i)) \\ &= d(g(x_i), g_0(x_i)) + d(f(g_0(x_i)), f_0(g_0(x_i))) < \epsilon, \end{aligned}$$

and so $f \circ g \in U(x_1, \ldots, x_n; U_1, \ldots, U_n)$. In particular the preimage of $U(x_1, \ldots, x_n; U_1, \ldots, U_n)$ under the group operation contains the open neighbourhood

$$U(g_0(x_1), \ldots, g_0(x_n); B_{\epsilon/2}(f_0 \circ g_0(x_1)), \ldots, B_{\epsilon/2}(f_0 \circ g_0(x_1))) \\ \times U(x_1, \ldots, x_n; B_{\epsilon/2}(g_0(x_1))), \ldots, B_{\epsilon/2}(g_0(x_n)))$$

of (f_0, g_0). Hence multiplication is continuous as a map $G^2 \to G$ and G is a topological group. □

Example 1.12 (Groups of unitary maps with the SOT)**.** For V an inner product space a **unitary map** is a map $\phi : V \to V$ such that $\langle \phi(v), \phi(w) \rangle = \langle v, w \rangle$ for all $v, w \in V$, and we write $\mathrm{U}(V)$ for the set of unitary linear maps $V \to V$ with unitary linear inverses.

V is, in particular, a metric space with metric $d(x, y) := \langle x - y, x - y \rangle^{1/2}$. The elements of $\mathrm{U}(V)$ are isometries w.r.t. this metric and so by Proposition 1.11 $\mathrm{U}(V)$ is a topological group with the topology of pointwise convergence. When V is a Hilbert space (*i.e.* when V is additionally complete in the metic d) then this topology on $\mathrm{U}(V)$ is the strong operator topology (SOT) restricted to $\mathrm{U}(V)$.

Between topologized and topological

To better understand topological groups we shall also look at some weaker structures with some axioms stripped away – centipede mathematics. These structures are also studied in their own right; for a much more detailed development including many examples and open problems see [AT08, Chapters 1 & 2].

Suppose that G is a topologized group written multiplicatively. We say that **left (resp. right) multiplication is continuous** if the maps $G \to G; y \mapsto xy$ (resp. $G \to G; y \mapsto yx$) are continuous for all $x \in G$. Such a group is said to be a **left-topological** (resp. **right-topological**) **group**. A group which is both a left-topological and a right-topological group is called a **semitopological group**.

Observation 1.13. Any Abelian left-topological group is semitopological since left multiplication by y is the same as right multiplication by y.

Example 1.14 (The coset topology)**.** For a group G and $H \leqslant G$, equipping G with the topology whose closed sets are unions of left cosets of H makes it into a left-topological group; we call this topology the **coset topology (on G generated by H)**. ⚠ This terminology is not completely standard.

The open (and closed) sets in G are exactly the unions of left cosets of H, hence if $S \subset G$ then $\overline{S} = SH$. Right multiplication is continuous (if and) only if H is normal in G: Indeed, if right multiplication is continuous then since H is closed, Hy^{-1} is closed for all y, so $Hy^{-1} = SH$ for some $S \subset G$. Let $x \in S$ be such that $y^{-1} \in xH$, whence

$y^{-1}H = xH \subset SH = Hy^{-1}$ and so H is normal in G. In particular there are left-topological groups that are not semitopological.

Proposition 1.15. *Suppose that X is a topological space and G is a group of homeomorphisms of X. Then G with the topology of pointwise convergence w.r.t. the evaluation action is a semitopological group.*

Proof. For $x_1, \dots, x_n \in X$ and $U_1, \dots, U_n$ open in X we have

$$U(x_1, \dots, x_n; U_1, \dots, U_n)g^{-1} = U(g.x_1, \dots, g.x_n; U_1, \dots, U_n)$$

so right multiplication is continuous. Furthermore,

$$g^{-1}U(x_1, \dots, x_n; U_1, \dots, U_n) = U(x_1, \dots, x_n; g^{-1}.U_1, \dots, g^{-1}.U_n),$$

so left multiplication is continuous since the sets $g^{-1}.U_1, \dots, g^{-1}.U_n$ are open because the action is by continuous functions. □

The next example is not central to the course but may be contrasted with Example 1.12.

Example 1.16 (Groups of continuous maps with continuous inverses with the SOT)**.** For V an inner product space we write $\mathrm{GL}(V)$ for the set of continuous linear maps $V \to V$ with continuous linear inverses.

With V given the norm topology, that is the topology induced by the metric d as in Example 1.12, $\mathrm{GL}(V)$ is a group of homeomorphisms of V. Hence if $\mathrm{GL}(V)$ is endowed with the topology of pointwise convergence w.r.t. the evaluation action, then $\mathrm{GL}(V)$ becomes a semitopological group; this topology is the strong operator topology restricted to $\mathrm{GL}(V)$.

By contrast with Example 1.12 if V is infinite dimensional then composition on $\mathrm{GL}(V)$ need not be continuous nor need it have a continuous inverse.

A topologized group in which the group operation is continuous (as a map from the product space G^2) is called a **paratopological group**.

Example 1.17 (The reals with the right order topology)**.** The set[2] $\{(a, \infty) : -\infty \leqslant a \leqslant \infty\}$ is a topology on $\mathbb{R}$ which we call the **right order topology (on $\mathbb{R}$)**; we denote this topologized group $\mathbb{R}_{\mathrm{RO}}$.

$\mathbb{R}_{\mathrm{RO}}$ is a paratopological group since for $a \in \mathbb{R}$,

$$\{(x, y) : x + y \in (a, \infty)\} = \bigcup_{b \in \mathbb{R}} (a - b, \infty) \times (b, \infty)$$

so that the preimage of the open set (a, ∞) is open in the product topology. Inversion on $\mathbb{R}_{\mathrm{RO}}$ is not continuous since $(-\infty, -a)$ is not open (for any $a \in \mathbb{R}$), and hence $\mathbb{R}_{\mathrm{RO}}$ is not a topological group.

[2]For the avoidance of doubt $(-\infty, \infty) := \mathbb{R}$ and $(\infty, \infty) := \varnothing$.

$\mathbb{R}_{\text{RO}}$ is not Hausdorff: Any two non-empty open sets contain all sufficient large reals and hence have non-empty intersection.

The sets $[x, \infty)$ are compact because any open cover has a set containing x, and that set must have the form (a, ∞) for some $a < x$. This set on its own is a cover of $[x, \infty)$ and hence the cover has a subcover with one set. It follows that $\mathbb{R}_{\text{RO}}$ is locally compact, since for $x \in \mathbb{R}$, $[x - 1, \infty)$ is a compact neighbourhood of x; and $\mathbb{R}_{\text{RO}}$ is σ-compact since $\mathbb{R} = \bigcup_{z \in \mathbb{Z}} [z, \infty)$, and $[z, \infty)$ is compact.

On the other hand, any set A with arbitrarily large negative elements is not compact since $\{(a, \infty) : a \in \mathbb{R}\}$ is an open cover of A, but any finite subset has a smallest element and so is not a cover. In particular if $A \subset \mathbb{R}$ is non-empty then $\overline{A} = (-\infty, \sup A]$, and so no non-empty closed set is compact, and $\mathbb{R}_{\text{RO}}$ itself is not compact.

Observation 1.18. Every paratopological group G is semitopological since the maps $G \to G^2; x \mapsto (x, y)$ (and $G \to G^2; x \mapsto (y, x)$) are continuous for all $y \in G$, and the composition of continuous maps is continuous.

A semitopological group in which inversion is continuous is called a **quasitopological** group.

Example 1.19 (The reals with the cofinite and cocountable topologies)**.** Write $\mathbb{R}_{\text{CF}}$ and $\mathbb{R}_{\text{CC}}$ for the additive group $\mathbb{R}$ equipped with the topology whose proper closed sets are the finite sets, and whose proper closed sets are the countable sets respectively. These are genuinely topologies and are called the **cofinite** and **cocountable** topologies respectively.

$\mathbb{R}_{\text{CF}}$ and $\mathbb{R}_{\text{CC}}$ are quasitopological groups because $-x + U = U + (-x)$ is finite (resp. countable) whenever U is finite (resp. countable), and $-U$ is finite (resp. countable) when U is finite (resp. countable).

If $U, V \subset \mathbb{R}$ are non-empty and open in the cofinite (resp. cocountable) topology, then $U + V = \mathbb{R}$: for $x \in \mathbb{R}$, $x - U$ is infinite (resp. uncountable) and V^c is finite (resp. countable) and so $x - U \not\subset V^c$, whence $x \in U + V$ and $U + V = \mathbb{R}$ as claimed. In particular, $\{(x, y) \in \mathbb{R}^2 : x + y \neq 0\}$, which is the preimage under addition of an open set in the cofinite (resp. cocountable) topology, cannot contain a sum of non-empty open sets. It follows that multiplication is not continuous and $\mathbb{R}_{\text{CF}}$ (resp. $\mathbb{R}_{\text{CC}}$) is *not* paratopological.

$\mathbb{R}_{\text{CF}}$ and $\mathbb{R}_{\text{CC}}$ are not Hausdorff: Any two non-empty open sets U and V have finite (resp. countable) complements, but $\mathbb{R}$ is infinite (resp. uncountable) and so U is infinite (resp. uncountable) and $U \not\subset V^c$ which is to say that $U \cap V \neq \emptyset$.

$\mathbb{R}_{\text{CF}}$ is compact: Indeed, the cofinite topology on any topological space is compact since if $\mathcal{U}$ is an open cover, then let $U_0 \in \mathcal{U}$ be non-empty. U_0^c is finite, say $U_0^c = \{x_1, \ldots, x_m\}$, and since $\mathcal{U}$ is a cover we may take $U_i \in \mathcal{U}$ such that $x_i \in U_i$. The set $\{U_0, \ldots, U_m\}$ is a finite subcover of $\mathcal{U}$ and our claim is proved.

$\mathbb{R}_{CC}$ is not σ-compact, nor is it locally compact: This follows from the fact that the only compact sets in $\mathbb{R}_{CC}$ are finite. To see this fact note that if X is infinite then it has a countably infinite subset C, and $\{(\mathbb{R}\backslash C) \cup \{x\} : x \in C\}$ is an open cover of X with no finite subcover.

Observation 1.20. Every left-topological group G with a continuous inverse is a quasitopological group, since for $y \in G$ the right multiplication map $G \to G; x \mapsto xy = (y^{-1}x^{-1})^{-1}$ is continuous since it is a composition of inversion, left multiplication by y^{-1}, and inversion again.

The following diagram summarises the foregoing. The implications without any text next to them follow *a fortiori* – *i.e.* by simply dropping hypotheses – and the missing implications and non-implications can all be deduced from transitivity of implication and the law of excluded middle.

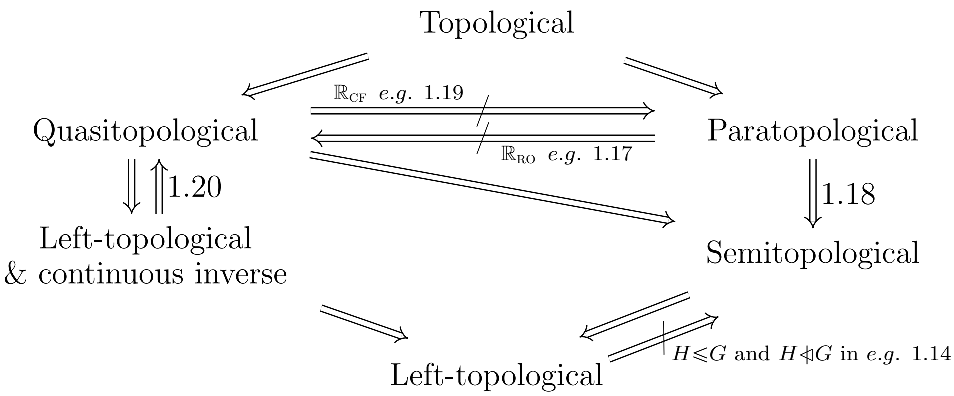

Figure 1: Relationships between types of topologized groups

Basic tools

In this section there are a few key lemmas (Lemmas 1.21, 1.23, 1.32,1.34, 1.36, & 1.41) which we highlight in red because they each capture a crucial technique or idea.

Lemma 1.21 (Key Lemma I)**.** *Suppose that G is a topologized group in which inversion is continuous. If U is a neighbourhood of 1_G then U contains a symmetric open neighbourhood of the identity and if S is symmetric then $\overline{S}$ is symmetric.*

Proof. If U is a neighbourhood of 1_G then U contains an open neighbourhood V of 1_G. Put $S := V \cap V^{-1}$ which is open and contains 1_G (since $1_G^{-1} = 1_G$) and moreover $S = S^{-1}$ so that S is a symmetric open neighbourhood of 1_G contained in U. For the second part, since inversion is continuous the preimage of $\overline{S}$ under inversion is the set $\overline{S}^{-1}$ and is closed and contains $S^{-1} = S$. It follows that $\overline{S} \subset \overline{S}^{-1}$. But $\overline{S}^{-1} \subset (\overline{S}^{-1})^{-1} = \overline{S}$, and we conclude that $\overline{S}^{-1} = \overline{S}$. □

Remark 1.22. Every conclusion of Lemma 1.21 may fail if 'topologized group with continuous inverse' is replaced by 'paratopological group': In $\mathbb{R}_{\mathrm{RO}}$, the only symmetric and open sets are $\varnothing$ and $\mathbb{R}$, hence $(-1,\infty)$ is a neighbourhood of the identity that does not contain a symmetric neighbourhood of the identity; and $\overline{\{0\}} = (-\infty, 0]$ which is not symmetric despite $\{0\}$ being symmetric.

Lemma 1.23 (Key Lemma II)**.** *Suppose that G is a left-topological (resp. right-topological) group, U is open, and V is any set. Then VU (resp. UV) is open; U is a neighbourhood of x if and only if $x^{-1}U$ (resp. Ux^{-1}) is a neighbourhood of the identity; and $\overline{xV} = x\overline{V}$ (resp. $\overline{Vx} = \overline{V}x$).*

Proof. First, $VU = \bigcup_{v\in V} vU$ which is a union of open sets since $G \to G; x \mapsto v^{-1}x$ is continuous. Secondly, if U is a neighbourhood of x then there is an open set $U_x \subset U$ containing x. Continuity of $G \to G; z \mapsto xz$ then says that $x^{-1}U_x$ is an open set containing 1_G and contained in $x^{-1}U$, which is to say $x^{-1}U$ is a neighbourhood of the identity. Similarly if $x^{-1}U$ is a neighbourhood of the identity then U is a neighbourhood of x by continuity of $G \to G; z \mapsto x^{-1}z$. Finally, since $G \to G; z \mapsto x^{-1}z$ is continuous, $x\overline{V}$ is closed and contains xV, hence $\overline{xV} \subset x\overline{V}$. Apply this with x replaced by x^{-1} and V replaced by xV to get $\overline{V} \subset x^{-1}\overline{xV}$, whence $\overline{xV} = x\overline{V}$. The parenthetical results follow *mutatis mutandis.* □

The ideas of the last two lemmas can be used to contain compact sets in 'nice' sets:

Lemma 1.24. *Suppose that G is a locally compact quasitopological group and K is a compact set. Then there is a symmetric open neighbourhood of the identity containing K and contained in a compact set.*

Proof. Since G is locally compact there is a compact neighbourhood of the identity L; let V be an open neighbourhood of the identity contained in L. The set $\{xV : x \in K\}$ is an open cover of K and so there are $x_1, \dots, x_m \in K$ such that $K \subset x_1V \cup \cdots \cup x_mV$; let $x_0 := 1_G$. $x_iV \subset x_iL$ and since left multiplication is continuous, x_iL is compact, and since inversion is continuous, $(x_iL)^{-1}$ is compact. $x_iV \cup (x_iV)^{-1}$ is symmetric by design and open since left multiplication and inversion are continuous. It follows that $V := \bigcup_{i=0}^{m} (x_iV) \cup (x_iV)^{-1}$ is a symmetric open set contained in a finite union of compact sets. A finite union of compact sets is compact and so V is contained in a compact set and by design $K \subset V$ and $1_G \in V$. □

Remark 1.25. We cannot replace 'quasitopological' by 'paratopological' above: $\mathbb{R}_{\mathrm{RO}}$ is a locally compact paratopological group, but the only set containing an open symmetric neighbourhood of the identity is $\mathbb{R}$ which is not compact.

We also cannot drop the local compactness requirement: $\mathbb{R}_{\mathrm{CC}}$ is a quasitopological group in which all the compact sets are finite while all neighbourhoods are infinite.

We can also use Lemma 1.23 to see what happens to subgroups under the process of topological closure. We begin with a technical lemma.

Lemma 1.26. *Suppose that G is a semitopological group and $H \subset G$. If H is closed under multiplication (*i.e. *$xy \in H$ whenever $x, y \in H$), then so is $\overline{H}$; if H is a union of conjugacy classes in G (*i.e. *$xH = Hx$ for $x \in G$) then so is $\overline{H}$.*

Proof. First, by Lemma 1.23 $h\overline{H} = \overline{hH} = \overline{H}$ for all $h \in H$. Hence $Hw \subset \overline{H}$ for all $w \in \overline{H}$. Again by Lemma 1.23 we have $\overline{H}w = \overline{Hw} \subset \overline{\overline{H}} = \overline{H}$, and hence $\overline{H}^2 \subset \overline{H}$ as required. Secondly, by Lemma 1.23, $x\overline{H} = \overline{xH} = \overline{Hx} = \overline{H}x$ for all $x \in G$. □

Remark 1.27. We cannot replace 'semitopological' with 'left-topological' above: Suppose that G is a group with subgroups H and K such that HK is not closed under multiplication. Then G with the coset topology generated by K is left-topological but has $\overline{H} = HK$, so that even though H is closed under multiplication, its topological closure is not.

Proposition 1.28. *Suppose that G is a quasitopological group and $H \leqslant G$. Then $\overline{H}$ is a subgroup of G. Furthermore, if H is normal then so is $\overline{H}$. In particular, $\overline{\{1_G\}}$ is a normal subgroup of G.*

Proof. By the first part of Lemma 1.26, $\overline{H}$ is closed under multiplication and by Lemma 1.21, $\overline{H}^{-1} = \overline{H}$. Since H is non-empty it follows that $\overline{H}$ is a group. If H is normal then $\overline{H}$ is normal by the second part of Lemma 1.26. Since $\{1_G\}$ is a normal subgroup of G we then get the last claim. □

Remark 1.29. We cannot replace 'quasitopological' by 'paratopological' above: $\{0\}$ is a subgroup of $\mathbb{R}_{\mathrm{RO}}$ but $\overline{\{0\}} = (-\infty, 0]$ which is not a subgroup.

