C3.4 Algebraic Geometry

Lecture 1: Introduction

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PRELIMINARY VERSION. WILL BE REWRITTEN.

With particular thanks to Balázs Szendrői, on whose lectures these lectures are based.
Thanks also to Alexander Ritter and Damian Rössler.

What is algebraic geometry about?

Algebraic geometry is the study of geometric spaces given by polynomial equations. More precisely, it is the study of geometric spaces given by the *vanishing* of polynomial equations of (Cartesian) coordinates.

Polynomials: very natural notion. As soon as we have "numbers" that we can add and multiply, we can take a bunch of variables and write down polynomials.

We get polynomial rings $S[x_1, x_2, \ldots]$ over rings S.

In this course,

- we will consider polynomial rings over fields k, and
- we will have a finite number of indeterminates (variables), so

$$R = k[x_1, \ldots, x_n].$$

We will think of variables $x_1, \ldots, x_n$ as Cartesian coordinates on affine space

$$p = (x_1, \ldots, x_n) \in \mathbb{A}^n = \mathbb{A}^n_k = k^n.$$

Familiar examples

You already know some examples from earlier studies!

- Fix constants $a_1, \ldots, a_n, c \in k$. Then

$$H = \left\{ \sum_{i=1}^{n} a_i x_i - c = 0 \right\} \subset \mathbb{A}^n$$

  is an *affine hyperplane*. If $c = 0$, then H is a *hyperplane* (codimension one linear subspace).

- More generally, if we take several such equations, we get *affine linear subspaces*, respectively (if all constants are zero) *linear subspaces*.
- Take $k = \mathbb{R}$. Then

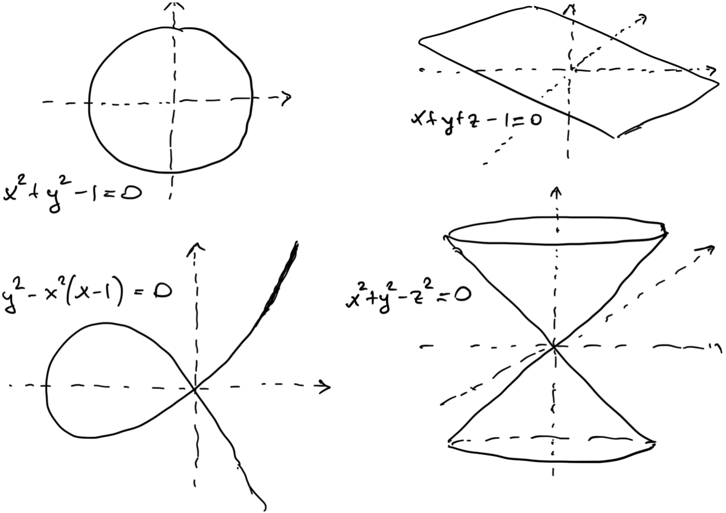
$$\{x^2 + y^2 - 1 = 0\} \subset \mathbb{A}^2_{\mathbb{R}}$$

  is a *circle*. More generally, any quadratic equation in (x, y) describes a (real) *plane conic* or *conic section*.

More familiar examples

- Take $k = \mathbb{C}$. Then
$$\{p(x,y) = 0\} \subset \mathbb{A}^2_{\mathbb{C}}$$
for any polynomial $p \in \mathbb{C}[x,y]$ is a (complex) *affine plane curve*, studied in the Oxford Part B course on Complex Algebraic Curves (and of course elsewhere).
- Take $k = \mathbb{R}$ again. Then
$$\{x^2 + y^2 - z^2 - c = 0\} \subset \mathbb{A}^3_{\mathbb{R}}$$
is a *hyperboloid*, with number of sheets depending on the sign of $c \neq 0$. For $c = 0$, we get the *quadric cone*.
- With $k = \mathbb{R}$ and arbitrary n, we have the (real) $(n-1)$-*sphere*
$$S^{n-1} = \left\{ \sum_{i=1}^{n} x_i^2 - 1 = 0 \right\} \subset \mathbb{A}^n_{\mathbb{R}}.$$

Familiar examples over $\mathbb{R}$ in pictures



General features

- A lot of the theory works for arbitrary fields k. We will assume
  - k has characteristic 0;
  - k is algebraically closed.
- Just take $k = \mathbb{C}$ if you wish!
- Number of variables will be arbitrary (finite!).
- Number of equations will also be arbitrary (finite! but now not a restriction).
- Drawing pictures remains a lot easier if $k = \mathbb{R}$ and $n \leq 3$...
- Don't forget other fields such as $\mathbb{F}_p, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t), \ldots$ in further studies.

Applications of algebraic geometry

- Within pure mathematics: interacts with many different fields!
  - Key role in Wiles' proof of Fermat's Last Theorem
- Recent prominent role in theoretical physics
  - Spacetime models in string theory from algebraic geometry via supersymmetry
- Prominent applications in other areas
  - Algebraic robotics: describe motion of automate constrained by polynomial conditions.
  - Cryptography: cryptosystems from geometry (elliptic curves, abelian varieties...)
  - Algebraic systems biology: describe equilibria of complicated polynomial interaction systems

Sources of information

- Lecture notes by Prof Ritter on course website
  - Will follow the same notation.
  - The material in lectures forms a subset of the notes; will ignore categorical aspects but feel free to read those sections for a different, important point of view.
- Books
  - Many books around. Reid: UAG is perhaps the most useful. Hartshorne: Algebraic geometry is the "bible" but is too advanced just for this course.
- Problem sheets
  - There will be 5 problem sheets in total. Sheet 0 is not for handing in.