On the other hand, paratopological groups in which the closure of every subgroup is a subgroup have been studied in [FT14].

Proposition 1.30. *Suppose that G is a compact semitopological group. Then $\overline{\{1_G\}}$ is a normal subgroup of G.*

Proof. Put $H := \overline{\{1_G\}}$ then by Lemma 1.26, $H^2 \subset H$ and $xH = Hx$ for all $x \in G$. Now consider $\mathcal{H} := \{xH : x \in H\}$. This is a set of closed subsets of H by Lemma 1.23, which has the finite intersection property: suppose $x_1H, \dots, x_nH \in \mathcal{H}$. Then $x_iH \supset x_i \cdots x_nH = Hx_i \cdots x_n \supset x_1 \cdots x_{i-1}Hx_i \cdots x_n = x_1 \cdots x_nH$ since $x_1 \cdots x_{i-1}, x_{i+1} \cdots x_n \in H$ and H is (multiplicatively) closed. Since G is compact, $V := \bigcap \mathcal{H}$ is non-empty.

V is closed and non-empty, so there is some $y \in V$. By Lemma 1.23 $yH = \overline{\{y\}} \subset V$, but then $y^2H \in \mathcal{H}$ and so $y^2H \supset V \supset yH$, and since G is a group, $yH \supset H$. Now $H \in \mathcal{H}$, and so $H \supset V \supset yH \supset H$ – in other words $V = H$. But then for all $x \in H$ we have $H \subset xH$, and since $1_G \in H$ we have some $y \in H$ such that $xy = 1_G$ and H is closed under inverses and hence a subgroup. □

Remark 1.31. In Exercise I.6 we give an example of a compact semitopological group that is not quasitopological, so this result does not just follow from Proposition 1.28.

We cannot relax 'semitopological' to 'left-topological': if G is a finite group with a non-normal subgroup H then G with the coset topology generated by H has $\overline{\{1_G\}} = H$ which is not normal, but it is compact since G is finite. Similarly, we cannot relax the compactness requirement to local compactness or σ-compactness in view of the group $\mathbb{R}_{\text{RO}}$ in which the closure of the identity is not even a group (see Remark 1.29).

Lemma 1.32 (Key Lemma III). *Suppose that G is a left-topological (resp. right-topological) group, S is a set and V is an open neighbourhood of the identity. Then $\overline{SV} \subset SVV^{-1}$ (resp. $\overline{VS} \subset V^{-1}VS$).*

Proof. Let $A := G\backslash(SVV^{-1})$ and $B := G\backslash(AV)$. B is closed since AV is open by Lemma 1.23. If $x \in AV$ then there is some $v \in V$ such that $xv^{-1} \in A$, so $xv^{-1} \notin SVV^{-1}$. Hence $SV \subset B$ and since B is closed $\overline{SV} \subset B$. Now if $x \in B$ then $x \notin A$ since $1_G \in V$, and hence $x \in SVV^{-1}$ as claimed. The parenthetical results follow *mutatis mutandis*. □

Corollary 1.33. *Suppose that G is a left-topological group and $H \leqslant G$. If H is a neighbourhood in G then H is open in G; if H is open in G then H is closed in G; if H is closed in G* and *of finite index then H is open in G.*

Proof. First, if H is a neighbourhood then there is a non-empty open set $U \subset H$. But then $H = HU$ is open by Lemma 1.23. For the second part, if H is open then by Lemma 1.32 $\overline{H} \subset HH^{-1} = H$ and so H is closed.

For the last part, since H is closed, every $W \in G/H$ is closed by Lemma 1.23. Since H is of finite index, $\bigcup\{W \in G/H : W \neq H\}$ is a finite union of closed sets and so closed. Finally, since G/H is a partition of G containing H, $H = G\backslash\bigcup\{W \in G/H : W \neq H\}$ is open as required. □

Lemma 1.34 (Key Lemma IV). *Suppose that G is a paratopological group and X is a neighbourhood of z. Then there is an open neighbourhood of the identity V such that $zV^2 \subset X$. Moreover, if G is a topological group then V may be taken to be symmetric.*

Proof. Let $U \subset X$ be an open neighbourhood of z. The map $(x, y) \mapsto xy$ is continuous and so $\{(x, y) : xy \in U\}$ is an open subset of $G \times G$. By definition of the product topology there is a set $\mathcal{S}$ of products of open sets in G such that

$$\{(x, y) : xy \in U\} = \bigcup\{S \times T : S \times T \in \mathcal{S}\}.$$

Since $z1_G = z \in U$, there is some $S \times T \in \mathcal{S}$ such that $(z, 1_G) \in S \times T$. Thus S is an open neighbourhood of z and T is an open neighbourhood of the identity, so by Lemma 1.23

$V := (z^{-1}S) \cap T$ is an open neighbourhood of the identity. Now $zV \subset S$ and $V \subset T$ and so $zV^2 \subset U$ as required. Moreover, if G is a topological group inversion is also continuous so by Lemma 1.21 V contains a symmetric open neighbourhood of the identity, and the conclusion follows by nesting. □

Remark 1.35. We cannot replace 'paratopological' by 'quasitopological' above: In $\mathbb{R}_{\text{CF}}$ (Example 1.19) if X is the complement of some $x \neq z$, then X is open but the sum of any two non-empty open sets is the whole of $\mathbb{R}$ and so cannot be contained in X.

Lemma 1.36 (Key Lemma V). *Suppose that G is a paratopological group and $K_1, \ldots, K_n$ are compact subsets of G. Then $K_1 \cdots K_n$ is compact. In particular, if K is compact then K^n is compact for all[3] $n \in \mathbb{N}_0$.*

Proof. The (topological) product of two compact sets is compact so if $K_1 \cdots K_{n-1}$ is compact and K_n is compact then $(K_1 \cdots K_{n-1}) \times K_n$ is compact. But then the continuous image of a compact set is compact and so $K_1 \cdots K_n = (K_1 \cdots K_{n-1})K_n$ is compact and the result follows by induction on n. □

Remark 1.37. Exercise I.5 gives an example of a quasitopological group where the conclusion above does not hold.

Corollary 1.38. *Suppose that G is a locally compact topological group. Then there is a σ-compact, locally compact open subgroup of G.*

Proof. Apply Lemma 1.24 to get a symmetric open neighbourhood of the identity S contained in a compact set L. Then $H := \langle S \rangle$ is a subgroup of G which is locally compact and open and closed by Corollary 1.33. It is contained in $\bigcup_{n \in \mathbb{N}_0} (L \cap H)^n$, and since H is closed $H \cap L$ is compact, and so this union is a countable union of compact (by Lemma 1.36) sets. The result is proved. □

Remark 1.39. In Exercise I.8 we ask for an example to show that the hypotheses of Corollary 1.38 may not be relaxed from 'topological' to 'paratopological', and in Exercise I.4 that the hypothesis 'locally compact' may not be changed to 'σ-compact'.

A cover $\mathcal{U}$ is a **refinement** of a cover $\mathcal{V}$ of a set X if $\mathcal{U}$ is a cover of X and each set in $\mathcal{U}$ is contained in some set in $\mathcal{V}$.

Observation 1.40. Refinements are transitive meaning that if $\mathcal{W}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$ then $\mathcal{W}$ is a refinement of $\mathcal{U}$.

[3] Note that $K^0 = \{1_G\}$ by definition and so is compact since it is finite.

Lemma 1.41 (Key Lemma VI). *Suppose that G is a paratopological group, $K \subset G^n = G \times \cdots \times G$ is compact for some $n \in \mathbb{N}$, and $\mathcal{U}$ is an open cover of K. Then there is an open neighbourhood of the identity $U \subset G$ such that $\{x_1U \times \cdots \times x_nU, Ux_1 \times \cdots \times Ux_n : x \in K\}$ is a refinement of $\mathcal{U}$. If G is a topological group then U may be taken to be symmetric.*

Proof. First, the structure of the product topology (and Lemma 1.23) means that we can pass to a refinement of $\mathcal{U}$ where for each $x \in K$ there are open neighbourhoods of the identity $U_1^{(x)}, \dots, U_n^{(x)}$ (our notation is a little clumsy here to make the x-dependence explicit) such that $x_1U_1^{(x)} \times \cdots \times x_nU_n^{(x)}$ is in this refinement. The set $\bigcap_{i=1}^n U_i^{(x)}$ is an open neighbourhood of the identity and so by Lemma 1.34 there is a open neighbourhood of the identity U_x such that $U_x^2 \subset U_i^{(x)}$ for all $1 \leqslant i \leqslant n$. In particular, $\mathcal{V} := \{x_1U_x \times \cdots \times x_nU_x : x \in K\}$ is an open cover of K and a refinement of $\mathcal{U}$.

By compactness of K there is a finite set $F \subset K$ such that $\mathcal{W} := \{x'_1U_{x'} \times \cdots \times x'_nU_{x'} : x' \in F\}$ is a cover of K. Let $U := \bigcap_{x' \in F} U_{x'}$ which is a finite intersection of open neighbourhoods of the identity and so a open neighbourhood of the identity. Since $\mathcal{W}$ is a cover of K, for each $x \in K$ there is some $x' \in F$ such that $x \in x'_1U_{x'} \times \cdots \times x'_nU_{x'}$, and hence

$$\begin{aligned} x_1U \times \cdots \times x_nU &\subset x'_1U_{x'}U \times \cdots \times x'_nU_{x'}U \\ &\subset x'_1U_{x'}^2 \times \cdots \times x'_nU_{x'}^2 \subset x'_1U_1^{(x')} \times \cdots \times x'_nU_n^{(x')} \end{aligned}$$

so that $\{x_1U \times \cdots \times x_nU : x \in K\}$ is a refinement of $\mathcal{V}$ which in turn is a refinement of $\mathcal{U}$.

Apply this to G^{op}, with the same topology, we get another neighbourhood of the identity U' such that $\{U'x_1 \times \cdots \times U'x_n : x \in K\}$ refines $\mathcal{U}$. Taking the intersection of U and U' gives the result. If G is topological then by Lemma 1.21 U contains a symmetric open neighbourhood of the identity and we may pass to this. □

Remark 1.42. The lemma above is not unrelated to the Generalised Tube Lemma from topology (see *e.g.* [Mun00, Lemma 26.8]), which is also known as Wallace's Theorem.

With the results we have now established we can explain the lack of examples of compact paratopological groups that are not topological groups:

Theorem 1.43. *Suppose that G is a compact paratopological group. Then G is a topological group.*

Proof. Suppose that $K \subset G$ is closed and $x \notin K^{-1}$. For $y \in K$, if $yx \in \overline{\{1_G\}}$ then by Proposition 1.30 $x^{-1}y^{-1} \in \overline{\{1_G\}}$ and so by Lemma 1.23, $x^{-1} \in \overline{\{1_G\}}y = \overline{\{y\}} \subset \overline{K} = K$, a contradiction. Hence $yx \notin \overline{\{1_G\}}$ and again, by Lemma 1.23 there is an open neighbourhood U_y of y such that $U_yx \cap \overline{\{1_G\}} = \varnothing$ and in particular $1_G \notin U_yx$.

Apply Lemma 1.41 to the cover $\{U_y : y \in K\}$ of K to get an open neighbourhood of the identity U such that for all $y \in K$ we have $yU \subset U_z$ for some $z = z(y) \in K$. It follows that $1_G \notin yUx$ for all $y \in K$, so $K^{-1} \cap Ux = \varnothing$. Thus K^{-1} is closed and the result is proved. □

Remark 1.44. We cannot replace 'paratopological' by 'quasitopological' in view of $\mathbb{R}_{\text{CF}}$ which is a compact quasitopological group that is not a topological group. We also cannot relax 'compact' to 'locally compact' since $\mathbb{R}_{\text{RO}}$ is a locally compact paratopological group that is not a topological group.

⚠ In [Rav15] it states that every locally compact paratopological group is a topological group. This does not contradict the above, it is simply using a different definition of local compactness in which every element is contained in a *closed* compact neighbourhood.

A topological space X is said to be **regular** if for all $x \in X$ every neighbourhood of x contains a closed neighbourhood of x. ⚠ The literature is inconsistent on the meaning of regular, and for some other authors a regular space is required to be Hausdorff.

Proposition 1.45. *Suppose that G is a topological group. Then G is regular.*

Proof. Let V be a neighbourhood of $x \in G$. By Lemma 1.34 there is a symmetric open neighbourhood of the identity U such that $xU^2 \subset V$, and so by Lemmas 1.23 & 1.32, $x\overline{U} \subset xUU^{-1} = xU^2 \subset V$ as required. □

Remark 1.46. The quasitopological group $\mathbb{R}_{\text{CF}}$ is not regular because the only closed neighbourhood is the whole of $\mathbb{R}$ which cannot be contained in any neighbourhood that is not the whole of $\mathbb{R}$; and the paratopological group $\mathbb{R}_{\text{RO}}$ is not regular for the same reasons.

There are also purely topological conditions that give rise to regularity:

Proposition 1.47. *Suppose that X is a locally compact Hausdorff topological space. Then X is regular.*

Proof. Let V be an open neighbourhood of $x \in X$, which by local compactness we may assume is contained in a compact neighbourhood U. For all $x \neq y \in X$ there is an open set U_y containing y which is disjoint from an open set V_y containing x. $\{U_y : y \in U \backslash V\}$ is an open cover of a closed subset of the compact set U and so has a finite subcover, say $U_{y_1}, \dots, U_{y_m}$. But U is a compact subset of a Hausdorff topological space, so $(U \backslash U_{y_1}) \cap \cdots \cap (U \backslash U_{y_m})$ is closed, contained in V, and contains $V_{y_1} \cap \cdots \cap V_{y_m}$ which is an open set containing x. □

In the above we used that compact subsets of Hausdorff topological spaces are closed, and for non-Hausdorff spaces that are regular the following lemma helps recover the situation:

Lemma 1.48. *Suppose that X is a regular space and $K \subset X$ is compact. Then $\overline{K}$ is compact.*

Proof. Suppose $\mathcal{U}$ is an open cover of $\overline{K}$. Then for each $x \in K$ there is an open neighbourhood of x in $\mathcal{U}$, call it U_x. By regularity there is an open neighbourhood of x, call it V_x, such that $\overline{V_x} \subset U_x$. The set $\{V_x : x \in K\}$ is an open cover of K and so by compactness has a finite subcover, say $K \subset V_{x_1} \cup \cdots \cup V_{x_k}$ and hence $\overline{K} \subset U_{x_1} \cup \cdots \cup U_{x_k}$. Thus $\mathcal{U}$ has a finite subcover of $\overline{K}$, and the result is proved. □

Remark 1.49. The reals with the cocountable topology has every compact set being finite, and also all finite sets are closed hence if K is compact then $\overline{K} = K$ is compact in this topology, but this space is not regular.

Since every topological group is regular the closure of a compact subset of any topological group is compact. We cannot relax 'topological' to 'paratopological' here since $\{0\}$ is a compact subset of $\mathbb{R}_{\text{RO}}$ whose closure is $(-\infty, 0]$ which is not compact. Similarly we cannot relax 'topological' to 'quasitopological': Exercise I.7 gives an example of a quasitopological group that is not locally compact but which has a compact dense subgroup. In particular, this means that there are paratopological and quasitopological groups that are not regular.

Lemma 1.50. *Suppose that X is a regular space and K is a compact set inside an open set B. Then there is an open set $C \supset K$ with $\overline{C} \subset B$. If X is, additionally, locally compact then $\overline{C}$ may be taken to be compact.*

Proof. Since B is open, for each $x \in K$ there is an open set U_x containing x and contained in B, and since X is regular, there is an open neighbourhood V_x of x with $\overline{V_x} \subset U_x$. Since K is compact $K \subset V_{x_1} \cup \cdots \cup V_{x_m}$ for some $x_1, \ldots, x_m \in K$. Put $C := V_{x_1} \cup \cdots \cup V_{x_m}$ and get the result. If X is locally compact then we may assume that U_x is compact and hence $\overline{C}$ is compact as claimed. □

2 Continuous homomorphisms

The maps which will concern us the most are continuous homomorphisms, and also continuous *open* homomorphisms, that is continuous homomorphisms in which the image of an open set is open.

Example 2.1. The map $\theta : \mathbb{R} \to S^1; x \mapsto \exp(2\pi i x)$ from the real line to the circle group is a surjective continuous open homomorphism.

Example 2.2. The maps $\mathbb{R} \to \mathbb{R}; x \mapsto \alpha x$ for $\alpha \in \mathbb{R}$ are continuous homomorphisms of the real line. For $\alpha = 0$ this map is *not* open; for $\alpha \neq 0$, this map has an inverse of the same form and so is open and in fact a homeomorphic isomorphism.

Example 2.3. If G is a topologized group with continuous inverse and G^{op} is given the same topology as G, then the map $G \to G^{\text{op}}; x \mapsto x^{-1}$ is a homeomorphic isomorphism, because it has a continuous homomorphic inverse $G^{\text{op}} \to G; x \mapsto x^{-1}$.

Example 2.4. Suppose that G is a semitopological group. Then for $a \in G$, conjugation by a, that is the map $G \to G; x \mapsto axa^{-1}$ is an isomorphism with inverse map $G \to G; x \mapsto a^{-1}xa$. Both these maps are homeomorphisms for fixed a since left and right multiplication is continuous and the composition of continuous maps is continuous. ⚠ We do *not* need inversion to be continuous.

Example 2.5. For a topologized group G, the identity map $\theta : G_{\mathrm{D}} \to G; x \mapsto x$ from G with its discrete topology to G with the given topology is a continuous homomorphism, because the identity map is a homomorphism and any map from a discrete space is continuous.

⚠ G_{D} is a topological group, but in general G need not even be a left-topological group.

In left-topological groups, the algebraic structure makes checking continuity and openness a little easier: First, recall that a **neighbourhood base** of a point x in a topological space X is a family $B = (B_i)_{i \in I}$ of neighbourhoods of x such that if U is an open set containing x then there is some $i \in I$ such that $B_i \subset U$.

Proposition 2.6. *Suppose that G and H are left-topological groups and $B = (B_i)_{i \in I}$ is a neighbourhood base of the identity in H. Then a homomorphism $\theta : G \to H$ is continuous if (and only if) $\theta^{-1}(B_i)$ is a neighbourhood of the identity for all $i \in I$; and a homomorphism $\theta : H \to G$ is open if (and only if) $\theta(B_i)$ is a neighbourhood of the identity for all $i \in I$.*

Proof. Suppose that $U \subset H$ is open and $\theta(y) \in U$. By Lemma 1.23 there is an open neighbourhood of the identity V_y such that $\theta(y)V_y \subset U$, and hence $i \in I$ such that $B_i \subset V_y$. Thus $\theta^{-1}(B_i) \subset \theta^{-1}(V_y)$ so $y\theta^{-1}(B_i) \subset \theta^{-1}(U)$ (using that θ is a homomorphism) and hence $\theta^{-1}(U)$ contains a neighbourhood of y *i.e.* $\theta^{-1}(U)$ is open. The parenthetical 'only if' follows since B_i contains an open neighbourhood of 1_H and $\theta(1_G) = 1_H$. The result for open maps follows similarly.

□

Corollary 2.7. *Suppose that G is a semitopological group and $B = (B_i)_{i \in I}$ is a neighbourhood base of the identity such that B_i^{-1} is a neighbourhood of the identity for all $i \in I$. Then G is quasitopological.*

Proof. Since G is semitopological the map $G \to G^{\mathrm{op}}; x \mapsto x^{-1}$ is a homomorphism between left-topological groups, and so Proposition 2.6 gives the result. □

The initial topology, subgroups, and product groups

Given a function $f : X \to Y$ into a topological space the **initial topology on X w.r.t. f** is the topology $\{f^{-1}(U) : U \text{ is open in } Y\}$. In words it is the weakest topology (meaning coarsest topology, or topology with the fewest open sets) on X making f continuous.

Proposition 2.8. *Suppose that G is a topologized group, $\theta : H \to G$ is a group homomorphism, and H is given the initial topology w.r.t. θ. Then (θ is continuous and)*

(i) if group inversion is continuous on G, then it is continuous on H;

(ii) if left (resp. right) multiplication is continuous on G, then it is continuous on H;

(iii) and if multiplication is continuous on G^2 then it is continuous on H^2.

In particular if G is a topological (resp. paratopological, quasitopological, semitopological, left-topological or right-topological) group then so is H.

Proof. Suppose U is an open set in H so that there is W, open in G, such that $U = \theta^{-1}(W)$.

For (i), note that $U^{-1} = (\theta^{-1}(W))^{-1} = \theta^{-1}(W^{-1})$, but W^{-1} is open in G and so U^{-1} is open in H. For (ii), given $x \in H$, $xU = x\theta^{-1}(W) = \theta^{-1}(\theta(x)W)$, but $\theta(x)W$ is open in G and so xU is open in H, so left multiplication is continuous. The result for right multiplication follows similarly. Finally, for (iii), let $\mathcal{S}$ be a set of products of open sets in G such that $\{(x, y) \in G^2 : xy \in W\} = \bigcup \mathcal{S}$. Then

$$\begin{aligned}\{(x, y) \in H^2 : xy \in U\} &= \{(x, y) \in H^2 : \theta(x)\theta(y) \in W\}\\ &= \{(x, y) \in H^2 : (\theta(x), \theta(y)) \in S \times T \text{ for some } S \times T \in \mathcal{S}\}\\ &= \bigcup \{\theta^{-1}(S) \times \theta^{-1}(T) : S \times T \in \mathcal{S}\},\end{aligned}$$

and this last set is a union of open sets and so open. The result is proved. □

We say that a topologized group H is a **topological subgroup** of a topologized group G if H is a subgroup of G and has the subspace topology *i.e.* has the initial topology w.r.t. the inclusion homomorphism $H \to G; x \mapsto x$.

Note from the proposition that if G is a topological (resp. paratopological, quasitopological, semitopological, left-topological or right-topological) group then so is H.

Example 2.9. A number of the topological groups from the introduction are examples of topological subgroups: $\mathbb{Q}$ (from Example 1.4) is a topological subgroup of the real line $\mathbb{R}$; and $\mathbb{R}_{>0}$ (from Example 1.6) and S^1 (from Example 1.7) are topological subgroups of $\mathbb{C}^*$.

Example 2.10 (p-divisor topology on $\mathbb{Z}$)**.** Write $\mathbb{Z}_{p-\text{DIV}}$ for the group $\mathbb{Z}$ and the initial topology w.r.t. the quotient map $\mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})_{\text{D}}; x \mapsto x + p\mathbb{Z}$, which is a topological group by Proposition 2.8. This is the same as the coset topology on $\mathbb{Z}$ generated by $p\mathbb{Z}$.

The group $\mathbb{Z}/p\mathbb{Z}$ is finite – this follows from the division algorithm – and so *all* topologies on $\mathbb{Z}/p\mathbb{Z}$ are finite and so in particular the topology on $\mathbb{Z}_{p-\text{DIV}}$ is finite.

It can be useful to have the initial topology with respect to multiple functions, and to this end we need the direct product of topologized groups:

Proposition 2.11. *Suppose that $(G_i)_{i \in I}$ is a family of topologized groups. Then $\prod_{i \in I} G_i$ is a topologized group when it is given the group structure of the product group, and the topological structure of the product topology, and for $j \in I$ the projection map $p_j : \prod_{i \in I} G_i \to G_j; x \mapsto x_j$ is a continuous open homomorphism. Furthermore, for each $j \in I$ there are continuous homomorphisms $\iota_j : G_j \to \prod_{i \in I} G_i$ such that G_j has the initial topology w.r.t. ι_j, and* *(i) if inversion is continuous on all of the G_is then it is continuous on $\prod_{i\in I} G_i$;*

(ii) if left (resp. right) multiplication is continuous on all the G_is then it is continuous on $\prod_{i\in I} G_i$;

(iii) and if multiplication is continuous on all of the G_i^2s then it is continuous on $(\prod_{i\in I} G_i)^2$.

In particular if G_i is a topological (resp. paratopological, quasitopological, semitopological, left-topological or right-topological) group for all $i \in I$ then so is $\prod_{i\in I} G_i$.

Proof. The first part is just a statement of the usual results concerning product groups and product topologies. The sets of the form $\prod_{i\in I} U_i$ where U_i is open for all $i \in I$, and $U_i = G_i$ for all but finitely many $i \in I$, form a base $\mathcal{B}$ for the product topology.

We define $\iota_j : G_j \to \prod_{i\in I} G_i$ by letting $\pi_j(\iota_j(x)) = x$ for all $x \in G_j$ and $\pi_i(\iota_j(x)) = 1_{G_i}$ for all $i \neq j$. This is a continuous homomorphism, and the initial topology on G_j w.r.t. ι_j is $\{\pi_j(U) : U \text{ is open in } \prod_{i\in I} G_i\}$ which is exactly the set of open subsets of G_j in its original topology. In other words the topology on G_j is initial as described.

For (i), by definition of inversion in the product group, $(\prod_{i\in I} U_i)^{-1} = \prod_{i\in I} U_i^{-1}$, and if $U_i = G_i$ then $U_i^{-1} = G_i$, so if inversion is continuous for all of the G_is and $\prod_{i\in I} U_i \in \mathcal{B}$ then $(\prod_{i\in I} U_i)^{-1} \in \mathcal{B}$ and the result is proved.

For (ii), given $x \in \prod_{i\in I} G_i$ and $\prod_{i\in I} U_i \in \mathcal{B}$ we have $x^{-1}\prod_{i\in I} U_i = \prod_{i\in I} x_i^{-1}U_i$ by definition of the group operation in the product group. $x_i^{-1}U_i$ is open for all $i \in I$ by Lemma 1.23, and if $U_i = G_i$ then $x_i^{-1}U_i = G_i$, so $x^{-1}\prod_{i\in I} U_i \in \mathcal{B}$. It follows that left multiplication by x is continuous. Similarly for right multiplication.

Finally, for (iii), if $\prod_{i\in I} U_i \in \mathcal{B}$ let $J \subset I$ be finite such that $U_i = G_i$ for all $i \notin J$. Then

$$V_i := \{(x_i, y_i) : x_i, y_i \in U_i\} = \bigcup\{S_i \times T_i : S_i \times T_i \in \mathcal{S}_i\}$$

for some set $\mathcal{S}_i$ of products of open sets in G_i, and if $U_i = G_i$ then $V_i = G_i \times G_i$ whence

$$\left\{(x, y) : xy \in \prod_{i\in I} U_i\right\} = \bigcup\left\{\prod_{i\in I} S_i \times \prod_{i\in I} T_i : \begin{array}{l} S_i \times T_i \in \mathcal{S}_i \text{ for all } i \in J \\ \text{and } S_i = T_i = G_i \text{ for all } i \notin J \end{array}\right\},$$

which is a union of sets in $\mathcal{B} \times \mathcal{B}$ and so open. The result is proved. □

We call the topologized group above the **topological direct product** of the groups $(G_i)_{i\in I}$, and given n topologized groups G, H, $\dots$, K, we write $G \times H \times \cdots \times K$ for $\prod_{i\in\{1,\dots,n\}} G_i$ where $G_1 := G$, $G_2 := H$, $\dots$, $G_n := K$.

Remark 2.12. Suppose that G is a group and $(G_i)_{i\in I}$ are copies of the group G with different topologies. Then we write G_Δ for the group G with initial topology w.r.t. the diagonal homomorphism $G \to \prod_{i\in I} G_i; x \mapsto (x, \dots, x)$. Concretely this is just the topology on G

generated by all the topologies together – that is the set of unions of sets of the form $\bigcap_{i\in J} U_i$ where $J \subset I$ is finite and U_i is open in G_i for all $i \in J$.

In particular, combining Propositions 2.8 & 2.11, if G_i is a topological (resp. paratopological, quasitopological, semitopological, left-topological or right-topological) group for all $i \in I$ then so is G_Δ.

Example 2.13 (Prime divisor topology on $\mathbb{Z}$)**.** By the preceding remark we may give $\mathbb{Z}$ the topology generated by the topological groups $\mathbb{Z}_{p-\text{DIV}}$ – that is $\mathbb{Z}$ with the p-divisor topology from Example 2.10 – as p ranges the primes. This makes $\mathbb{Z}$ into a topological group denoted $\mathbb{Z}_{\text{PD}}$.

The topology of $\mathbb{Z}_{\text{PD}}$ is Hausdorff: if $x \neq y$ then without loss of generality $(x+1)-y > 1$. Every natural number bigger than 1 has a smallest factor bigger than 1, and this factor will be prime, so there is a prime p with $x+1-y \in p\mathbb{Z}$. Then $x+p\mathbb{Z}\cap y+p\mathbb{Z} = (y-1)+p\mathbb{Z}\cap y+p\mathbb{Z}$. Since $p \nmid 1$ we have $y-1+p\mathbb{Z} \neq y+p\mathbb{Z}$ and since the intersection of two cosets is either a coset or empty we have that the open sets $x+p\mathbb{Z}$ and $y+p\mathbb{Z}$ are disjoint giving the claimed result.

Quotient groups

For G a topologized group and $H \leqslant G$, the **quotient topology** on left cosets G/H has $U \subset G/H$ open if and only if $\bigcup U$ is open in G; or, equivalently, $C \subset G/H$ closed if and only if $\bigcup C$ is closed in G.

Remark 2.14. This topology is the final topology on G/H w.r.t. the quotient map $q : G \to G/H; x \mapsto xH$ – it is the strongest topology (meaning finest topology, or topology with the most open sets) on G/H making q continuous.

Lemma 2.15. *Suppose that G is a left-topological group and $H \leqslant G$ is dense. Then G/H with the quotient topology is indiscrete.*

Proof. Suppose that $C \subset G/H$ is closed and non-empty. Then $\bigcup C$ is closed and contains xH for some $x \in G$. However $\overline{xH} = x\overline{H} = xG = G$ by Lemma 1.23, and so $\bigcup C = G$ and hence the only non-empty closed set in G/H is G/H and the result is proved. □

Proposition 2.16. *Suppose that G is a right-topological group and $H \leqslant G$. Then for G/H with the quotient topology, the quotient map $q : G \to G/H$ is open.*

Proof. If U is open in G then UH is open by Lemma 1.23 (for right-topological groups). But $\bigcup q(U) = UH$, and so $q(U)$ is open in G/H by definition. □

Quotient maps are not necessarily closed:

Example 2.17. $\mathbb{Q}$ is a dense subgroup of the real line $\mathbb{R}$ and so Lemma 2.15 tells us that $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology. The quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Q}; x \mapsto x + \mathbb{Q}$ is *not* closed since $q(\{0\}) = \{\mathbb{Q}\}$ is not closed in $\mathbb{R}/\mathbb{Q}$.

Corollary 2.18. *Suppose that G is a topologized group, and $H \leqslant G$. Then G/H with the quotient topology is compact (resp. σ-compact) if G is compact (resp. σ-compact); and if G is right-topological and locally compact, then G/H with the quotient topology is locally compact.*

Proof. The quotient map q is continuous and the continuous image of a compact set is compact, so if G is compact then so is G/H, and if G is σ-compact so that $G = \bigcup_{n\in\mathbb{N}_0} K_n$ for compact sets K_n, then $G/H = q(G) = \bigcup_{n\in\mathbb{N}_0} q(K_n)$ is σ-compact.

Now let $xH \in G/H$, and suppose G is locally compact. There is an open set U containing x and contained in a compact set K. Since q is open (Proposition 2.16), $q(U)$ is an open set containing xH and contained in $q(K)$. The latter is compact since q is continuous and so G/H is locally compact as claimed. □

Proposition 2.19. *Suppose that G is a topologized group and H is a normal subgroup of G. Then*

(i) if group inversion on G is continuous, then it is continuous on G/H;

(ii) if left (resp. right) multiplication is continuous on G, then it is continuous on G/H;

(iii) and if multiplication is continuous on G^2 then it is also on $(G/H)^2$.

In particular, if G is a topological (resp. paratopological, quasitopological, semitopological, left-topological or right-topological) group then so is G/H.

Proof. Suppose that $U \subset G/H$ is open. If inversion is continuous on G then

$$\bigcup U^{-1} = \bigcup \{(xH)^{-1} : xH \in U\} = \bigcup \{x^{-1}H : xH \in U\} = \left\{x^{-1} : x \in \bigcup U\right\} = \left(\bigcup U\right)^{-1}$$

and so U^{-1} is open in G/H by definition since $\bigcup U$ is open in G. If left multiplication on G is continuous, then for $x \in G$,

$$\bigcup (xH)^{-1}U = \bigcup \{(x^{-1}H)(yH) : yH \in U\} = \bigcup \{x^{-1}yH : yH \in U\} = x^{-1}\bigcup U,$$

and so $(xH)^{-1}U$ is open in G/H and hence left multiplication by xH is continuous. Similarly for right multiplication.

Finally suppose multiplication on G is continuous. Define

$$W := \{(zH, wH) \in (G/H)^2 : (zH)(wH) \in U\} \text{ and } V := \left\{(z, w) \in G^2 : zw \in \bigcup U\right\}.$$

Suppose that $(xH, yH) \in W$. Then $xy \in (xH)(yH) \subset \bigcup U$ so $(x, y) \in V$ and since V is open there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$, and $S \times T \subset V$. If $s \in S$ and $t \in T$, then $st \in \bigcup U$, and since the latter is a union of cosets of H we have $(st)H \subset \bigcup U$. Since H is normal we have $(sH)(tH) = (st)H \subset \bigcup U$, and so $SH \times TH \subset V$.

By Lemma 1.23, SH and TH are open sets, and so the sets $S' := \{sH : s \in S\}$ and $T' := \{tH : t \in T\}$ are open in G/H; $xH \in S'$ and $yH \in T'$; and $S' \times T' \subset W$. It follows that W is open, and multiplication on G/H is continuous. □

Example 2.20 (The real line modulo 1)**.** The real line $\mathbb{R}$ (Example 1.3) has a (normal) subgroup $\mathbb{Z}$ and so the group $\mathbb{R}/\mathbb{Z}$ may be given the quotient topology making it into a topological group by Proposition 2.19.

⚠In the literature on topological spaces (though not in these notes) the notation $\mathbb{R}/\mathbb{Z}$ is sometimes used to refer to a different space, called the adjunction space in which all the integers in $\mathbb{R}$ are identified but the rest of $\mathbb{R}$ remains the same. In other language this is a countably infinite bouquet of circles all connected at the point $\mathbb{Z}$.

Example 2.21 (The reals with the circle topology)**.** By Proposition 2.8 $\mathbb{R}$ is a topological group (which we shall denote $\mathbb{R}_C$) when endowed with the initial topology w.r.t. the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ where $\mathbb{R}/\mathbb{Z}$ is the reals $\pmod 1$ (Example 2.20). We call this the **circle topology** on $\mathbb{R}$. The open sets in the circle topology have the form $U + \mathbb{Z}$ where $U \subset \mathbb{R}$ is open in the real line.

Since $\mathbb{R}_C$ has the initial topology, a set $A \subset \mathbb{R}_C$ is compact if (and only if) $q(A)$ is compact in $\mathbb{R}/\mathbb{Z}$: Indeed, if $\mathcal{U}$ is an open cover of A, we can write $\mathcal{U} = \{q^{-1}(V) : V \in \mathcal{V}\}$ for some set $\mathcal{V}$ of open subsets of $\mathbb{R}/\mathbb{Z}$. Now, if $q(A)$ is compact then there are $V_1, \dots, V_n \in \mathcal{V}$ such that $q(A) \subset V_1 \cup \cdots \cup V_n$, and hence $A \subset q^{-1}(q(A)) \subset q^{-1}(V_1) \cup \cdots \cup q^{-1}(V_n)$ and so $\{q^{-1}(V_1), \dots, q^{-1}(V_n)\}$ is a finite subcover of $\mathcal{U}$.

⚠In particular, $A := [0, 1]$ and $B := [0, 1/2) \cup [3/2, 2]$ are compact, but $A \cap B = [0, 1/2)$ is not compact. This phenomenon of the intersection of two compact sets not being compact cannot happen in a Hausdorff space where every compact set is closed, and hence where the intersection of two compact sets is an intersection of a closed set with a compact set which is, therefore, compact.