Commutative algebra in algebraic geometry

k denotes a field, algebraically closed and of characteristic 0, with unit $1 \in k$.
R a finitely generated, unital, commutative k-algebra: finitely generated as a commutative ring, has multiplication by elements of k, also has unit $1 \in R$. For example,

$$R = k[x_1, \ldots, x_n].$$

We will consider ideals $I \lhd R$, their intersections, products, quotient rings, etc. Also ring/algebra homomorphisms, kernels, images, etc.
I will quote results from Commutative Algebra. They can be taken without proof in this course; the Part B course Commutative algebra proves most of these results.

A simple but important proposition

Proposition 1.1

Let k be a field, S a finitely generated commutative k-algebra. Then

$$S \cong k[x_1, \ldots, x_n]/I$$

for some n and an ideal $I \lhd k[x_1, \ldots, x_n]$.

Proof.

Let $s_1, \ldots s_n \in S$ be a set of k-algebra generators of S. Consider the ring homomorphism

$$\varphi \colon k[x_1, \ldots, x_n] \to S$$

defined by $\varphi(x_i) = s_i$. Then φ is surjective, since s_i generate S. Considering

$$I = \ker \varphi \lhd k[x_1, \ldots, x_n],$$

we get indeed

$$S \cong k[x_1, \ldots, x_n]/I$$

by the Isomorphism Theorem for rings. □

Vanishing sets

We are working in the space $k^n = \{a = (a_1, \ldots, a_n) : a_j \in k\}$.
This space corresponds to the polynomial ring $R = k[x_1, \ldots, x_n]$.
$X \subset k^n$ is an *affine (algebraic) variety*, if $X = \mathbb{V}(I)$ for some ideal $I \subset R$, where

$$\mathbb{V}(I) = \{a \in k^n : f(a) = 0 \text{ for all } f \in I\} \subset k^n.$$

Example 1.2

- For the zero ideal, $\mathbb{V}(0) = k^n$.
- For the ideal $\langle 1 \rangle = R$ generated by the identity, $\mathbb{V}(R) = \emptyset$.
- For some nonconstant $f \in R \setminus k$ generating principal ideal $\langle f \rangle \lhd R$, we get

$$V_f = \mathbb{V}(\langle f \rangle) = \{a \in k^n : f(a) = 0\},$$

  the *hypersurface* defined by f.

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Lecture 2

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Recollection: vanishing sets

We are working in the space $k^n = \{a = (a_1, \ldots, a_n) : a_j \in k\}$.
This space corresponds to the polynomial ring $R = k[x_1, \ldots, x_n]$.
$X \subset k^n$ is an *affine (algebraic) variety*, if $X = \mathbb{V}(I)$ for some ideal $I \subset R$, where

$$\mathbb{V}(I) = \{a \in k^n : f(a) = 0 \text{ for all } f \in I\} \subset k^n.$$

An easy but important example

Let $a = (a_1, \ldots, a_n) \in k^n$ and consider

$$\mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle \lhd R.$$

The following are all easy to check:

- $\mathbb{V}(\mathfrak{m}_a) = \{a\} \subset k^n$.
- The ideal $\mathfrak{m}_a \lhd R$ is the kernel of the evaluation homomorphism

  $$\mathrm{ev_a} \colon R \to k$$

  defined by $f \mapsto f(a)$.
- The ideal $\mathfrak{m}_a \lhd R$ is a maximal ideal of R.

Recall that an ideal $\mathfrak{m} \lhd R$ of a ring is *maximal* if it is not equal to R, nor is properly contained in another proper ideal of R. Remember that $\mathfrak{m} \lhd R$ is maximal if and only if the quotient $R/\mathfrak{m}$ is a field.

Some basic properties of vanishing sets

1. $I \subset J \Rightarrow \mathbb{V}(I) \supset \mathbb{V}(J)$.
2. $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cdot J) = \mathbb{V}(I \cap J)$.
3. $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I + J)$. (Note: $\langle I \cup J \rangle = I + J$.)
4. If I, J are relatively prime (i.e. $I + J = \langle 1 \rangle$), then $\mathbb{V}(I), \mathbb{V}(J)$ are disjoint.

The proofs are easy exercises.
Assuming that k is algebraically closed, the converse to (4) also holds; to prove this, one needs to use Corollary 2.8 below.

Hilbert's Basis Theorem

Theorem 2.1 (Hilbert's Basis Theorem)

$R = k[x_1, \ldots, x_n]$ *is a Noetherian ring. In other words, it satisfies the following equivalent conditions.*

1. *Every ideal is* **finitely generated** *(f.g.)*

$$I = \langle f_1, \ldots, f_m \rangle = Rf_1 + \cdots + Rf_m.$$

2. **ACC** *(Ascending Chain Condition) on ideals:*

$$I_1 \subset I_2 \subset \cdots \text{ ideals } \Rightarrow I_N = I_{N+1} = \cdots$$

$$\text{eventually all become equal.}$$

Equations of affine varieties

Corollary 2.2

Any vanishing set $V = \mathbb{V}(I)$ is the common zero locus in k^n of a finite *number of polynomials:*

$$V = \{a \in \mathbb{A}^n \colon f_1(a) = \ldots = f_m(a) = 0\}.$$

Proof.

Use the Hilbert Basis Theorem: take a set of generators $f_1, \ldots, f_m$ of the ideal I. So

$$I = \langle f_1, \ldots, f_m \rangle \lhd R.$$

Then clearly $f(a) = 0$ for all $f \in I$ if and only if $f_i(a) = 0$ for all $i = 1, \ldots, m$. □

We will often refer to $f_1 \ldots, f_m$ as the "equations of V", even though the set of equations is not really well defined.

On Noetherian rings

Easy Proposition 2.3

If R is Noetherian, any quotient of R is also Noetherian.

Corollary 2.4

A finitely generated k-algebra S is Noetherian.

Easy Proposition 2.5

If R is Noetherian, any ideal of I is contained in a maximal ideal $\mathfrak{m}$.

Proof.

Keep adding elements; eventually you must get to a maximal ideal by the ACC. $\square$

This statement is true in fact in arbitrary rings, but the proof is harder and requires Zorn's Lemma.

Hilbert's Weak Nullstellensatz

Theorem 2.6 (Weak Nullstellensatz)

Assume that k is algebraically closed. Then every maximal ideal of the ring $R = k[x_1, \ldots, x_n]$ is of the form $\mathfrak{m}_a \lhd R$ for some $a = (a_1, \ldots, a_n) \in k^n$.

This fails over fields that are not algebraically closed.

Example 2.7

Let $k = \mathbb{R}$, $R = \mathbb{R}[x]$, and $I = \langle x^2 + 1 \rangle$.
Then $I = \ker \psi$ for $\psi \colon R \to \mathbb{C}$ given by $f \mapsto f(i)$.
So R/I is a field, and in particular $I \lhd R$ is maximal. But clearly I is a principal ideal not generated by degree one polynomial(s).

Corollary 2.8

$\mathbb{V}(I) = \emptyset \Leftrightarrow 1 \in I \Leftrightarrow I = R.$

Proof.

If $1 \notin I$ then I is a proper ideal, so it lies inside some maximal ideal $\mathfrak{m}$. By the Weak Nullstellensatz $\mathfrak{m} = \mathfrak{m}_a$ for some $a \in \mathbb{A}^n$.
But $I \subset \mathfrak{m}_a$ implies $\mathbb{V}(I) \supset \mathbb{V}(\mathfrak{m}_a) = \{a\}$. □

The Zariski topology on k^n

The *Zariski topology* on k^n is defined by declaring that the closed sets are the sets of the form $\mathbb{V}(I)$.

Easy Proposition 2.9

This is indeed a topology.

Proof.

Use easy properties of vanishing sets listed above. $\square$

The open sets of the topology look as follows:

$$\begin{aligned} U_I &= k^n \setminus \mathbb{V}(I) = k^n \setminus (\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_m)) \\ &= (k^n \setminus \mathbb{V}(f_1)) \cup \cdots \cup (k^n \setminus \mathbb{V}(f_m)) \\ &= D_{f_1} \cup \cdots \cup D_{f_m}, \end{aligned}$$

where $I = \langle f_1, \ldots, f_m \rangle$ and the D_{f_i} are called the *basic open sets*

$$D_f = k^n \setminus \mathbb{V}(f) = \{a \in k^n : f(a) \neq 0\}.$$

Let $\mathbb{A}^n$ be *affine n-space*, the topological space k^n with the Zariski topology.

The Zariski topology on $\mathbb{A}^1$

Example 2.10

$\mathbb{A}^1 = k$ has the following closed sets: $\emptyset, \mathbb{A}^1$, and all finite subsets of $\mathbb{A}^1$.

Proof.

Let $I \lhd R = k[x]$. As R is a PID, we have $I = \langle f \rangle$ for some $f \in R$. If f is not constant, it has a finite set of roots.
Conversely, any finite subset of k is clearly the root set of some polynomial f. $\square$

Correspondingly, the open sets in $\mathbb{A}^1$ are $\emptyset, \mathbb{A}^1$, and the complement of any finite set of points.
Some things to observe:

- This topology is not Hausdorff, since any two non-empty open sets intersect.
- The open sets are dense, as the only closed set with infinitely many points is $\mathbb{A}^1$ (note k is infinite).

The Zariski topology on $\mathbb{A}^2$

The following statement is not obvious; see Problem Sheet 1.

Example 2.11

$\mathbb{A}^2 = k^2$ has the following closed sets: $\emptyset, \mathbb{A}^2$, and finite unions of

- plane curves given by equations $p(x, y) = 0$, and
- points of $\mathbb{A}^2$.

The Zariski topology on general affine varieties

Recall that the Zariski topology on $\mathbb{A}^n = k^n$ is defined by declaring that the closed sets are the sets of the form $\mathbb{V}(I)$ for ideals $I \lhd R = k[x_1, \ldots, x_n]$.