The open mapping theorem

Example 2.5 shows that there are continuous bijective group homomorphisms that are not homeomorphic isomorphisms. This is by way of contrast with the purely algebraic situation where any bijective group homomorphism is a group isomorphism (*i.e.* has an inverse that is a homomorphism), but in alignment with the topological situation where continuous bijections need not be homeomorphisms. With a few mild conditions on the topology we can recover the algebraic situation:

Theorem 2.22 (Open Mapping Theorem)**.** *Suppose that G is a σ-compact left-topological group, H is a locally compact Hausdorff left-topological group, and $\pi : G \to H$ is a continuous bijective homomorphism. Then π is a homeomorphic isomorphism.*

Proof. Since the inverse of a bijective group homomorphism is a group isomorphism, it suffices to show that $\pi(C)$ is closed whenever C is closed in G. Let K_n be compact in G such that $G = \bigcup_{n\in\mathbb{N}} K_n$.

Claim. *There is some $n \in \mathbb{N}$ such that $\pi(K_n)$ is a neighbourhood.*

Proof. We use a Baire Category argument, though no familiarity with these is assumed. We construct a nested sequence of closed neighbourhoods inductively: Let U_0 be a compact (and so closed since H is Hausdorff) neighbourhood in H, and for $n \in \mathbb{N}$ let $U_n \subset \pi(K_n)^c \cap U_{n-1}$ be a closed neighbourhood.

This is possible since (by the inductive hypothesis) U_{n-1} is a neighbourhood and so contains an open neighbourhood V_{n-1}. But then $\pi(K_n)^c \cap V_{n-1}$ is open and non-empty since otherwise $\pi(K_n)$ contains a neighbourhood. It follows that $\pi(K_n)^c \cap U_{n-1}$ contains an open neighbourhood and so it contains a closed neighbourhood since H is locally compact and regular.

Now by the finite intersection property of the compact space U_0, the set $\bigcap_n U_n$ is non-empty. This contradicts surjectivity of π since $G = \bigcup_{n\in\mathbb{N}} K_n$ and the claim is proved. □

Claim. *If $X \subset H$ is compact then $\pi^{-1}(X)$ is compact.*

Proof. By the previous claim $\pi(K_n)$ contains a neighbourhood (and hence so does $x\pi(K_n)$ by Lemma 1.23) and the set $\{x\pi(K_n) : x \in H\}$ covers X, so by compactness of X there are elements $x_1, \ldots, x_m$ such that $X \subset \bigcup_{i=1}^m x_i\pi(K_n)$ and hence $\pi^{-1}(X) \subset \bigcup_{i=1}^m \pi^{-1}(x_i)K_n$ (by injectivity of π). $\pi^{-1}(x_i)K_n$ is compact by Lemma 1.23, and since a finite union of compact sets is compact it follows that $\pi^{-1}(X)$ is contained in a compact set. Finally, X is closed so $\pi^{-1}(X)$ is closed and a closed subset of a compact set is compact as required. □

Finally, suppose that $C \subset G$ is closed, and y is a limit point of $\pi(C)$. H is locally compact so y has a compact neighbourhood X. Now $\pi^{-1}(X)$ is compact and so $\pi^{-1}(X) \cap C$ is compact. But then $X \cap \pi(C)$ is compact since π is continuous, and hence closed since H is Hausdorff. But by design $y \in \overline{X \cap \pi(C)} = X \cap \pi(C) \subset \pi(C)$. □

Corollary 2.23. *Suppose that G is a countable locally compact Hausdorff topological group. Then G is discrete. In particular, if G is a compact Hausdorff topological group then G is either finite or uncountable.*

Proof. G_{D} is a σ-compact semitopological group by Example 1.2, and the identity map $G_{\mathrm{D}} \to G$ is a continuous bijective homomorphism (Example 2.5). Hence by the Open

Mapping Theorem this is a homeomorphism and so G is discrete. Finally, if G is compact and countable then it is compact and discrete and so finite. □

Remark 2.24. None of the hypotheses may be dropped: The real line is an example of an uncountable locally compact Hausdorff topological group that is not discrete (since singletons are not open), and the rationals as a topological subgroup are an example of a countable Hausdorff topological group that is not discrete. Finally, $\mathbb{Q}_{\mathrm{I}}$, that is $\mathbb{Q}$ with the indiscrete topological, is a countable (locally) compact topological group that is not discrete.

Remark 2.25. Furstenberg [Fur55] gave a proof that there are infinitely many primes in topological language and here we can dress this up in terms of topologized groups:

Suppose, for a contradiction, that there were finitely many primes. Then the topology of $\mathbb{Z}_{\mathrm{PD}}$ (from Example 2.13) is generated by a finite collection of finite topologies and so is itself a finite topology and hence compact. But we saw in Example 2.13 that it is Hausdorff so by Corollary 2.23 $\mathbb{Z}_{\mathrm{PD}}$ is finite giving the contradiction claimed.

3 Continuous complex-valued functions on topological groups

For a topological space X we write $C(X)$ for the set of continuous functions $X \to \mathbb{C}$. This is closed under pointwise addition and multiplication of functions and contains the constant functions, so it is a $\mathbb{C}$-algebra.

Remark 3.1. For any indiscrete space X, the space $C(X)$ contains *only* the constant functions; and for any discrete space X, the space $C(X)$ contains *all* functions $X \to \mathbb{C}$.

Constructing non-constant continuous functions

In view of the preceding remark it is not always possible to construct non-trivial continuous functions, but for topologies that are sufficiently rich in open sets we have a chance: First, the dyadic rationals in $[0,1]$ are the set $D := \bigcup_{i=0}^{\infty} D_i$, where

$$D_0 := \{0/1, 1/2\}, D_1 := \{0/2, 1/2, 2/2\}, D_2 := \{0/4, 1/4, 2/4, /3/4, 4/4\}, \&c.$$

In particular D is dense in $[0,1]$; we have the nesting $D_0 \subset D_1 \subset \dots$; and every element of $D_{i+1} \setminus D_i$ can be written in the form $\frac{1}{2}(q + q')$ where q and q' are *consecutive* elements of D_i.

Write S° for the interior of S, that is the set of $x \in S$ that are contained in an open set contained in S. In particular S° is an open subset of S, and $S^\circ = S$ if and only if S is open.

Example 3.2 (The real line contd., Example 1.3)**.** ⚠The interior of the closure of B may be very different to B, even if B is open: there are open dense subsets S of $[0,1]$ of arbitrarily small measure, so that $\overline{S}^\circ = (0,1)$, but S itself is much smaller.

Proposition 3.3. *Suppose that G is a paratopological group, $A, B \subset G$ with A compact and $\overline{A} \subset \overline{B}^{\circ}$. Then there are sets $(U_q)_{q \in D}$ with $U_0 = A$, $U_1 = B$, and $\overline{U_q} \subset \overline{U_{q'}}^{\circ}$ whenever $q, q' \in D$ have $q < q'$.*

Proof. By Lemma 1.41 there is an open neighbourhood of the identity U such that $AU \subset B$, and by Lemma 1.34 there are open neighbourhoods V_i of the identity such that $V_{i+1}^2 \subset V_i$ for all $i \in \mathbb{N}_0$ and $V_0^2 \subset U$.

We set $U_0 := A$ and $U_1 := B$ and define U_q for $q \in D_{i+1} \backslash D_i$ iteratively for $i \in \mathbb{N}_0$. Suppose that at step i, for all consecutive pairs $q < q'$ in D_i we have $\overline{U_q} V_i \subset \overline{U_{q'}}$ – this is certainly true for $i = 0$. For $q < q'$ consecutive elements of D_i set $U_{\frac{1}{2}(q+q')} := \overline{U_q} V_{i+1}$ so that a) $\overline{U_q} V_{i+1} \subset \overline{U_{\frac{1}{2}(q+q')}}$; and b) $\overline{U_{\frac{1}{2}(q+q')}} V_{i+1} \subset \overline{\overline{U_q} V_{i+1}} V_{i+1} \subset \overline{\overline{U_q} V_{i+1}^2} \subset \overline{\overline{U_{q'}}} = \overline{U_{q'}}$ by Lemma 1.23. Every element of $D_{i+1} \backslash D_i$ is the average of two consecutive elements of D_i, and the result is proved. □

Proposition 3.4. *Suppose that X is a locally compact regular topological space, $A, B \subset X$ with A compact and $\overline{A} \subset \overline{B}^{\circ}$ Then there are open sets $(U_q)_{q \in D}$ with $A \subset U_0$, $U_1 \subset B$, and $\overline{U_q} \subset \overline{U_{q'}}^{\circ}$ whenever $q, q' \in D$ have $q < q'$.*

Proof. We set $U_0 := A$ and $U_1 := B$ and define U_q for $q \in D_{i+1} \backslash D_i$ iteratively for $i \in \mathbb{N}_0$. Suppose that at step i, for all consecutive pairs $q < q'$ in D_i we have $\overline{U_q} \subset \overline{U_{q'}}^{c} irc$ and $\overline{U_q}$ compact – this is certainly true for $i = 0$ by Lemma 1.48. For $q < q'$ consecutive elements of D_i, apply Lemma 1.50 to get an open set $U_{\frac{1}{2}(q+q')}$ containing $\overline{U_q}$ and contained in $\overline{U_{q'}}^{\circ}$, with compact closure, and the result is proved. □

Lemma 3.5. *Suppose that X is a topological space, and $(U_q)_{q \in D}$ are such that $\overline{U_q} \subset \overline{U_{q'}}^{\circ}$ whenever $q < q'$. Then there is a continuous function $g : X \to [0, 1]$ with $g(x) = 0$ for $x \in \overline{U_0}$, and $g(x) = 1$ for $x \notin \overline{U_1}^{\circ}$.*

Proof. For $x \in G$ let $S(x) := \{q \in D : x \in \overline{U_q}^{\circ}\}$ and define $g : G \to [0, 1]$ by $g(x) := \inf S(x) \cup \{1\}$. This certainly maps into $[0, 1]$. If $x \in \overline{U_0}$ then $q \in S(x)$ for all $q > 0$ by nesting, and hence $g(x) = 0$; if $x \notin \overline{U_1}^{\circ}$, then $S(x) = \varnothing$ by nesting, and so $g(x) = 1$. It remains to establish that g is continuous: First, for $\alpha \leqslant 1$ we have $g^{-1}([0, \alpha)) = \bigcup \{\overline{U_q}^{\circ} : q < \alpha\}$ is open. The harder case is showing for $\alpha \geqslant 0$ that $g^{-1}((\alpha, 1])$ is open; suppose that $g(x_0) > \alpha$. Then there is $q \in D$ with $g(x_0) > q > \alpha$, and hence $x_0 \notin \overline{U_q}^{\circ}$. Now, if $z \notin \overline{U_q}$ then by nesting $g(z) \geqslant q > \alpha$, and hence $x_0 \in G \backslash \overline{U_q} \subset g^{-1}((\alpha, 1])$. Thus every element of $g^{-1}((\alpha, 1])$ is contained in an open subset of $g^{-1}((\alpha, 1])$, and so $g^{-1}((\alpha, 1])$ itself must be open. Finally, the half-open sets $(\alpha, 1]$ and $[0, \alpha)$ for $\alpha \in [0, 1]$ form a base for the topology on $[0, 1]$ and hence g is continuous as required. □

Corollary 3.6. *Suppose that G is a regular paratopological group, and $A \subset B$ are compact and open sets respectively. Then there is a continuous function $g : G \to [0, 1]$ with $g(x) = 0$ for all $x \in A$ and $g(x) = 1$ for all $x \notin B$.*

Proof. Since the topology is regular and A is compact there is an open set C containing A with $\overline{C} \subset B$. Apply Proposition 3.3 and then Lemma 3.5 to get $g : G \to [0,1]$ continuous with $g(x) = 0$ for all $x \in \overline{A}$, and $g(x) = 1$ for all $x \notin \overline{C}^{\circ}$. The result follows since $\overline{C}^{\circ} \subset B$. □

Remark 3.7. We know from Proposition 1.45 that every topological group is regular, hence every topological group is a regular paratopological group so in particular the above corollary applies to all topological groups. It is sometimes called the 'complete regularity of topological groups'.

Example 3.8. For the rationals (Example 1.4), the function $g : \mathbb{Q} \to \mathbb{C}$ with $g(x) = 0$ if $x^2 < 2$ and $g(x) = 1$ if $x^2 > 2$ is continuous because for every $x \in \mathbb{Q}$ there is an interval on which it is constant (on the rationals in that interval), and it has $g(0) = 0$ and $g(x) = 1$ for all x not in the open set $\{y \in \mathbb{Q} : y^2 > 2\}$. This is not the sort of function that is constructed by the argument of Corollary 3.6, and suffers from not having a continuous extension to the reals because it is not *uniformly* continuous.

Remark 3.9. ⚠Corollary 3.6 does *not* assume that G is not indiscrete so that there may not be *any* non-constant continuous functions. Exercise II.9 asks for a proof that if G is a topological group and the only continuous functions are constant then G is indiscrete, and also for examples to show how things differ for quasitopological and paratopological groups.

Corollary 3.10. *Suppose that X is a locally compact regular topological space, and $A \subset B$ are compact and open sets respectively. Then there is a continuous function $g : G \to [0,1]$ with $g(x) = 0$ for all $x \in A$ and $g(x) = 1$ for all $x \notin B$.*

Proof. By exactly the same argument as Corollary 3.6 with Proposition 3.4 replacing Proposition 3.3. □

Compactly supported continuous functions

Given a topological space X the **support** of a (not necessarily continuous) function $f : X \to \mathbb{C}$, denoted $\operatorname{supp} f$, is the set of $x \in X$ such that $f(x) \neq 0$; f is said to be **compactly supported** if its support is contained in a compact set. We write $C_c(X)$ for the subset of functions in $C(X)$ that are compactly supported. ⚠As we have defined it the support of a function that is compactly supported need not actually *be* a compact set; it is simply contained in one.

Remark 3.11. The set $C_c(X)$ is a subalgebra of $C(X)$ since the union of two compact sets is compact and the support of the sum of two functions is contained in the union of their supports; and the support of the product of two functions is the intersection of their supports which is certainly contained in a compact set if one is. More than this, the function

$$\|f\|_\infty := \sup \{|f(x)| : x \in X\}$$

is a norm on $C_c(X)$. It is well-defined since every continuous (complex-valued) function on a compact set is bounded, and the axioms of a norm are easily checked.

⚠In general $\|\cdot\|_\infty$ is *not* a norm on $C(X)$ since we are not assuming the elements of $C(X)$ are bounded.

⚠In general $C_c(X)$ is *not* complete despite the fact that the uniform limit of continuous functions is continuous since this limit function may not be compactly supported.

Remark 3.12. ⚠If $f, g \in C(X)$ then the support of g is open and there is a continuous function $h : \operatorname{supp} g \to \mathbb{C}$ such that $f = gh$, but in general this need[4] *not* have a continuous extension to the whole of X. By way of contrast, if $f, g \in C_c(X)$ and $\overline{\operatorname{supp} f} \subset \operatorname{supp} g$ then there is $h \in C_c(X)$ such that $f = gh$.

Proposition 3.13. *Suppose that G is a left-topological group and $C_c(G)$ contains a function that is not identically zero. Then G is locally compact.*

Proof. Suppose that $f \in C_c(G)$ is not identically zero. Then $\operatorname{supp} f$ is open (since f is continuous), non-empty and contained in a compact set K (since f is compactly supported). It follows that K is a compact neighbourhood of some point $x \in G$, and by Lemma 1.23 $yx^{-1}K$ is then a compact neighbourhood of y for $y \in G$ as required. □

Example 3.14. ⚠For the rationals (Example 1.4) we have $C_c(\mathbb{Q}) = \{0\}$.

For us Corollary 3.6 will be crucial in providing a supply of compactly supported functions in locally compact topological groups.

Corollary 3.15. *Suppose that G is a locally compact topological group and $K \subset G$ is compact. Then there is a continuous compactly supported $f : G \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$.*

Proof. Since G is locally compact it contains a compact neighbourhood of the identity L; let $H \subset L$ be an open neighbourhood of the identity. KH is open by Lemma 1.23. Apply Corollary 3.6 to get a continuous $f : G \to [0, 1]$ with $f(x) = 1$ for all $x \in K$ and $\operatorname{supp} f \subset KH \subset KL$ which is compact by Lemma 1.36. □

Furthermore, we can produce continuous partitions of unity:

Corollary 3.16. *Suppose that G is a locally compact topological group, $F : G \to [0, 1]$ is continuous, K is a compact set containing the support of F, and $\mathcal{U}$ is an open cover of K. Then there is some $n \in \mathbb{N}$ and continuous compactly supported functions $f_1, \dots, f_n : G \to [0, 1]$ such that $F = f_1 + \cdots + f_n$; and for each $1 \leqslant i \leqslant n$ there is $U_i \in \mathcal{U}$ such that* $\operatorname{supp} f_i \subset U_i$.

[4]Consider, for example, the functions $f(x) = x$ and $g(x) = x^2$ in $C(\mathbb{R})$. Then $h(x) = 1/x$ for all $x \in \operatorname{supp} g$ but h has no continuous extension to $\mathbb{R}$.

Proof. Since $\mathcal{U}$ is an open cover of K, for each $x \in K$ there is an open neighbourhood of x, call it $U_x \in \mathcal{U}$, and by Proposition 1.45 there is a closed neighbourhood $V_x \subset U_x$ of x. Since each V_x is a neighbourhood and $\{V_x : x \in K\}$ is a cover of K, compactness tells us that there are elements $x_1, \dots, x_n$ such that $K \subset V_{x_1} \cup \dots \cup V_{x_n}$. By Lemma 1.48 $\overline{K}$ is compact and so for each i the set $V_{x_i} \cap \overline{K}$ is a closed subset of a compact set and so compact. Apply Corollary 3.6 to $V_{x_i} \cap \overline{K} \subset U_{x_i}$ to get a continuous function $g_i : G \to [0,1]$ such that $g_i(x) = 1$ for all $x \in V_{x_i} \cap \overline{K}$ and $\operatorname{supp} g_i \subset U_{x_i}$.