Let $X = \mathbb{V}(I)$ be a vanishing set defined by an ideal $I \lhd R$.

Definition

The **Zariski topology** on $X \subset \mathbb{A}^n$ is the subspace topology inherited from $\mathbb{A}^n$.

A vanishing set $X = \mathbb{V}(I) \subset \mathbb{A}^n$ together with its Zariski topology is called an *affine variety*.

Note that thus the closed sets in X are $\mathbb{V}(I + J) = X \cap \mathbb{V}(J)$ for any ideal $J \subset R$, or equivalently, $\mathbb{V}(S)$ for ideals $I \subset S \subset R$.

Definition

An *affine subvariety* $Y \subset X$ is a closed subset of X.

Vanishing ideal

For any subset $X \subset \mathbb{A}^n$, let

$$\mathbb{I}(X) = \{f \in R : f(a) = 0 \text{ for all } a \in X\}.$$

Example 2.12

1. $\mathbb{I}(\{a\}) = \mathfrak{m}_a = \{f \in R : f(a) = 0\}$.
2. $\mathbb{I}(\mathbb{V}(x^2)) = (x) \subset k[x]$, so $\mathbb{I}(\mathbb{V}(I)) \neq I$ in general.

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Lecture 3

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Vanishing ideal

Recall: for any subset $X \subset \mathbb{A}^n$, let

$$\mathbb{I}(X) = \{f \in R : f(a) = 0 \text{ for all } a \in X\}.$$

Easy properties of the vanishing ideal

1. $X \subset Y \Rightarrow \mathbb{I}(X) \supset \mathbb{I}(Y)$.
2. $I \subset \mathbb{I}(\mathbb{V}(I))$.

Lemma 3.1

$\mathbb{V}(\mathbb{I}(\mathbb{V}(I))) = \mathbb{V}(I)$.

Proof.

We have $I \subset \mathbb{I}(\mathbb{V}(I))$ so $\mathbb{V}(I) \supset \mathbb{V}(\mathbb{I}(\mathbb{V}(I)))$.
Conversely, let $a \in \mathbb{V}(I) \setminus \mathbb{V}(\mathbb{I}(\mathbb{V}(I)))$.
Then there is an $f \in \mathbb{I}(\mathbb{V}(I))$ with $f(a) \neq 0$.
But such an f vanishes on $\mathbb{V}(I)$, and $a \in \mathbb{V}(I)$. Contradiction. $\square$

Corollary 3.2

$\mathbb{V}(\mathbb{I}(X)) = X$ for any affine variety X.

Dictionary between ideals and affine varieties

What we are doing here is building up a dictionary between ideals

$$I \lhd R = k[x_1, \ldots, x_n]$$

and affine varieties $X \subset \mathbb{A}^n$.

- From ideals to varieties: $I \mapsto \mathbb{V}(I)$.
- From varieties to ideals: $X \mapsto \mathbb{I}(X)$.
- One-sided inverse: $\mathbb{V}(\mathbb{I}(X)) = X$ for any affine variety X.
- The mappings are inclusion-reversing: subvarieties $Y \subset X$ correspond to over-ideals $J \supset I$.
- Maximal ideals $\mathfrak{m} \lhd R$ correspond to minimal (closed) subvarieties: points $p \in \mathbb{A}^n$.

Reducible and irreducible varieties

An affine variety X is *reducible* if $X = X_1 \cup X_2$ for proper closed subsets X_i ($X_i \subsetneq X$). Otherwise, we call X *irreducible*.

Examples and properties

1. $\mathbb{V}(x_1x_2) = \mathbb{V}(x_1) \cup \mathbb{V}(x_2)$ is reducible.
2. If X irreducible, then any non-empty open subset is dense. (Exercise)
3. If X irreducible, then any two non-empty open subsets intersect. (Exercise)

Theorem 3.3

An affine variety $X = \mathbb{V}(I) \neq \emptyset$ is irreducible if and only if its ideal $\mathbb{I}(X) \subset R$ is a prime ideal*.*

Note this is a statement about the ideal $\mathbb{I}(X) \subset R$, not about I that was used to define X on the left hand side!

Think about the example $I = \langle x^2 \rangle \lhd k[x]$ above.

Irreducible varieties and prime ideals

Theorem 3.4

An affine variety $X = \mathbb{V}(I) \neq \emptyset$ is irreducible if and only if its ideal $\mathbb{I}(X) \subset R$ is a prime ideal.

Proof.

If $\mathbb{I}(X)$ is not prime, then pick f_1, f_2 satisfying $f_1 \notin \mathbb{I}(X)$, $f_2 \notin \mathbb{I}(X)$, $f_1 f_2 \in \mathbb{I}(X)$. Then

$$X \subset \mathbb{V}(f_1 f_2) = \mathbb{V}(f_1) \cup \mathbb{V}(f_2)$$

so take $X_i = X \cap \mathbb{V}(f_i) \neq X$ (since $f_i \notin \mathbb{I}(X)$).
Conversely, if X is not irreducible, $X = X_1 \cup X_2$, $X_i \neq X$, so there are $f_i \in \mathbb{I}(X_i) \setminus \mathbb{I}(X)$ but $f_1 f_2 \in \mathbb{I}(X)$, so $\mathbb{I}(X)$ is not prime. (Here we used $\mathbb{V}(\mathbb{I}(X)) = X$ for X_i). $\square$

Irreducible varieties: some examples

Here are some examples of irreducible varieties.

1. The variety $\mathbb{A}^n$ itself is irreducible. Proof: it corresponds to the zero ideal $I = 0$, which is prime since R is an integral domain.
2. $\mathbb{V}(\langle x_n \rangle) \subset \mathbb{A}^n$ is irreducible, since $I = \langle x_n \rangle$ is a prime ideal, as
$$R/I \cong k[x_1, \ldots, x_{n-1}]$$
is an integral domain.
3. More generally, if f is an irreducible polynomial in R, then $V_f = \mathbb{V}(\langle f \rangle)$ is an irreducible variety. Indeed, $I = \langle f \rangle$ is a principal ideal in the UFD $k[x_1, \ldots, x_n]$ and as f is irreducible, it generates a prime ideal inside it.

Extending the dictionary between ideals and affine varieties

Dictionary between ideals $I \triangleleft R = k[x_1, \ldots, x_n]$ and affine varieties $X \subset \mathbb{A}^n$.

- Maps: $I \mapsto \mathbb{V}(I)$ and $X \mapsto \mathbb{I}(X)$.
- One-sided inverse: $\mathbb{V}(\mathbb{I}(X)) = X$ for any affine variety X.
- Inclusion-reversing: subvarieties $Y \subset X$ correspond to over-ideals $J \supset I$.
- Maximal ideals $\mathfrak{m} \triangleleft R$ correspond to points $p \in \mathbb{A}^n$.
- Prime ideals $I \triangleleft R$ correspond to irreducible subvarieties $X \subset \mathbb{A}^n$.

Decomposition into irreducible components

Theorem 3.5

An affine variety $X \subset \mathbb{A}^n$ can be decomposed into **irreducible components***: there exists a decomposition*

$$X = X_1 \cup X_2 \cup \cdots \cup X_N,$$

where $X_i \subset \mathbb{A}^n$ are irreducible affine varieties, and the decomposition is unique up to reordering if we ensure $X_i \not\subset X_j$ for all $i \neq j$.

Proof.

We show existence of the decomposition.

Suppose X is not irreducible. Then we can write $X = Y_1 \cup Y_1'$ for proper subvarieties Y_1, Y_1'. Suppose one of these is not irreducible, say $Y_1 = Y_2 \cup Y_2'$. Keep going, assume that the process does not end in a finite number of steps. We obtain a sequence $X \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots$. But then

$$\mathbb{I}(X) \subsetneq \mathbb{I}(Y_1) \subsetneq \mathbb{I}(Y_2) \subsetneq \cdots$$

This contradicts the ACC on ideals of R.

Proof of uniqueness: see notes. □

An example

Consider $I = \langle xy, xz \rangle \lhd k[x, y, z]$. Let us work out the irreducible decomposition of $X = \mathbb{V}(I)$.
We have $I = \langle x \rangle . \langle y, z \rangle$.
This means that if we let $I_1 = \langle x \rangle$, $I_2 = \langle y, z \rangle$ and $X_i = \mathbb{V}(I_i)$ then

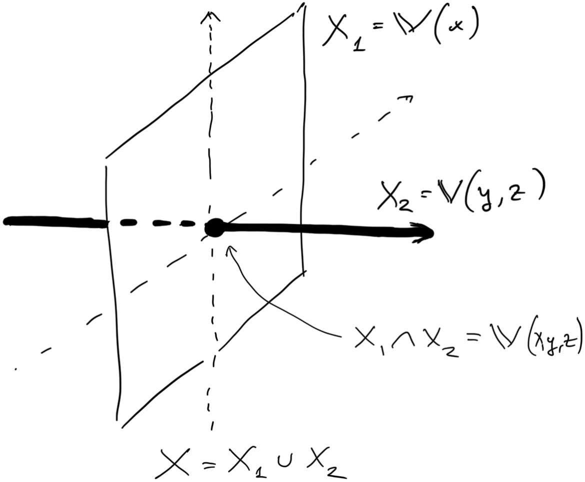
$$X = X_1 \cup X_2.$$

Clearly X_1, X_2 are not contained in each other. Also both X_i are irreducible, since $k[x, y, z]/I_i$ are both integral domains (easy check). So this is an irreducible decomposition of X.
Geometrically:

- X_1 is the (y, z) plane $\{x = 0\}$.
- X_2 is the x-axis $\{y = z = 0\}$.
- $X_1 \cap X_2 = X_3$ is given by the ideal $I_3 = I_1 + I_2 = \langle x, y, z \rangle$: the origin.

An example

Here $I = \langle xy, xz \rangle \lhd k[x,y,z]$ and $X = \mathbb{V}(I)$.