Since the sets $V_{x_1}, \dots, V_{x_n}$ are closed, $\overline{K} \subset V_{x_1} \cup \dots \cup V_{x_n}$, and so since the g_is are non-negative we have

$$\overline{\operatorname{supp} F} \subset \overline{K} \subset (V_{x_1} \cap \overline{K}) \cup \dots \cup (V_{x_n} \cap \overline{K}) \subset \operatorname{supp}(g_1 + \dots + g_n).$$

Thus (see Remark 3.12) there is $h \in C_c(G)$ such that $F = h(g_1 + \dots + g_n)$ and since F maps into $[0,1]$ and $g_1(x) + \dots + g_n(x) \geqslant 1$ on the support of F, we conclude that h maps into $[0,1]$; for $1 \leqslant i \leqslant n$ put $f_i = g_i h$.

It remains to check the properties of the f_is. First, f_i is a continuous function $G \to [0,1]$ by design of h and g_i. Secondly, $F = f_1 + \dots + f_n$ by design. Finally, $\operatorname{supp} f_i \subset \operatorname{supp} g_i \subset U_{x_i} \in \mathcal{U}$. Moreover, since the f_is are non-negative $\operatorname{supp} f_i \subset K$ so f_i has compact support. The result is proved. □

Integrals of continuous functions

We say that a complex-valued function f from a set X is **non-negative** if $f(x) \geqslant 0$ for all $x \in X$; we say a linear functional $\int$ from a complex vector space of complex-valued functions V is **non-negative** if $\int f \geqslant 0$ whenever f is non-negative.

Our motivating example of an integral is the Riemann integral:

Example 3.17. The set R of Riemann integrable functions $\mathbb{R} \to \mathbb{C}$ has some basic properties often established in first courses on analysis *e.g.* [Gre20]. In particular, R is a complex vector space under point-wise addition and scalar multiplication of functions, and

$$\int : R \to \mathbb{C}; f \mapsto \int_{-\infty}^{\infty} f(x)dx$$

is a non-negative linear map. Furthermore, $C_c(\mathbb{R})$ is a subspace of R, and $\int$ restricted to $C_c(\mathbb{R})$ is non-trivial (meaning not identically zero).

Remark 3.18. ⚠ We are only concerned with proper integrals, and though the integral in $\int$ appears to be improper we are restricting attention to compactly supported functions so the integrals are, in fact, proper.

Remark 3.19. Non-triviality of $\int$ when *restricted* to $C_c(\mathbb{R})$ is important; see Exercise III.7 for a contrasting situation.

Given a topological space X if $f, g \in C_c(X)$ are both real-valued then we write $f \geqslant g$ if $f - g$ is non-negative, and $C_c^+(X)$ for the set of $f \in C_c(G)$ such that $f \geqslant 0$, where 0 is the constant 0 function.

Remark 3.20. The functions $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Re} z$, $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Im} z$, $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{x, 0\}$ and $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{-x, 0\}$ are continuous and so any $f \in C_c(X)$ can be written as $f = f_1 - f_2 + if_3 - if_4$ for $f_1, f_2, f_3, f_4 \in C_c^+(X)$, and this decomposition is unique. We shall frequently have call to understand elements of $C_c(X)$ through this linear combination of elements of $C_c^+(X)$.

Remark 3.21. If $f, g \in C_c(X)$ are real-valued with $f \geqslant g$ and $\int$ is a non-negative linear functional $C_c(X) \to \mathbb{C}$ then $\int f \geqslant \int g$; and if $f \in C_c(G)$ then $|\int f| \leqslant \int |f|$.

Remark 3.22. The decomposition in Remark 3.20 can be used to show that if $\int$ is a non-negative linear functional then $\overline{\int f} = \int \overline{f}$ for all $f \in C_c(X)$.

Remark 3.23. We think of non-negative linear functionals as integrals and in fact the Riesz-Markov-Kakutani Representation Theorem tells us that if X has a sufficiently nice topology then *every* non-negative linear map $C_c(X) \to \mathbb{C}$ arises as an integral against a suitably well-behaved measure on X.

Remark 3.24. Suppose that $f \in C(X)$. By the triangle inequality if $\Delta := \{z \in \mathbb{C} : |z| < \epsilon/2\}$ and $f(x), f(y) \in z + \Delta$ then $|f(x) - f(y)| < \epsilon$ and hence $\mathcal{U} := \{f^{-1}(z + \Delta) : z \in \mathbb{C}\}$ is an open cover of X such that if $U \in \mathcal{U}$ and $x, y \in U$ then $|f(x) - f(y)| < \epsilon$.

Given a further topological space Y and $F : X \times Y \to \mathbb{C}$ and $x \in X$, we write $\int_y F(x, y)$ for the functional $\int : C_c(Y) \to \mathbb{C}$ applied to the function $Y \to \mathbb{C}; y \mapsto F(x, y)$ (assuming this function is continuous and compactly supported), and similarly for $y \in Y$ and $\int_x F(x, y)$. It will be crucial for us that the order of integration can be interchanged and this is what the next result concerns:

Theorem 3.25 (Fubini's Theorem for continuous compactly supported functions)**.** *Suppose that G is a locally compact topological group, $\int$ and $\int'$ are non-negative linear functionals $C_c(G) \to \mathbb{C}$, and $F \in C_c(G \times G)$. Then the map $x \mapsto \int'_y F(x, y)$ is continuous and compactly supported, so that $\int_x \int'_y F(x, y)$ exists. Similarly $y \mapsto \int_x F(x, y)$ is continuous and compactly supported, so that $\int'_y \int_x F(x, y)$ exists and moreover*

$$\int_x \int'_y F(x, y) = \int'_y \int_x F(x, y).$$

Proof. In view of the decomposition in Remark 3.20 and linearity of $\int$ and $\int'$ it is enough to establish the result for F non-negative.

Since $F \in C_c^+(G \times G)$ has support contained in a compact set K, and since the coordinate projection maps $G \times G \to G$ are continuous (and the union of two compact sets is compact) there is a compact set L such that $K \subset L \times L$. It follows that the maps $x \mapsto F(x, y)$ for $y \in G$ and $y \mapsto F(x, y)$ for $x \in G$ are continuous and have support in the compact set L.

We also need an auxiliary 'dominating function' which is a compactly supported continuous function on whose support all of the 'action' happens. For those familiar with the theory of integration, the Dominated Convergence Theorem may come to mind. Concretely, by Corollary 3.15 there is a continuous function $f : G \to [0, 1]$ with $f(x) = 1$ for all $x \in L$ supported in a compact set M.

For $\epsilon > 0$ (by Remark 3.24) let $\mathcal{U}$ be an open cover of $G \times G$ such that $|F(x, y) - F(x', y')| < \epsilon$ for all $(x, y), (x', y') \in U \in \mathcal{U}$. $M \times M$ is compact and so by Lemma 1.41 there is a symmetric open neighbourhood of the identity U in G such that $\mathcal{U}' := \{xU \times yU : x, y \in M\}$ is a refinement of $\mathcal{U}$ (as a cover of $M \times M$ *not* of $G \times G$). First, the support of $\int'_y F(x, y)$ is contained in the (compact) set L and if $x' \in xU$ then by design and non-negativity of $\int'$ we have

$$\int'_y F(x', y) = \int'_y F(x', y)f(y) \leqslant \int'_y (F(x, y) + \epsilon)f(y) = \int'_y F(x, y) + \epsilon \int' f.$$

Since U is symmetric we have $x \in x'U$ and similarly $\int'_y F(x, y) \leqslant \int'_y F(x', y) + \epsilon \int' f$ and hence $|\int'_y F(x', y) - \int'_y F(x, y)| \leqslant \epsilon \int' f$. Since ϵ is arbitrary (and $\int' f$ does not depend on ϵ) it follows that $x \mapsto \int'_y F(x, y)$ is continuous (and compactly supported) and similarly for $y \mapsto \int_x F(x, y)$.

By Corollary 3.16 applied to f supported on the compact set M with the open cover $\{xU : x \in M\}$, there are continuous compactly supported $f_1, \ldots, f_n : G \to [0, 1]$ such that $f_1 + \cdots + f_n = f$ and $\operatorname{supp} f_i \subset x_iU$ for some $x_i \in M$. Now, $F(x, y) = F(x, y)f(x)f(y)$ and $f = f_1 + \cdots + f_n$, so

$$F(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{n} F(x, y)f_i(x)f_j(y) \text{ for all } x, y \in G.$$

By design of $\mathcal{U}'$ and $\mathcal{U}$, for $1 \leqslant i, j \leqslant n$ there is $\lambda_{i,j} \geqslant 0$ such that $|F(x, y) - \lambda_{i,j}| < \epsilon$ for all $(x, y) \in \operatorname{supp} f_i \times \operatorname{supp} f_j$. We conclude that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} f_i(x)f_j(y) - \epsilon f(x)f(y) \leqslant F(x, y) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} f_i(x)f_j(y) + \epsilon f(x)f(y).$$

Since $\int$ and $\int'$ are non-negative linear functionals, we conclude that

$$\left|\int_x \int'_y F(x, y) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \int f_i \int' f_j\right| \leqslant \epsilon \int f \int' f$$

and

$$\left|\int'_y \int_x F(x, y) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \int f_i \int' f_j\right| \leqslant \epsilon \int f \int' f.$$

The result is proved by the triangle inequality since ϵ is arbitrary (and $\int f$ and $\int' f$ do not depend on ϵ). □

Remark 3.26. ⚠ It is *not* enough to assume that $F : G \times G \to \mathbb{C}$ is such that the maps $G \to \mathbb{C}; x \mapsto \int'_y F(x, y)$ and $G \to \mathbb{C}; y \mapsto \int_x F(x, y)$ are well-defined, continuous, and compactly supported. Exercise III.4 asks for an example.

4 The Haar integral

We now turn to one of the most beautiful aspects of the theory of topological groups. This describes the way the topology and the algebra naturally conspire to produce an integral. Given a topological group G and a function $f \in C(G)$ we write

$$\lambda_x(f)(z) := f(x^{-1}z) \text{ for all } x, z \in G.$$

Remark 4.1. $\lambda_x(f) \in C(G)$ for all $f \in C(G)$ and $x \in G$ (since left multiplication is continuous and the composition of continuous functions is continuous), and λ is a left action meaning $\lambda_{xy}(f) = \lambda_x(\lambda_y(f))$ for all $x, y \in G$ and $\lambda_{1_G}(f) = f$, and the maps λ_x are linear on the vector space $C(G)$. ⚠ Without inversion this is naturally a right action.

Remark 4.2. For a topological group G, λ restricts to an action on the space $C_c(G)$ and this action is isometric with respect to $\|\cdot\|_\infty$ *i.e.* $\|\lambda_x(f)\|_\infty = \|f\|_\infty$ for all $x \in G$.

Proposition 4.3. *Suppose that G is a topological group and $f \in C_c(G)$. Then $G \to C_c(G); x \mapsto \lambda_x(f)$ is continuous.*

Proof. Let $W \subset C_c(G)$ be open and suppose $x \in G$ has $\lambda_x(f) \in W$. Since W is open there is $\epsilon > 0$ such that $\lambda_{x'}(f) \in W$ whenever $\|\lambda_{x'}(f) - \lambda_x(f)\|_\infty < \epsilon$. We shall show that there is an open neighbourhood of the identity V such that $\|\lambda_{x'}(f) - \lambda_x(f)\|_\infty < \epsilon$ for all $x' \in xV$ from which the result follows.

Let K be a compact set containing the support of f. As in Remark 3.24 let $\mathcal{U}$ be an open cover of G such that $|f(y) - f(y')| < \epsilon$ for all $y, y' \in U \in \mathcal{U}$. By Lemma 1.41 there is a symmetric open neighbourhood of the identity V such that $\{Vy : y \in K\}$ is a refinement of $\mathcal{U}$ (as a cover of K).

Suppose that $v \in V$ and $y \in G$ is such that $\lambda_v(f)(y) - f(y) \neq 0$. Then either $f(y) \neq 0$ so $y \in K$, but then $V^{-1}y = Vy$ is a subset of an element of $\mathcal{U}$ and so $|\lambda_v(f)(y) - f(y)| < \epsilon$; or $\lambda_v(f)(y) \neq 0$ so $v^{-1}y \in K$, but then $V(v^{-1}y)$ is a subset of an element of $\mathcal{U}$ and so again $|\lambda_v(f)(y) - f(y)| < \epsilon$. Since $\lambda_v(f) - f$ is continuous and compactly supported it attains its bounds so $\|\lambda_v(f) - f\|_\infty < \epsilon$. Finally, since λ is an action, the map λ_x is linear, and this action is isometric (Remark 4.2) we have

$$\|\lambda_{xv}(f) - \lambda_x(f)\|_\infty = \|\lambda_x(\lambda_v(f) - f)\|_\infty = \|\lambda_v(f) - f\|_\infty < \epsilon.$$

The result is proved. □

Given a topological group G we say that $\int : C_c(G) \to \mathbb{C}$ is a **(left) Haar integral** on G if $\int$ is a non-trivial (meaning not identically zero) non-negative linear map with

$$\int \lambda_x(f) = \int f \text{ for all } x \in G \text{ and } f \in C_c(G).$$

We sometimes call this last property **(left) translation invariance**.

Remark 4.4. Our definition of Haar integral requires $C_c(G)$ to be non-trivial and hence (*c.f.* Proposition 3.13) for G to support a Haar integral it must be locally compact. It will turn out in Theorem 4.11 that this is enough to guarantee that there is a Haar integral.

Remark 4.5. There is an analogous notion of right Haar integral which we shall not pursue here.

Example 4.6 (The Riemann Integral)**.** The map $\int$ in Example 3.17 restricted to $C_c(\mathbb{R})$ is a Haar integral, with the only property not already recorded being translation-invariance.

Example 4.7. If G is a discrete group then it supports a left Haar integral:

$$\int : C_c(G) \to \mathbb{C}; f \mapsto \sum_{x \in G} f(x).$$

Remark 4.8. See Exercise III.1 for a partial converse.

The integral of a non-negative continuous function that is not identically 0 is positive, and this already follows from the axioms of a Haar integral. To establish this we begin with a lemma on the comparability of functions:

Lemma 4.9. *Suppose that G is a topological group, $f, g \in C_c^+(G)$ and f is not identically zero. Then there is $n \in \mathbb{N}$, $c_1, \dots, c_n \geqslant 0$ and $y_1, \dots, y_n \in G$ such that*

$$g(x) \leqslant \sum_{i=1}^n c_i \lambda_{y_i}(f)(x) \text{ for all } x \in G.$$

Proof. Since f is not identically zero there is some $x_0 \in G$ such that $f(x_0) > 0$ and hence (by Lemma 1.23) an open neighbourhood of the identity U such that $f(x_0 y) > f(x_0)/2$ for all $y \in U$. Let K be compact containing the support of g. Then $\{xU : x \in K\}$ is an open cover of K and so there are elements $x_1, \dots, x_n$ such that $x_1U, \dots, x_nU$ covers K. But then

$$g(x) \leqslant 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^n f(x_0 x_i^{-1} x) = 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^n \lambda_{x_i x_0^{-1}}(f)(x) \text{ for all } x \in G,$$

and the result is proved. □

Corollary 4.10. *Suppose that G is a topological group, $\int$ is a left Haar integral on G, and $f \in C_c^+(G)$ has $\int f = 0$. Then f is identically zero.*

Proof. We suppose, for a contradiction, that f is not identically zero. Then by Lemma 4.9 for $g \in C_c^+(G)$ we have $g \leqslant \sum_{i=1}^n c_i \lambda_{y_i}(f)$ for $c_1, \ldots, c_n \geqslant 0$ and $y_1, \ldots, y_n \in G$. By linearity, non-negativity, and translation invariance of the Haar integral

$$\int g \leqslant \sum_{i=1}^n c_i \int \lambda_{y_i}(f) = \sum_{i=1}^n c_i \int f = 0.$$

Since $g \geqslant 0$, non-negativity of the Haar integral implies $\int g \geqslant 0$, and hence $\int g = 0$.

Now, in view of Remark 3.20 we have that $\int h = 0$ for all $h \in C_c(G)$ *i.e.* $\int$ is identically zero contradicting the non-triviality of the Haar integral. The lemma follows. □

Existence of a Haar Integral

Our first main aim is to establish the following.

Theorem 4.11 (Existence of a Haar integral)**.** *Suppose that G is a locally compact topological group. Then there is a left Haar integral on G.*

We begin by defining a sort of approximation: for $f, \phi \in C_c^+(G)$ with ϕ not identically 0 put

$$(f;\phi) := \inf \left\{ \sum_{j=1}^n c_j : n \in \mathbb{N}; c_1, \ldots, c_n \geqslant 0; y_1, \ldots, y_n \in G; \text{ and } f \leqslant \sum_{j=1}^n c_j \lambda_{y_j^{-1}}(\phi) \right\}. \quad (4.1)$$

We think of this as a sort of 'covering number' and begin with some basic properties:

Lemma 4.12. *Suppose that $f, g, \phi, \psi \in C_c^+(G)$ with ϕ and ψ are not identically* 0*. Then*

(i) $(f;\phi)$ *is well-defined;*

(ii) $(\phi;\phi) \leqslant 1$*;*

(iii) $(f;\phi) \leqslant (g;\phi)$ *whenever* $f \leqslant g$*;*

(iv) $(f+g;\phi) \leqslant (f;\phi) + (g;\phi)$*;*

(v) $(\mu f;\phi) = \mu(f;\phi)$ *for* $\mu \geqslant 0$*;*

(vi) $(\lambda_x(f);\phi) = (f;\phi)$ *for all* $x \in G$*;*

(vii) $(f;\psi) \leqslant (f;\phi)(\phi;\psi)$*.*

Proof. Lemma 4.9 shows that the set on the right of (4.1) is non-empty; it has 0 as a lower bound. (i) follows immediately. For (ii)[5] note that $\phi \leqslant 1.\lambda_{1_G^{-1}}(\phi)$ so that $(\phi;\phi) \leqslant 1$. (iii), (iv), (v), and (vi) are all immediate. Finally, for (vii) suppose $c_1, \ldots, c_n \geqslant 0$ are such that $f \leqslant \sum_{j=1}^n c_j \lambda_{y_j^{-1}}(\phi)$, so that by (iii), (iv), (v), and (vi) we have $(f;\psi) \leqslant \sum_{j=1}^n c_j(\phi;\psi)$. The result follows on taking infima. □

[5]As it happens it is easy to prove equality here but we do not need it.

To make use of $(\cdot\,;\cdot)$ we need to fix a non-zero reference function $f_0 \in C_c^+(G)$ and for $\phi \in C_c^+(G)$ not identically zero we put

$$I_\phi(f) := \frac{(f;\phi)}{(f_0;\phi)} \leqslant (f; f_0), \tag{4.2}$$

where the inequality follows from Lemma 4.12 (vii).

Many of the properties of Lemma 4.12 translate into properties of I_ϕ. In particular, we have $I_\phi(f_1 + f_2) \leqslant I_\phi(f_1) + I_\phi(f_2)$; for suitable ϕ we also have the following converse.

Lemma 4.13. *Suppose that G is a locally compact topological group, $f_1, f_2 \in C_c^+(G)$ and $\epsilon > 0$. Then there is a symmetric open neighbourhood of the identity V such that if $\phi \in C_c^+(G)$ is not identically 0 and has support in V then $I_\phi(f_1) + I_\phi(f_2) \leqslant I_\phi(f_1 + f_2) + \epsilon$.*

Proof. Let K be a compact set containing the support of both f_1 and f_2 (possible since the union of two compact sets is compact) and apply Corollary 3.15 to get $F : G \to [0, 1]$ continuous, compactly supported, and with $F(x) = 1$ for all $x \in K$.

For $j \in \{1, 2\}$ let g_j be continuous such that $(f_1 + f_2 + \epsilon F)g_j = f_j$ (possible in view of Remark 3.12 and use that $\overline{\operatorname{supp} f_i} \subset \overline{K} \subset F^{-1}(\{1\}) \subset \operatorname{supp} F$). By Remark 3.24 (and the fact that the intersection of two open covers is an open cover) there is an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$ then $|g_j(x) - g_j(y)| < \epsilon$ for $j \in \{1, 2\}$. K is compact; apply Lemma 1.41 to $\mathcal{U}$ to get a symmetric open neighbourhood of the identity V such that $\{yV : y \in K\}$ refines $\mathcal{U}$ (as a cover of K).

Now suppose that $\phi \in C_c^+(G)$ is not identically 0 and has support in V, and that $c_1, \ldots, c_n \geqslant 0$ and $y_1, \ldots, y_n \in G$ are such that

$$f_1(x) + f_2(x) + \epsilon F(x) \leqslant \sum_{i=1}^n c_i \phi(y_i x) \text{ for all } x \in G.$$

If $\phi(y_i x)g_j(x) \neq 0$ then $x \in K$ and $y_i^{-1} \in xV$ (using $V = V^{-1}$), but xV is a subset of a set in $\mathcal{U}$ so $g_j(x) \leqslant g_j(y_i^{-1}) + \epsilon$ and hence

$$f_j(x) \leqslant \sum_{i=1}^n c_i \phi(y_i x) g_j(x) \leqslant \sum_{i=1}^n c_i (g_j(y_i^{-1}) + \epsilon)\phi(y_i x) \text{ for all } x \in G, j \in \{1, 2\}.$$

By Lemma 4.12 (ii),(iii), (iv),(v) & (vi) we have

$$(f_j; \phi) \leqslant \sum_{i=1}^n c_i (g_j(y_i^{-1}) + \epsilon) \text{ for all } j \in \{1, 2\},$$

but $g_1(y^{-1}) + g_2(y^{-1}) \leqslant 1$ for all $y \in G$, so

$$(f_1; \phi) + (f_2; \phi) \leqslant \sum_{i=1}^n c_i (1 + 2\epsilon).$$

Taking infima and then applying Lemma 4.12 (iv) and (v) and the inequality in (4.2) we get

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) &\leqslant (1+2\epsilon)I_\phi(f_1+f_2+\epsilon F)\\ &\leqslant (1+2\epsilon)(I_\phi(f_1+f_2)+\epsilon I_\phi(F))\\ &\leqslant I_\phi(f_1+f_2) + (2(f_1+f_2;f_0)+(F;f_0)+2\epsilon(F;f_0))\epsilon. \end{aligned}$$

The result follows since $\epsilon > 0$ was arbitrary and F, f_1, f_2 and f_0 do not depend on ϵ. □

With these lemmas we can turn to the main argument.

Proof of Theorem 4.11. By Corollary 3.15 (applied with $K = \{1_G\}$) there is $f_0 \in C_c^+(G)$ with f_0 not identically 0. Write F for the set of functions $I : C_c^+(G) \to \mathbb{R}_{\geqslant 0}$ with $I(f) \leqslant (f; f_0)$ for all $f \in C_c^+(G)$ endowed with the product topology *i.e.* the weakest topology such that the maps $F \to [0,(f;f_0)]; I \mapsto I(f)$ are continuous for all $f \in C_c^+(G)$. Since the closed interval $[0,(f;f_0)]$ is compact, F is a product of compact spaces and so compact. Let X be the set of $I \in F$ such that

$$I(f_0) = 1 \tag{4.3}$$

$$I(\mu f) = \mu I(f) \text{ for all } \mu \geqslant 0, f \in C_c^+(G), \tag{4.4}$$

and

$$I(\lambda_x(f)) = I(f) \text{ for all } x \in G, f \in C_c^+(G). \tag{4.5}$$

The set X is closed as an intersection of the preimage of closed sets. Moreover, by Lemma 4.12 $I_\phi \in X$ for any $\phi \in C_c^+(G)$ that is not identically zero: the fact that $I(f) \in [0,(f;f_0)]$ follows from the inequality in (4.2); (4.3) by design; (4.4) by (v); and (4.5) by (vi).

This almost gives us a Haar integral (on non-negative functions) except that in general the elements of X are not additive, meaning we do not in general have $I(f+f') = I(f)+I(f')$. To get this we introduce some further sets: for $\epsilon > 0$ and $f, f' \in C_c^+(G)$ define

$$B(f,f';\epsilon) := \{I \in X : |I(f+f') - I(f) - I(f')| \leqslant \epsilon\}.$$

As with X, the sets $B(f,f';\epsilon)$ are closed. We shall show that any finite intersection of such sets is non-empty: For any $f_1, f_1', f_2, f_2', \dots, f_n, f_n' \in C_c^+(G)$ and $\epsilon_1, \dots, \epsilon_n > 0$, by Lemma 4.13 there are symmetric open neighbourhoods of the identity $V_1, \dots, V_n$ such that if $\phi \in C_c^+(G)$ is not identically 0 and is supported in V_i then

$$|I_\phi(f_i + f_i') - I_\phi(f_i) - I_\phi(f_i')| < \epsilon_i. \tag{4.6}$$

Since G is locally compact by Lemma 1.24 there is a symmetric open neighbourhood of the identity H contained in a compact set L; set $V := H \cap \bigcap_{i=1}^n V_i$ which is also a symmetric open neighbourhood of the identity and by Corollary 3.6 there is $\phi \in C^+(G)$ that is not identically 0 with support contained in V, and hence in the compact set L which is to say it has compact support. I_ϕ enjoys (4.6) for all $1 \leqslant i \leqslant n$, and we noted before that $I_\phi \in X$, hence $I_\phi \in \bigcap_{i=1}^n B(f_i, f_i', \epsilon_i)$. We conclude that $\{B(f, f'; \epsilon) : f, f' \in C_c^+(G), \epsilon > 0\}$ is a set of closed subsets of F with the finite intersection property, but F is compact and so there is some I in all of these sets. Such an I is additive since $|I(f + f') - I(f) - I(f')| < \epsilon$ for all f, f' and $\epsilon > 0$. It remains to define $\int : C_c(G) \to \mathbb{C}$ by putting

$$\int f := I(f_1) - I(f_2) + iI(f_3) - iI(f_4) \text{ where } f = f_1 - f_2 + if_3 - if_4 \text{ for } f_1, f_2, f_3, f_4 \in C_c^+(G).$$

This decomposition of functions in $C_c(G)$ is unique (noted in Remark 3.20) and so this is well-defined. Moreover, $\int$ is linear since I is additive and enjoys (4.4); it is non-negative since I is non-negative (and $I(0) = 0$); it is translation invariant by (4.5); and it is non-trivial by (4.3). The result is proved. □

Uniqueness of the Haar integral

Our second main aim is to establish the following result.

Theorem 4.14 (Uniqueness of the Haar Integral)**.** *Suppose that G is a locally compact topological group and $\int$ and $\int'$ are left Haar integrals on G. Then there is some $\lambda > 0$ such that $\int = \lambda \int'$.*

For this we introduce a little more notation: Given a topological group G and $f \in C_c(G)$ we write $\widetilde{f}(x) = \overline{f(x^{-1})}$.

Remark 4.15. $\widetilde{\cdot}$ is a conjugate-linear multiplicative involution on $C_c(G)$, since complex conjugation and $x \mapsto x^{-1}$ are both continuous (and continuous images of compact sets are compact).

Proof of Theorem 4.14. Suppose that $f_0, f_1 \in C_c^+(G)$ are not identically 0 and write K for a compact set containing the support of f_0 and f_1 (which exists since finite unions of compact sets are compact). By Lemma 1.24 there is a symmetric open neighbourhood of the identity, H, contained in a compact set L.

First, by Corollary 3.15 there is a continuous compactly supported function $F : G \to [0, 1]$ with $F(x) = 1$ for all $x \in KL$ (this set is compact by Lemma 1.36, and hence the corollary applies).

Now, suppose $\epsilon > 0$ and use Remark 3.24 (and the fact that intersections of open covers are open covers) to get an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$ then $|f_i(x) - f_i(y)| < \epsilon$ for $i \in \{0, 1\}$. By Lemma 1.41 applied to $\mathcal{U}$ and the compact set KL there is a symmetric open neighbourhood of the identity V such that $\{xV : x \in KL\}$ is a refinement of $\mathcal{U}$ (as a cover of KL), and by Corollary 3.6 there is a continuous function $h : G \to [0, 1]$ that is not identically zero and is supported in $V \cap H$, and in particular supported in L so it has compact support.

For $x \in G$, translation invariance of $\int'$ (and Remark 3.22) tells us that

$$\int_y' h(y^{-1}x) = \int_y' \overline{\widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(y)} = \int' \overline{\widetilde{h}}.$$

For $i \in \{0,1\}$, the map $x \mapsto \int_y' f_i(x)h(y^{-1}x) = f_i(x)\int' \overline{\widetilde{h}}$ is continuous and is supported in K and so is compactly supported and $\int_x \int_y' f_i(x)h(y^{-1}x)$ exists and equals $\int f_i \int' \overline{\widetilde{h}}$ (by linearity of $\int$ and $\int\int'$). On the other hand the map $(x,y) \mapsto f_i(x)h(y^{-1}x)$ is continuous and supported on $K \times L$ and so is compactly supported and hence by Fubini's Theorem (Theorem 3.25), $y \mapsto \int_x f_i(x)h(y^{-1}x)$ exists, and (using translation invariance of $\int$) we have

$$\int f_i \int' \overline{\widetilde{h}} = \int_x \int_y' f_i(x)h(y^{-1}x) = \int_y' \int_x f_i(x)h(y^{-1}x) = \int_y' \int_x f_i(yx)h(x).$$

Since $\{yV : y \in KL\}$ refines $\mathcal{U}$ (as a cover of KL) we have $|f_i(yx) - f_i(y)| < \epsilon$ for $x \in V$ and $y \in KL$; and for $x \in H$ and $f_i(yx) \neq 0$ or $f_i(y) \neq 0$ we have $y \in KH$ whence $F(y) = 1$. It follows that

$$f_i(y)h(x) - \epsilon F(y)h(x) \leqslant f_i(yx)h(x) \leqslant f_i(y)h(x) + \epsilon F(y)h(x) \text{ for all } x, y \in G,$$

and so by non-negativity and linearity of $\int$ and $\int'$ we have

$$\int_y' \int_x f_i(y)h(x) - \int_y' \int_x \epsilon F(y)h(x) \leqslant \int_y' \int_x f_i(yx)h(x) \leqslant \int_y' \int_x f_i(y)h(x) + \int_y' \int_x \epsilon F(y)h(x).$$

It follows (using linearity of $\int$) that $|\int' f_i \int h - \int f_i \int' \overline{\widetilde{h}}| \leqslant \epsilon \int' F \int h$, and hence by the triangle inequality (and division, which is valid since $\int f_0, \int f_1 \neq 0$ by Corollary 4.10 as f_0 and f_1 are not identically zero) that

$$\left|\frac{\int' f_0}{\int f_0} - \frac{\int' f_1}{\int f_1}\right| \leqslant \left|\frac{\int' f_0}{\int f_0} - \frac{\int' \overline{\widetilde{h}}}{\int h}\right| + \left|\frac{\int' \overline{\widetilde{h}}}{\int h} - \frac{\int' f_1}{\int f_1}\right| \leqslant \epsilon \int' F\left(\frac{1}{\int f_0} + \frac{1}{\int f_1}\right).$$

Since ϵ was arbitrary (and in particular f_0, f_1, and F do not depend on it) it follows that $\int' f / \int f$ is a constant λ for all $f \in C_c^+(G)$ not identically zero. This constant must be non-zero since $\int'$ is non-trivial, and it must be positive since $\int'$ and $\int$ are non-negative. The result follows from the usual decomposition (Remark 3.20), and the fact that $\int 0, \int' 0 = 0$. □

5 The Peter-Weyl Theroem

Suppose that G is a topological group, and for an inner product space V recall the definition of $\mathrm{U}(V)$ from Example 1.12. A **finite dimensional unitary representation of** G is a continuous homomorphism $G \to \mathrm{U}(V)$ for some finite dimensional complex inner product space V.

A function $f : G \to \mathbb{C}$ is said to be a **matrix coefficient** if there is a finite dimensional unitary representation $\pi : G \to \mathrm{U}(V)$, and elements $v, w \in V$ such that $f(x) = \langle \pi(x)v, w\rangle$ for all $x \in G$.

Example 5.1. Suppose that $\pi : G \to \mathrm{U}(V)$ is a finite dimensional unitary representation of a topological group G and $e_1, \dots, e_n$ is an orthonormal basis for V. If we write $A_{i,j} := \langle \pi(x)e_i, e_j\rangle$ and suppose that $\lambda \in \mathbb{C}^n$ is the vector for $v \in V$ written w.r.t. the basis $e_1, \dots, e_n$ (*i.e.* $\lambda_i = \langle v, e_i\rangle$), then λA – the matrix A pre-multiplied by the row vector λ – is $\pi(x)v$ written w.r.t. the basis $e_1, \dots, e_n$. This is the reason for the terminology 'matrix coefficient'.

Remark 5.2. All matrix coefficients are continuous, since continuity of $\pi : G \to \mathrm{U}(V)$ and the definition of the topology on $\mathrm{U}(V)$ means that $x \mapsto \pi(x)v$ is continuous for all $v \in V$, and the projections $v \mapsto \langle v, w\rangle$ are continuous for all $w \in V$, so the resulting composition is also continuous.

Lemma 5.3. *Suppose that G is a compact topological group. Then there is a unique left Haar integral $\int$ on G with $\int 1 = 1$ such that*

$$\langle f, g\rangle := \int f\overline{g} \text{ for all } f, g \in C(G)$$

is an inner product on $C(G)$ and for each $x \in G$, λ_x is unitary w.r.t. this inner product. Furthermore, $\|f\|_2 := \langle f, f\rangle^{1/2}$ and $\|f\|_1 := \int |f|$ define norms on $C(G)$ and

$$\|f\|_1 \leqslant \|f\|_2 \leqslant \|f\|_\infty \text{ for all } f \in C(G).$$

Proof. By Theorem 4.11 there is a left Haar integral $\int'$ on G. Since G is compact the constant function 1 is compactly supported and so by Corollary 4.10, $\int' 1 > 0$. Diving by this positive constant we get a left Haar integral $\int$ with $\int 1 = 1$. Now suppose that $\int'$ is another left Haar integral with $\int' 1 = 1$. By Theorem 4.14 $\int' = \lambda \int$ for some $\lambda > 0$, but since $\int 1 = 1 = \int' 1$ we conclude that $\lambda = 1$ and $\int = \int'$ giving the claimed uniqueness.

Linearity in the first argument and conjugate-symmetry of $\langle \cdot, \cdot\rangle$ follow from linearity of the Haar integral and Remark 3.22 respectively. $\langle f, f\rangle \geqslant 0$ for all $f \in C(G)$ since $\int$ is non-negative and $\langle \cdot, \cdot\rangle$ is then positive definite by Corollary 4.10.

The Haar integral is left-invariant so

$$\langle f, g\rangle = \int f\overline{g} = \int \lambda_x(f\overline{g}) = \int \lambda_x(f)\overline{\lambda_x(g)} \text{ for all } f, g \in C(G),$$

and the first part is proved.

For any inner product $f \mapsto \langle f, f\rangle^{1/2}$ is a norm, so $\|\cdot\|_2$ is a norm. Absolute homogeneity of $\|\cdot\|_1$ follows from the fact that the modulus of a complex number is multiplicative and $\int$

is linear, and the triangle inequality follows from, non-negativity, linearity and the triangle inequality for the modulus of a complex number. $\|f\|_1 \geqslant 0$ by non-negativity of $\int$, and finally $\|\cdot\|_1$ is positive definite by Corollary 4.10.

Since G is compact the constant functions 1 and $\|f\|_\infty^2$ are both in $C(G)$. By the Cauchy-Schwarz inequality (which holds for all inner products) we have

$$\|f\|_1 = \int |f| = \langle 1, |f| \rangle \leqslant \|1\|_2 \||f|\|_2 = \|f\|_2 \text{ for all } f \in C(G);$$

and by non-negativity of $\int$ we have

$$\|f\|_2^2 = \int |f|^2 \leqslant \int \|f\|_\infty^2 = \|f\|_\infty^2 \text{ for all } f \in C(G).$$

The result is proved. □

Remark 5.4. For the remainder of this section we write $\int$ for the unique Haar integral in Lemma 5.3, and use the notation $\langle \cdot, \cdot \rangle$, $\|\cdot\|_2$ and $\|\cdot\|_1$ as in this lemma.

Remark 5.5. Convergence in $\|\cdot\|_\infty$ is called convergence in L_∞ or **uniform convergence**; convergence in $\|\cdot\|_2$ is called convergence in L_2; and convergence in $\|\cdot\|_1$ is called convergence in L_1.