Another example

Consider $J = \langle x^2 - 4 \rangle \lhd k[x, y]$. Let us work out the irreducible decomposition of $Y = \mathbb{V}(J)$.
This polynomial is reducible, so we have $J = \langle x - 2 \rangle . \langle x + 2 \rangle$. So if we let $J_1 = \langle x - 2 \rangle$, $J_2 = \langle x + 2 \rangle$ and $Y_i = \mathbb{V}(I_i)$ then

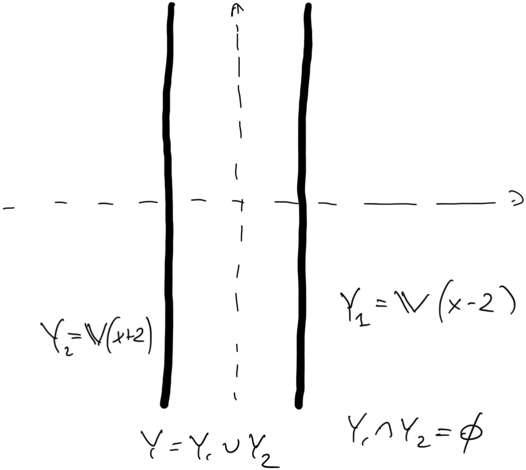
$$Y = Y_1 \cup Y_2$$

a union of two lines in $\mathbb{A}^2$. Indeed, both components are irreducible, as the corresponding ideals are prime (check!).
In this case, the intersection is given by the ideal $J_3 = J_1 + J_2 = \langle 1 \rangle = R$, so

$$Y_3 = Y_1 \cap Y_2 = \emptyset.$$

Another example

Here $J = \langle x^2 - 4\rangle \lhd k[x,y]$ and $Y = \mathbb{V}(J)$.



Ideals and varieties: completing the dictionary

We have our dictionary between ideals $I \lhd R = k[x_1, \ldots, x_n]$ and affine varieties $X \subset \mathbb{A}^n$. Maps: $I \mapsto \mathbb{V}(I)$ and $X \mapsto \mathbb{I}(X)$, and

$$\mathbb{V}(\mathbb{I}(X)) = X.$$

We also know that $\mathbb{I}(\mathbb{V}(I))$ may be different from I. Example: $I = \langle x^2 \rangle$.
Key observation: for any $X \subset \mathbb{A}^n$, $\mathbb{I}(X)$ has the property that if $f^m \in \mathbb{I}(X)$ then $f \in \mathbb{I}(X)$, as $\mathbb{I}(X)$ is defined by a *vanishing condition*.

Definition

An ideal $I \lhd R$ of a ring is called a *radical ideal*, if $r^m \in I$ for some $m \geq 1$ implies $r \in I$.
Given any ideal $I \lhd R$, its *radical* is the set

$$\sqrt{I} = \{f \in R : f^m \in I \text{ for some } m \geq 1\}.$$

Radicals and radical ideals

Easy Lemma 3.6

1. *$\sqrt{I} \lhd R$ is an ideal of R.*
2. *$I \subset R$ is radical if and only if the quotient R/I has no nilpotent elements.*
3. *$\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.*
4. *For any $X \subset \mathbb{A}^n$, $\mathbb{I}(X) \lhd R$ is a radical ideal.*

Example 3.7

$\sqrt{\langle x^2 \rangle} = \langle x \rangle$.

Expectation: we can only hope for $\mathbb{I}(\mathbb{V}(I)) = I$ if the right hand side is radical, since the left hand side is.

Caveat: condition still not sufficient!

Example 3.8

Let $k = \mathbb{R}$, and $I = \langle x^2 + 1 \rangle \lhd \mathbb{R}[x]$. Then $\mathbb{V}(I) = \emptyset$, so $\mathbb{I}(\mathbb{V}(I)) = \mathbb{R}[x]$ still does not equal I.

Strong Nullstellensatz, completion of the dictionary

Theorem 3.9 (Hilbert's Nullstellensatz)

Assume that k is algebraically closed. *For any ideal* $I \lhd R = k[x_1, \ldots, x_n]$, *we have*

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

In particular, if I *is radical then* $\mathbb{I}(\mathbb{V}(I)) = I$.

Corollary 3.10

There are order-reversing bijections between affine varieties $X \subset \mathbb{A}^n$ *and radical ideals* $I \lhd R = k[x_1, \ldots, x_n]$.

$$\begin{array}{rcl} X & \mapsto & \mathbb{I}(X) \\ \mathbb{V}(I) & \leftarrow\!\shortmid & I \\ \{varieties\} & \leftrightarrow & \{radical\ ideals\} \\ \{irreducible\ varieties\} & \leftrightarrow & \{prime\ ideals\} \\ \{points\} & \leftrightarrow & \{maximal\ ideals\} \end{array}$$

Towards proof of the Strong Nullstellensatz

We are going to assume the Weak Nullstellensatz, and deduce the strong version. Let us start with

Lemma 3.11

For any proper ideal $I \lhd R$, we have $\mathbb{V}(I) \neq \emptyset$.

Proof.

Pick a maximal ideal $I \subset \mathfrak{m} \subset R$.
By the Weak Nullstellensatz, $\mathfrak{m} = \mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a \in k^n$.
Hence $\mathbb{V}(I) \supset \mathbb{V}(\mathfrak{m}_a) = \{a\} \supset \mathbb{V}(R) = \emptyset$. □

Remember this statement already needs k algebraically closed; otherwise it is false.