The second inequality in Lemma 5.3 tells us that uniform convergence implies convergence in L_2, and the first that convergence in L_2 implies convergence in L_1.

For $f, g \in C(G)$ we define their **convolution** to be the function

$$x \mapsto f * g(x) := \int_y f(y)g(y^{-1}x) = \langle f, \lambda_x(\widetilde{g}) \rangle.$$

Lemma 5.6 (Basic properties of convolution)**.** *Suppose that G is a compact topological group. Then*

(i) $C(G) \to C(G); g \mapsto g * f$ *is well-defined and linear for all* $f \in C(G)$*;*

(ii) $h * (g * f) = (h * g) * f$ *for all* $f, g, h \in C(G)$*;*

(iii) $\lambda_x(g * f) = \lambda_x(g) * f$ *for all* $x \in G$, $f, g \in C(G)$*;*

(iv) $\langle g * f, h \rangle = \langle g, h * \widetilde{f} \rangle$ *for all* $f, g, h \in C(G)$ *(recall* $\widetilde{f}$ *from just before Remark 4.15);*

(v) $\|h * f\|_\infty \leqslant \min\{\|h\|_1 \|f\|_\infty, \|h\|_2 \|\widetilde{f}\|_2\}$ *for all* $f, h \in C(G)$*.*

Proof. By the first part of Fubini's Theorem (Theorem 3.25) the function $g * f \in C(G)$ since $(x, y) \mapsto g(x)f(x^{-1}y)$ is continuous and compactly supported. Since $\int_x$ is linear, $g \mapsto g * f$ is well-defined and linear giving (i).

For (ii) we apply λ_y to the integrand $z \mapsto g(z)f(z^{-1}y^{-1}x)$ using that $\int_z$ is a left Haar integral; then Fubini's Theorem (Theorem 3.25) since $(z,y) \mapsto h(y)g(y^{-1}z)f(z^{-1}x)$ is continuous; and finally linearity of $\int_y$ to see that

$$\begin{aligned} h * (g * f)(x) &= \int_y h(y) \int_z g(z)f(z^{-1}y^{-1}x) \\ &= \int_y h(y) \int_z g(y^{-1}z)f(z^{-1}x) = \int_z \left(\int_y h(y)g(y^{-1}z) \right) f(z^{-1}x) = (h * g) * f(x) \end{aligned}$$

as claimed.

For (iii) note that $\lambda_t(g * f)(x) = g * f(t^{-1}x) = \langle g, \lambda_{t^{-1}x}(\widetilde{f})\rangle = \langle g, \lambda_{t^{-1}}(\lambda_x(\widetilde{f}))\rangle = \langle \lambda_t(g), \lambda_x(\widetilde{f})\rangle = \lambda_t(g) * f(x)$ since λ_t is unitary w.r.t. $\langle \cdot, \cdot \rangle$ by Lemma 5.3.

For (iv), since the function $(x,y) \mapsto g(x)f(x^{-1}y)\overline{h(y)}$ is continuous and compactly supported, by Fubini's Theorem (Theorem 3.25) and linearity of $\int_y$; and then Remark 3.22 we have

$$\begin{aligned} \langle g * f, h\rangle &= \int_y \int_x g(x)f(x^{-1}y)\overline{h(y)} \\ &= \int_x g(x) \int_y f(x^{-1}y)\overline{h(y)} = \int_x g(x) \int_y \overline{h(y)\widetilde{f}(y^{-1}x)} = \langle g, h * \widetilde{f}\rangle, \end{aligned}$$

as required.

Finally, (v) follows on the one hand since

$$|h * f(x)| \leqslant \int_y |h(y)||f(y^{-1}x)| \leqslant \int |h|\|f\|_\infty = \|h\|_1\|f\|_\infty,$$

and on the other since $|h * f(x)| = |\langle h, \lambda_x(\widetilde{f})\rangle| \leqslant \|h\|_2\|\lambda_x(\widetilde{f})\|_2 = \|h\|_2\|\widetilde{f}\|_2$. The result is proved. □

Remark 5.7. As usual, in view of the associativity in (ii) there is no ambiguity in omitting parentheses when writing expressions like $h * g * f$.

Remark 5.8. The linearity of the maps in (i) and inequality (v) mean that convolution maps convergence in L_1 to uniform convergence *c.f.* Remark 5.5.

Before beginning our main argument we need one more tool which will deal with the fact our inner product spaces are not in general complete.

Remark 5.9. A complete inner product space is called a Hilbert space and the results of this section are usually developed with respect to these. ⚠ In particular, a unitary representation is usually a continuous group homomorphism $\pi : G \to \mathrm{U}(H)$ for a complex Hilbert space H, not merely a complex inner product space. Every finite dimensional complex inner product space is complete and so a Hilbert space, and so our definition at the start of the section is not at variance with this, but in general care is warranted.

Proposition 5.10. *Suppose that G is a compact topological group G, $f \in C(G)$ and $(g_n)_{n\in\mathbb{N}}$ is a sequence of elements of $C(G)$ with $\|g_n\|_1 \leqslant 1$. Then there is a subsequence $(g_{n_i})_{i\in\mathbb{N}}$ such that $g_{n_i} * f$ converges uniformly to some element of $C(G)$ as $i \to \infty$.*

Proof. For each $j \in \mathbb{N}$, Remark 3.24 gives us an open cover $\mathcal{U}_j$ of G such that if $x, y \in U \in \mathcal{U}_j$ then $|f(x) - f(y)| < 1/j$. Since G is compact apply Lemma 1.41 to get an open neighbourhood of the identity U_j such that $\{xU_j : x \in G\}$ refines $\mathcal{U}_j$; and by compactness again there is a finite cover $\{x_{1,j}U_j, \dots, x_{k(j),j}U_j\}$ which refines $\{xU_j : x \in G\}$. By Lemma 5.3 (v) $g_n * f(x) \in [-\|f\|_\infty, \|f\|_\infty]$. The interval $[-\|f\|_\infty, \|f\|_\infty]$ is sequentially compact, meaning every sequence has a convergent subsequence. A countable product of sequentially compact spaces is sequentially compact[6] so there is a subsequence $(n_i)_i$ such that $g_{n_i} * f(x_{k,j})$ converges, say to $h(x_{k,j})$, as $i \to \infty$ for all $1 \leqslant k \leqslant k(j)$ and $j \in \mathbb{N}$.

Suppose $\epsilon > 0$ and let $j := \lceil 3\epsilon^{-1} \rceil$. For all $1 \leqslant k \leqslant k(j)$ let M_k be such that $|g_{n_i} * f(x_{k,j}) - h(x_{k,j})| < \epsilon/6$ for all $i \geqslant M_k$; let $M := \max\{M_k : 1 \leqslant k \leqslant k(j)\}$ and suppose that $i, i' \geqslant M$.

For $x \in G$ there is some $1 \leqslant k \leqslant k(j)$ such that $x \in x_{k,j}U_j$ and hence for all $y \in G$ we have $y^{-1}x, y^{-1}x_{k,j} \in y^{-1}x_{k,j}U_j$ which is a subset of an element of $\mathcal{U}_j$, so $|f(y^{-1}x) - f(y^{-1}x_{k,j})| < 1/j$. Thus for $g \in C(G)$ with $\|g\|_1 \leqslant 1$ we have

$$\begin{aligned}|g * f(x) - g * f(x_{k,j})| &= |\langle g, \lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\rangle| \\ &\leqslant \|g\|_1 \|\lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\|_\infty \leqslant \sup_{y\in G} |f(y^{-1}x) - f(y^{-1}x_{j,k})| \leqslant \frac{1}{j} \leqslant \epsilon/3.\end{aligned}$$

In particular this holds for $g = g_{n_i}$ and $g = g_{n_{i'}}$, so that

$$\begin{aligned}|g_{n_i} * f(x) - g_{n_{i'}} * f(x)| \leqslant{}& |g_{n_i} * f(x) - g_{n_i} * f(x_{k,j})| + |g_{n_i} * f(x_{k,j}) - h(x_{k,j})| \\ &+ |h(x_{k,j}) - g_{n_{i'}} * f(x_{k,j})| + |g_{n_{i'}} * f(x_{k,j}) - g_{n_{i'}} * f(x)| < \epsilon.\end{aligned}$$

Since $x \in G$ was arbitrary it follows that the sequence of functions $(g_{n_i} * f)_i$ is uniformly Cauchy and so converges to a continuous function on G. The result is proved. □

We say that $V \leqslant C(G)$ is **invariant** if $\lambda_x(v) \in V$ for all $v \in V$.

Example 5.11. Suppose that $V \leqslant C(G)$ is invariant and finite dimensional. Then $\pi : G \to \mathrm{U}(V); x \mapsto (V \to V; v \mapsto \lambda_x(v))$ is a finite dimensional unitary representation.

For any $V \leqslant C(G)$ write $V^\perp$ for the set of $w \in C(G)$ such that $\langle v, w\rangle = 0$ for all $v \in V$.

Proposition 5.12. *Suppose that G is a compact group and $f \in C(G)$. Then there is an invariant space $W \leqslant C(G)$ with $\dim W \leqslant \epsilon^{-2}\|f\|_2^2$ such that if $g \in W^\perp$ then $\|g * f\|_2 \leqslant \epsilon\|g\|_2$.*

[6] The proof of this is just Cantor's diagonal argument.

Proof. Let V be the set of vectors of the form

$$h_1 + \cdots + h_n \text{ where } n \in \mathbb{N}_0, h_i * \widetilde{f} * f = \lambda_i h_i \text{ and } \lambda_i \geqslant \epsilon^2 \text{ for all } 1 \leqslant i \leqslant n. \tag{5.1}$$

This is an invariant space by Lemma 5.6 (iii). For $v \in V$ we shall write $v = h_1 + \cdots + h_n$ to mean a decomposition as in (5.1) with the additional requirements that h_i is not identically zero (so $\|h_i\|_2^2 \neq 0$ since h_i is continuous), and $\lambda_i \neq \lambda_j$ for $i \neq j$, which is possible since the map $T : C(G) \to C(G); h \mapsto h * \widetilde{f} * f$ is linear. (The zero vector is represented as a sum with no terms.)

In fact T is positive definite and so the h_is, which are eigenvectors with corresponding eigenvalues λ_i, are perpendicular for different eigenvalues. In our language the relevant parts of this follow since if $h_i * \widetilde{f} * f = \lambda_i h_i$ and $h_j * \widetilde{f} * f = \lambda_j h_j$, then

$$\lambda_i \langle h_i, h_j \rangle = \langle \lambda_i h_i, h_j \rangle = \langle h_i * \widetilde{f} * f, h_j \rangle = \langle h_i, h_j * \widetilde{f} * f \rangle = \langle h_i, \lambda_j h_j \rangle = \overline{\lambda_j} \langle h_i, h_j \rangle.$$

Applying this identity with $j = i$ for some $h_j \neq 0$ we see that λ_i is real. Then applying it again with $\lambda_i \neq \lambda_j$ we have $\langle h_i, h_j \rangle = 0$. In particular, if $v = h_1 + \cdots + h_n$ in the way discussed after (5.1) then

$$\|v * \widetilde{f}\|_2^2 = \langle v * \widetilde{f} * f, v \rangle = \sum_{i=1}^{n} \lambda_i \|h_i\|_2^2 \geqslant \epsilon^2 \sum_{i=1}^{n} \|h_i\|_2^2 = \epsilon^2 \|v\|_2^2. \tag{5.2}$$

If V contains n linearly independent vectors, then by the Gram-Schmidt process[7] there are orthonormal vectors $v_1, \ldots, v_n \in V$. For $x \in G$, by Bessel's inequality[8]

$$\sum_{i=1}^{n} |\langle v_i, \lambda_x(f) \rangle|^2 \leqslant \|\lambda_x(f)\|_2^2 = \|f\|_2^2.$$

[7]Given $e_1, e_2, \ldots$ linearly independent, the Gram-Schmidt process in an inner product space defines

$$u_i := e_i - \sum_{k=1}^{i-1} \langle e_i, v_k \rangle v_k \text{ and } v_i := u_i / \|u_n\|.$$

It can be shown by induction that $v_1, v_2, \ldots$ is an orthonormal sequence.

[8]Bessel's inequality is the fact that if $v_1, v_2, \ldots$ is an orthonormal sequence in an inner product space then $\sum_{i=1}^{n} |\langle v_i, v \rangle|^2 \leqslant \|v\|^2$ for all v. To prove it note that because the v_is are orthonormal we have

$$\left\| \sum_{i=1}^{n} \langle v_i, v \rangle v_i \right\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, v \rangle \overline{\langle v_j, v \rangle} \langle v_i, v_j \rangle = \sum_{i=1}^{n} |\langle v_i, v \rangle|^2.$$

Hence by the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} |\langle v_i, v \rangle|^2 \right)^2 = \left| \left\langle v, \sum_{i=1}^{n} \langle v_i, v \rangle v_i \right\rangle \right|^2 \leqslant \|v\|^2 \left\| \sum_{i=1}^{n} \langle v_i, v \rangle v_i \right\|^2 = \|v\|^2 \left(\sum_{i=1}^{n} |\langle v_i, v \rangle|^2 \right).$$

Cancelling gives the inequality.

Integrating against x and using (5.2) we have

$$n\epsilon^2 \leqslant \sum_{i=1}^n \int_x |v_i * \widetilde{f}(x)|^2 = \int_x \sum_{i=1}^n |\langle v_i, \lambda_x(f)\rangle|^2 \leqslant \int_x \|f\|_2^2 = \|f\|_2^2.$$

It follows that $\dim V \leqslant \epsilon^{-2}\|f\|_2^2$.

Write $W := \{k * \widetilde{f} : k \in V\}$, which is invariant by Lemma 5.6 (iii) and the fact V is invariant. Let $M := \sup\{\|g * f\|_2 : g \in W^\perp \text{ and } \|g\|_2 \leqslant 1\}$. We shall be done if we can show that $M^2 \leqslant \epsilon^2$.

Claim. *If $h \in V^\perp$ then $\|h * \widetilde{f}\|_2 \leqslant M\|h\|_2$.*

Proof. First, $h * \widetilde{f} \in W^\perp$: To see this, for $v \in V$ write $v = h_1 + \cdots + h_n$ to mean a decomposition as in (5.1). Then

$$\langle h * \widetilde{f}, v * \widetilde{f}\rangle = \sum_{i=1}^n \langle h, h_i * \widetilde{f} * f\rangle = \sum_{i=1}^n \lambda_i \langle h, h_i\rangle = 0.$$

Now let $k \in W^\perp$ have $\|k\|_2 = 1$ such that $\|h * \widetilde{f}\|_2 = \langle h * \widetilde{f}, k\rangle = \langle h, k * f\rangle \leqslant \|h\|_2 \|k * f\|_2 \leqslant M\|h\|_2$ as claimed. □

Let $g_n \in W^\perp$ have $\|g_n * f\|_2 \to M$ and $\|g_n\|_2 \leqslant 1$. By Cauchy-Schwarz we have $\|g_n\|_1 \leqslant 1$ and we may apply Proposition 5.10 to pass to a subsequence which converges uniformly. Hence by relabelling we may now additionally assume that $g_n * f \to h$ uniformly for some $h \in C(G)$. In particular, $\|g_n * f\|_2 \to \|h\|_2$ and $\langle h, g_n * f\rangle \to \|h\|_2^2$ and hence $\|h\|_2 = M$. Moreover, if $v \in V$ then $\langle g_n * f, v\rangle = \langle g_n, v * \widetilde{f}\rangle = 0$, and the former converges to $\langle h, v\rangle$, whence $h \in V^\perp$.

Combining this with the claim above we have

$$\begin{aligned}\|h * \widetilde{f} - M^2 g_n\|_2^2 &= \|h * \widetilde{f}\|_2^2 - 2M^2 \operatorname{Re}\langle h * \widetilde{f}, g_n\rangle + M^4\|g_n\|_2^2 \\ &\leqslant M^2\|h\|_2^2 - 2M^2 \operatorname{Re}\langle h, g_n * f\rangle + M^4 \to 0.\end{aligned}$$

Hence $M^2 g_n \to h * \widetilde{f}$ in $\|\cdot\|_2$, and since convergence in $\|\cdot\|_2$ is mapped to uniform convergence by convolution operations we have $M^2 g_n * f \to h * \widetilde{f} * f$. Uniqueness of limits then ensures $M^2 h = h * \widetilde{f} * f$. If $M^2 \geqslant \epsilon^2$ then $h \in V$, but then since $h \in V^\perp$ we see h is not identically zero. In that case $M = \|h\|_2 = 0$ and certainly $M^2 \leqslant \epsilon^2$ as required. The result is proved. □

Theorem 5.13 (The Peter-Weyl Theorem)**.** *Suppose that G is a compact topological group. Then matrix coefficients are dense in $C(G)$ with the uniform norm.*

Proof. Suppose that $f \in C(G)$ and let $\epsilon > 0$. Remark 3.24 gives us an open cover $\mathcal{U}_j$ of G such that if $x, y \in U \in \mathcal{U}_j$ then $|\widetilde{f}(x) - \widetilde{f}(y)| < \epsilon/2$. Since G is compact, by Lemma 1.41 there is an open neighbourhood of the identity U such that $\{xU : x \in G\}$ refines $\mathcal{U}$, and by Lemma 1.34 there is an open set V such that $V^2 \subset U$. By Corollary 3.6, there is $g \in C(G)$ non-negative and not identically 0 such that $\operatorname{supp} g \subset V$. By rescaling g we may assume that $\int g = 1$. The support of $g * g$ is contained in $V^2 \subset U$ and by Fubini's Theorem (Theorem 3.25) we therefore have $\int g * g = 1$. But then

$$|g * g * \overline{f}(x) - \overline{f}(x)| = \left|\int_y g * g(y)\overline{f}(y^{-1}x) - \overline{f}(x)\right| = \left|\int_y g * g(y)(\widetilde{f}(x^{-1}y) - \widetilde{f}(x^{-1}))\right| \leqslant \epsilon,$$

for all $x \in G$ and so $\|\overline{f} - g * g * \overline{f}\|_\infty \leqslant \epsilon/2$.

Let $\delta < \epsilon\|g\|_2^{-1}\|\widetilde{f}\|_2^{-1}/2$ for reasons which will be come clear shortly. By Proposition 5.12 there is a finite dimensional invariant space $W \leqslant C(G)$ such that $\|h * g\|_2 \leqslant \delta\|h\|_2$ for all $h \in W^\perp$. Write $\pi_W : C(G) \to C(G)$ for the map projecting onto W. Then $g - \pi_W(g) \in W^\perp$ and so $\|g * g - \pi_W(g) * g\|_2 \leqslant \delta\|g - \pi_W(g)\|_2 \leqslant \delta\|g\|_2$. By Lemma 5.6 (v) we have

$$\|g * g * \overline{f} - \pi_W(g) * g * \overline{f}\|_\infty \leqslant \delta\|g\|_2\|\widetilde{f}\|_2.$$

By the triangle inequality we have $\|\overline{f} - \pi_W(g) * g * \overline{f}\|_\infty < \epsilon$. Finally, writing $k := (g * \overline{f})^\sim$ we have by definition; since λ_x is unitary; since W is invariant; since π_W is self-adjoint (meaning $\langle \pi_W v, w\rangle = \langle v, \pi_W w\rangle$ for all $v, w \in C(G)$); and again since λ_x is unitary, that

$$\begin{aligned}\pi_W(g) * g * \overline{f}(x) = \langle \pi_W(g), \lambda_x(k)\rangle &= \langle \lambda_{x^{-1}}(\pi_W(g)), k\rangle \\ &= \langle \pi_W(\lambda_{x^{-1}}(\pi_W(g))), k\rangle \\ &= \langle \lambda_{x^{-1}}(\pi_W(g)), \pi_W(k)\rangle \\ &= \langle \pi_W(g), \lambda_x(\pi_W(k))\rangle = \overline{\langle \lambda_x(\pi_W(k)), \pi_W(g)\rangle}.\end{aligned}$$

Hence $\overline{\pi_W(g) * g * \overline{f}(x)}$ is a matrix coefficient. Since $\epsilon > 0$ was arbitrary the result is proved. □

Remark 5.14. ⚠ There are other important parts to the Peter-Weyl Theorem which we have not included here.

6 The dual group

Suppose that G is a topological group. We write $\widehat{G}$ for the set of continuous homomorphisms $G \to S^1$ (where S^1 is as in Example 1.7), and call the elements of $\widehat{G}$ **characters**.

Remark 6.1. ⚠ While characters are (by definition) elements of $C(G)$, they are *not* in $C_c(G)$ unless G is compact.

We endow the set $\widehat{G}$ with the **compact-open topology**, that is the topology generated by the sets $\gamma U(K, \epsilon)$ where $\gamma \in \widehat{G}$,

$$U(K, \epsilon) := \{\lambda \in \widehat{G} : |\lambda(x) - 1| < \epsilon \text{ for all } x \in K\}$$

and $\epsilon > 0$ and K is a compact subset of G.

Proposition 6.2. *Suppose that G is a topological group. Then $\widehat{G}$ is a Hausdorff Abelian topological group with multiplication and inversion defined by*

$$(\gamma, \gamma') \mapsto (x \mapsto \gamma(x)\gamma'(x)) \text{ and } \gamma \mapsto (x \mapsto \overline{\gamma(x)}),$$

and identity the character taking the constant value 1. Moreover, $(U(K,\delta))_{K,\delta}$ as K ranges compact subsets of G and $\delta > 0$ is a neighbourhood base of the identity.

Proof. The fact that $\widehat{G}$ is an Abelian group is an easy check since S^1 is an Abelian group under multiplication and $z^{-1} = \overline{z}$ when $z \in S^1$.

Since $|\gamma(x) - 1| = |\overline{\gamma(x)} - 1|$ the inversion is certainly continuous. Now suppose that $\gamma\lambda \in \mu U(K,\epsilon)$ for some $\mu \in \widehat{G}$. Since $\gamma\lambda\overline{\mu}$ is continuous and K is compact $|\gamma\lambda\overline{\mu} - 1|$ achieves its bounds on K and hence there is some $\delta > 0$ such that $|(\gamma\lambda\overline{\mu})(x) - 1| < \epsilon - \delta$ for all $x \in K$. But then if $\gamma' \in \gamma U(K, \delta/2)$ and $\lambda' \in \lambda U(K, \delta/2)$ we have

$$\begin{aligned}|(\gamma'\lambda'\overline{\mu})(x) - 1| &\leqslant |(\gamma'\lambda'\overline{\mu})(x) - (\gamma\lambda'\overline{\mu})(x)| + |(\gamma\lambda'\overline{\mu})(x) - (\gamma\lambda\overline{\mu})(x)| + |(\gamma\lambda\overline{\mu})(x) - 1| \\ &< \delta/2 + \delta/2 + \epsilon - \delta = \epsilon.\end{aligned}$$

It follows that $\gamma'\lambda' \in \mu U(K,\epsilon)$ and so the preimage of $\gamma\lambda$ contains a neighbourhood of (γ, λ) in $\widehat{G} \times \widehat{G}$ *i.e.* multiplication is continuous. Finally, the topology is Hausdorff since if $\gamma \neq \lambda$ then there is some $x \in G$ such that $\gamma(x) \neq \lambda(x)$; put $\epsilon := |\gamma(x) - \lambda(x)|/2$ and note that $\gamma U(\{x\},\epsilon)$ and $\lambda U(\{x\},\epsilon)$ are disjoint open sets containing γ and λ respectively. □

We call the group $\widehat{G}$ endowed with the compact-open topology the **dual group** of G, so that the above proposition tells us that if G is a topological group then its dual group is a Hausdorff Abelian topological group.

We call the identity, denoted $1_{\widehat{G}}$, the **trivial character**.

Proposition 6.3. *Suppose that G is a compact topological group. Then $\widehat{G}$ is discrete.*

Proof. Suppose that $\gamma \neq 1_{\widehat{G}}$ so there is $x \in G$ such that $\gamma(x) \neq 1$. Let $y \in G$ be such that $|\gamma(y) - 1|$ is maximal (which exists since G is compact and $x \mapsto |\gamma(x) - 1|$ is continuous) and note that by assumption this is positive. If $|\gamma(y) - 1| < 1$ then we have

$$\begin{aligned}|\gamma(y^2) - 1| = |\gamma(y)^2 - 1| &= |(2 + (\gamma(y) - 1))||\gamma(y) - 1| \\ &\geqslant (2 - |\gamma(y) - 1|)|\gamma(y) - 1| > |\gamma(y) - 1|.\end{aligned}$$

This is a contradiction, whence $\gamma \notin U(G,1)$ and $\{1_{\widehat{G}}\}$ is open so the topology is discrete. □

Example 6.4. Suppose that G is a finite cyclic group endowed with the discrete topology. Since G is cyclic it is generated by some element x, and the map

$$\phi : G \to \widehat{G}; x^r \mapsto (G \to S^1; x^l \mapsto \exp(2\pi i r l/|G|))$$

is a well-defined homeomorphic isomorphism. To see this note that ϕ is well-defined in the sense that different representations of an element in the domain produce the same image: since $x^r = x^{r'}$ implies $|G| \mid r - r'$ and hence $\exp(2\pi irl/|G|) = \exp(2\pi ir'l/|G|)$; and ϕ is well-defined in the sense that $\phi(x^r)$ as defined is genuinely an element of $\widehat{G}$: $x^l = x^{l'}$ implies $|G| \mid l - l'$ and hence $\exp(2\pi irl/G|) = \exp(2\pi irl'/|G|)$ so that $\phi(x^r)$ is itself a well-defined function; it is continuous since G is discrete; and it is a homomorphism since $\exp(2\pi ir(l + l')/|G|) = \exp(2\pi irl/|G|)\exp(2\pi irl'/|G|)$.

ϕ is a homomorphism since $\exp(2\pi i(r + r')l/|G|) = \exp(2\pi irl/|G|)\exp(2\pi ir'l/|G|)$. ϕ is injective since if $\exp(2\pi irl/|G|) = 1$ for all l then $|G| \mid r$ so $x^r = 1_G$. ϕ is surjective since if $\gamma : G \to S^1$ is a homomorphism then $\gamma(x)^{|G|} = 1$ so $\gamma(x) = \exp(2\pi ir/|G|)$ for some $r \in \mathbb{Z}$, and $\gamma = \phi(x^r)$.

We conclude that $\phi : G \to \widehat{G}$ is a bijective group homomorphism and hence ϕ^{-1} is a group homomorphism. Since G is finite, G is compact and so $\widehat{G}$ is discrete by Proposition 6.3 and hence ϕ^{-1} is continuous as required.

Example 6.5. When G is a group with the indiscrete topology the only continuous functions are constant and so $\widehat{G}$ is the trivial group with one character taking the constant value 1 (and there is only one topology on a set with one element) so that we have completely determined the topological group $\widehat{G}$.

Example 6.5 gave topological reasons for the dual group being trivial, but there can also be algebraic reasons:

Example 6.6 (Non-Abelian finite simple groups)**.** Suppose that G is a non-Abelian finite simple[9] topological group.

Suppose that $\gamma : G \to S^1$ is a homomorphism. Since G is non-Abelian there are elements $x, y \in G$ with $xy \neq yx$, but then $xyx^{-1}y^{-1} \neq 1_G$ while

$$\gamma(xyx^{-1}y^{-1}) = \gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1} = 1$$

since S^1 *is* Abelian. We conclude that the kernel of γ is non-trivial, but all kernels are normal subgroups and since G is simple it follows that $\ker\gamma = G$ *i.e.* γ is trivial. In other words $\widehat{G} = \{1_{\widehat{G}}\}$.

The topology on G and $\widehat{G}$ are quite closely related: if G is compact then $\widehat{G}$ is discrete (Proposition 6.3), and the other way round we have the following:

Proposition 6.7. *Suppose that G is a discrete topological group. Then $\widehat{G}$ is compact.*

[9]A **simple group** is a group whose only normal subgroups are the trivial group and the whole group *e.g.* A_n, the alternating group on n elements, when $n \geqslant 5$. (The Abelian finite simple groups are the cyclic groups of prime order and their dual groups are described in Example 6.4.)

Proof. The set $\widehat{G}$ is a subset of the topological space M of functions $G \to S^1$ endowed with the product topology, which itself is compact by Tychonoff's theorem (*c.f.* the set F considered in the proof of Theorem 4.11.). Since G is discrete the only compact sets in G are finite and hence the topology on $\widehat{G}$ is the subspace topology induced by viewing it as a subspace of M. It remains to check that $\widehat{G}$ is closed at which point it follows that it is compact. To see it is closed, note that the sets $\{f : G \to S^1 : f(xy) = f(x)f(y)\}$ are closed for each $x, y \in G$, and hence

$$\bigcap \{\{f : G \to S^1 : f(xy) = f(x)f(y)\} : x, y \in G\}$$

is closed. This is the set of all homomorphisms $G \to S^1$, but *every* homomorphism is continuous since G is discrete and hence this set equals $\widehat{G}$. □

We can make use of the Haar integral we have developed to show that if G is a locally compact topological group then the dual group is also locally compact. To do this we need a lemma.

Lemma 6.8. *Suppose that G is a locally compact topological group supporting a Haar integral $\int$, $f_0 \in C_c^+(G)$ has $\int f_0 \neq 0$, and $\kappa, \delta > 0$. Then there is an open neighbourhood of the identity $L_{\delta,\kappa}$ such that if $|\int f_0\gamma| \geqslant \kappa$ then $|1 - \gamma(y)| < \delta$ for all $y \in L_{\delta,\kappa}$.*

Proof. Write K for a compact set containing the support of f_0 and U for a compact neighbourhood of the identity. UK is compact by Lemma 1.36. Apply Corollary 3.15 to get a continuous compactly supported $F : G \to [0, 1]$ such that $F(x) = 1$ for all $x \in UK$.

By Proposition 4.3 there is an open neighbourhood of the identity $L_{\delta,\kappa}$ (which we may assume is contained in U since U is a neighbourhood and so contains an open neighbourhood of the identity) such that $\|\lambda_y(f_0) - f_0\|_\infty < \delta\kappa / \int F$ for all $y \in L_{\delta,\kappa}$. (Note $\int F > 0$ by Corollary 4.10.) For $y \in L_{\delta,\kappa}$, the support of $\lambda_y(f_0) - f_0$ is contained in UK (since $L_{\delta,\kappa} \subset U$) and so

$$\int |\lambda_y(f_0) - f_0| \leqslant \|\lambda_y(f_0) - f_0\|_\infty \int F < \delta\kappa.$$

Now, if $y \in L_{\delta,\kappa}$ then

$$\begin{aligned} |1 - \gamma(y)|\kappa \leqslant \left|(\gamma(y) - 1)\int f_0\gamma\right| &= \left|\int f_0\lambda_{y^{-1}}(\gamma) - \int f_0\gamma\right| \\ &= \left|\int \lambda_y(f_0)\gamma - \int f_0\gamma\right| \leqslant \int |\lambda_y(f_0) - f_0| < \delta\kappa. \end{aligned}$$

Dividing by κ gives the claim. □

Theorem 6.9. *Suppose that G is a locally compact topological group. Then $\widehat{G}$ is locally compact.*

Proof. Let $\int$ be a left Haar integral on G (which exists by Theorem 4.11). Since $\int$ is non-trivial there is $f_0 \in C_c^+(G)$ such that $\int f_0 \neq 0$ and we may rescale so that $\int f_0 = 1$. Write K for a compact set containing the support of f_0 and define

$$V := \{\gamma \in \widehat{G} : |\gamma(x) - 1| \leqslant 1/4 \text{ for all } x \in K\},$$

so that V certainly contains, $U(K, 1/4)$, an open neighbourhood of the identity.

As in the proof of Proposition 6.7 we write M for the set of maps $G \to S^1$ endowed with the product topology so that M is compact. The set $\widehat{G}$ is contained in in the set M, but the compact-open topology on $\widehat{G}$ is *not*, in general, the same as that induced on $\widehat{G}$ as a subspace of M. Our aim is to make use of the compactness on M to show that $\widehat{G}$ is locally compact in the compact-open topology.

First we restrict to homomorphisms: write H for the set of homomorphisms $G \to S^1$, which is a closed subset of M since it is the intersection over all pairs $x, y \in G$ of the set of $f \in M$ such that $f(xy) = f(x)f(y)$. Write

$$C := \bigcap_{\delta > 0, x \in L_{\delta,3/4}} \{f \in H : |f(x) - 1| \leqslant \delta\}$$

which is also closed as an intersection of closed sets. By Proposition 2.6 as sets we have $C \subset \widehat{G}$ since the sets $\{z \in S^1 : |1 - z| \leqslant \delta\}$ form a neighbourhood base of the identity in S^1, and if $f \in C$ then $f^{-1}(\{z \in S^1 : |1 - z| \leqslant \delta\}) \supset L_{\delta,3/4}$ which is a neighbourhood of the identity in G.

If $\gamma \in V$ then $\left|1 - \int f_0\gamma\right| \leqslant \int f_0|1 - \gamma| \leqslant 1/4$, so by the triangle inequality $|\int f_0\gamma| \geqslant 3/4$ and hence the claim tells us that $\gamma \in C$. Thus (as sets) $V \subset C \subset \widehat{G}$ and so

$$V = \bigcap_{x \in K} \{f \in C : |f(x) - 1| \leqslant 1/4\},$$

which is again a closed subset of M.

Our aim is to show that V is compact in the compact-open topology on $\widehat{G}$. This follows if every cover of the form $\mathcal{U} = \{\gamma U(K_\gamma, \delta_\gamma) : \gamma \in V\}$ (where K_γ is compact and $\delta_\gamma > 0$) has a finite subcover. Write $L_\gamma := L_{\delta_\gamma/2,1/2}$ and note that by compactness of K_γ there is a finite set T_γ such that $K_\gamma \subset T_\gamma L_\gamma$. Write

$$U_\gamma := \{f \in M : |f(x) - 1| < \delta_\gamma/2 \text{ for all } x \in T_\gamma\}$$

which is an open set in M since T_γ is finite. Suppose that $\lambda \in (\gamma U_\gamma) \cap V$. Then since $\gamma, \lambda \in V$, the triangle inequality gives

$$\begin{aligned}\left|1 - \int f_0\overline{\gamma}\lambda\right| \leqslant \int f_0|1 - \overline{\gamma}\lambda| &= \int f_0|1 - \overline{\gamma} + \overline{\gamma} - \overline{\gamma}\lambda| \\ &\leqslant \int f_0|1 - \gamma| + \int f_0|1 - \lambda| \leqslant 1/2.\end{aligned}$$

Hence $\left|\int f_0\overline{\gamma}\lambda\right| \geqslant 1/2$ by the triangle inequality again. The claim gives $|1-\overline{\gamma(y)}\lambda(y)| < \delta_\gamma/2$ for all $y \in L_\gamma$. But $\overline{\gamma}\lambda \in U_\gamma$ so we also have $|1-\overline{\gamma(z)}\lambda(z)| < \delta_\gamma/2$ for all $z \in T_\gamma$. Thus, if $x \in K_\gamma$ then there is $z \in T_\gamma$ and $y \in L_\gamma$ such that $x = zy$ and

$$\begin{aligned}|1-\overline{\gamma(x)}\lambda(x)| &\leqslant |1-\overline{\gamma(z)}\lambda(z)| + |\overline{\gamma(z)}\lambda(z) - \overline{\gamma(zy)}\lambda(zy)|\\ &= |1-\overline{\gamma(z)}\lambda(z)| + |1-\overline{\gamma(y)}\lambda(y)| < \delta_\gamma.\end{aligned}$$

We conclude that $\gamma U_\gamma \cap V \subset \gamma U(K_\gamma, \delta_\gamma) \cap V$. Finally $\{\gamma U_\gamma : \gamma \in V\}$ is a cover of V by sets that are open in M. M is compact and V is closed as a subset of M so V is compact as a subset of M, and hence $\{\gamma U_\gamma : \gamma \in V\}$ has a finite subcover which leads to a finite subcover of our original cover $\mathcal{U}$. The result is proved. □

Remark 6.10. The above shows that the dual of a locally compact Hausdorff Abelian topological group is a locally compact Hausdorff Abelian topological group. Pontryagin duality is a powerful strengthening of this in which a crucial part is showing that characters separate points. This can be deduced from the Peter-Weyl Theorem.

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