Proof of NSZ, easy direction

We showed above that $\mathbb{I}(\mathbb{V}(I))$ is always radical, also $I \subset \mathbb{I}(\mathbb{V}(I))$, so

$$\sqrt{I} \subset \mathbb{I}(\mathbb{V}(I)).$$

Conclusion of the proof of the Strong Nullstellensatz

We want to show $\mathbb{I}(\mathbb{V}(I)) \subset \sqrt{I}$. Let $I = \langle f_1, \ldots, f_N \rangle$, and take $g \in \mathbb{I}(\mathbb{V}(I))$.

Trick: let $I' = \langle I, yg - 1 \rangle \subset k[x_1, \ldots, x_n, y]$.

Observe that $\mathbb{V}(I') = \emptyset \subset \mathbb{A}^{n+1}$.

By Lemma 3.11, $I' = k[x_1, \ldots, x_n, y]$.

So $1 \in I'$, giving an equality

$$1 = G_0(x_1, \ldots, x_n, y) \cdot (yg - 1) + \sum G_i(x_1, \ldots, x_n, y) \cdot f_i$$

for some polynomials G_j.

Multiplying this with a large power of g, for some large ℓ we can turn this into

$$g^\ell = F_0(x_1, \ldots, x_n, gy) \cdot (yg - 1) + \sum F_i(x_1, \ldots, x_n, gy) \cdot f_i$$

for some polynomials F_j.

Finally substitute gy by 1 here, to get

$$g^\ell = \sum F_i(x_1, \ldots, x_n, 1) \cdot f_i \in I.$$

So $g \in \sqrt{I}$ as required. $\square$

The coordinate ring of an affine variety

Let $X \subset \mathbb{A}^n$ be a (nonempty) affine variety. Let $R = k[x_1, \ldots, x_n]$ as usual.

Definition

The *coordinate ring* of X is the quotient ring

$$k[X] = R/\mathbb{I}(X).$$

Interpretation

The ring $k[X]$ is the ring of polynomial functions *on the variety* X. Polynomial functions on $\mathbb{A}^n$ which are zero everywhere along X do not change the value of such a function *on* X.

C3.4 Algebraic Geometry

Lecture 4

Dominic Joyce, Oxford University, MT 2026

The coordinate ring of an affine variety

Let $X \subset \mathbb{A}^n$ be a (nonempty) affine variety. Let $R = k[x_1, \ldots, x_n]$ as usual.

Definition

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Interpretation

The ring $k[X]$ is the ring of polynomial functions *on the variety* X. Polynomial functions on $\mathbb{A}^n$ which are zero everywhere along X do not change the value of such a function *on* X.

Properties:

- $k[X]$ is a *finitely generated, so Noetherian, k-algebra*.
- As discussed before, $\mathbb{I}(X)$ is always a radical ideal, so $k[X]$ is a *reduced ring* (it has no nilpotent elements).
- $k[X]$ is an *integral domain* if and only if X is irreducible.
- $k[X]$ is a *field* if and only if $X = \{a\}$ a point (k is algebraically closed!).

The coordinate ring of an affine variety: examples

Let $X \subset \mathbb{A}^n$ be an affine variety. Let $R = k[x_1, \ldots, x_n]$. The *coordinate ring* of X is the quotient ring

$$k[X] = R/\mathbb{I}(X).$$

Example 4.1

- For $X = \mathbb{A}^n$, we have $\mathbb{I}(X) = 0$ so $k[X] \cong R$.
- For $a \in \mathbb{A}^n$ a point, we have $\mathbb{I}(a) = \mathfrak{m}_a$, the maximal ideal at a, and as we saw before,
$$k[a] = R/\mathfrak{m}_a \cong k.$$
- For $X = V_f$ a hypersurface defined by an irreducible $f \in R$ (or at least one without repeated factors), we get
$$k[V_f] = R/\langle f \rangle,$$
the quotient of R by the principal ideal generated by f.

The ideal-subvariety correspondence for an affine variety

Consider a ring R and an ideal $I \lhd R$. Set $S = R/I$ and let $q\colon R \to S$ be the quotient map. Easy result in commutative algebra:

Proposition 4.2

There there is an inclusion-preserving one-to-one correspondence between ideals $I \subset \tilde{J} \lhd R$ and ideals $J \lhd S$, given by $\tilde{J} = q^{-1}(J)$.

Since the correspondence is inclusion-reversing, it preserves maximal ideals. We thus recover a more general version of the ideal-variety correspondence.

Corollary 4.3

There are order-reversing bijections between affine subvarieties $Y \subset X$ and radical *ideals $J \lhd k[X]$.*

$$\begin{array}{rcl}
(Y \subset X) & \mapsto & (\mathbb{I}_X(Y) \lhd k[X]) \\
(\mathbb{V}(\tilde{J}) \subset X) & \leftarrow\!\shortmid & (J \lhd k[X]) \\
\{irred.\ subvarieties\ Y \subset X\} & \leftrightarrow & \{prime\ ideals\ \mathfrak{p} \lhd k[X]\} \\
\{points\ a \in X\} & \leftrightarrow & \{maximal\ ideals\ \mathfrak{m}_a \lhd k[X]\}
\end{array}$$

Subvarieties and quotient algebras

It is worth spelling this correspondence out one more time from another point of view.
For any subvariety $Y \subset X \subset \mathbb{A}^n$, we have
$\mathbb{I}(X) \subset \mathbb{I}(Y) \lhd R = k[x_1, \ldots, x_n]$.
We also have coordinate rings

$$k[X] = R/\mathbb{I}(X)$$

and

$$k[Y] = R/\mathbb{I}(Y).$$

From $\mathbb{I}(X) \subset \mathbb{I}(Y)$, we get a surjection

$$k[X] \twoheadrightarrow k[Y],$$

with kernel $\mathbb{I}_X(Y) \lhd k[X]$, the ideal of *functions on* X *that vanish on* Y.
So closed affine subvarieties (closed inclusions of affine varieties) $Y \subset X$ correspond to surjective ring homomorphisms between their coordinate rings.

A basis for the Zariski topology on general affine varieties

We emphasize once more a basic aspect of the Zariski topology on a general affine variety $X \subset \mathbb{A}^n$.

Take $f \in k[X]$, then we get, by the correspondence above, a closed subvariety $\mathbb{V}(f) \subset X \subset \mathbb{A}^n$, and hence a Zariski open set, a so-called *basic open set*

$$D_f = X \setminus \mathbb{V}(f) = \{p \in X \colon f(p) \neq 0\}.$$

Proposition 4.4

The basic open sets form a basis of the Zariski topology on X: any open set is a finite *union of basic open sets.*

Proof.

Any open set in X is of the following form, for $I = \langle f_1, \ldots, f_m \rangle \lhd k[X]$:

$$\begin{aligned} U = X \setminus \mathbb{V}(I) &= X \setminus (\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_m)) \\ &= (X \setminus \mathbb{V}(f_1)) \cup \cdots \cup (X \setminus \mathbb{V}(f_m)) \\ &= D_{f_1} \cup \cdots \cup D_{f_m}. \end{aligned}$$

□

Basic open sets: a restatement

It seems worth spelling out the following, equivalent form of this. Let $X \subset \mathbb{A}^n$ be an affine variety.

Corollary 4.5

Let $p \in X$ and $p \in U \subset X$ a Zariski neighbourhood of p. Then there exists $f \in k[X]$ such that

$$p \in D_f \subset U$$

is also a Zariski neighbourhood of p.

Proof.

As proved above, any open set is covered by (finitely many) such. □

While simple, this is often very useful: when considering neighbourhoods of points in affine varieties, we can restrict to basic affine opens.

Morphisms between affine varieties

A function $F : \mathbb{A}^n \to \mathbb{A}^m$ is a *morphism* (or *polynomial map*) if it is defined by polynomials:

$$F(a) = (f_1(a), \ldots, f_m(a)) \quad \text{for some } f_1, \ldots, f_m \in R = k[x_1, \ldots, x_n].$$

We sometimes colloquially write $y_j = f_j(x_i)$ for the polynomial map F.
Given two affine varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, a *morphism of affine varieties* $F : X \to Y$ is defined by the restriction of a morphism $\mathbb{A}^n \to \mathbb{A}^m$, given by

$$F(a) = (f_1(a), \ldots, f_m(a)) \quad \text{for some } f_1, \ldots, f_m \in k[X].$$

Note that indeed, $f_i \in k[X]$ is enough information to define F *on* X.
We require that for all $a \in X$, the image point $F(a) = (f_1(a), \ldots, f_m(a)) \in Y$.
$F : X \to Y$ is an *isomorphism*, if F is a morphism and there is an inverse morphism $G : Y \to X$ with $F \circ G = \mathrm{id}$, $G \circ F = \mathrm{id}$.

Examples of morphisms between affine varieties

(E1) *Linear projections.* Let $n \geq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_m).$$

(E2) *Inclusions of linear subspaces.* Let $n \leq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0, \ldots, 0).$$

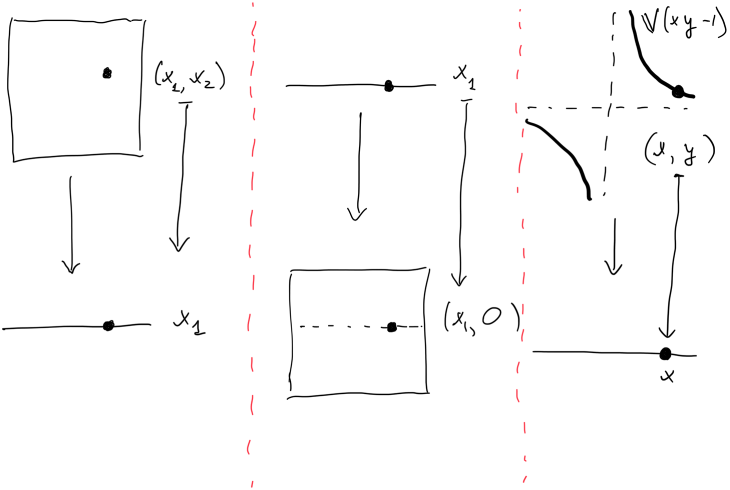
These can of course also be applied to subvarieties of $\mathbb{A}^n$. For example,

(E3) *Projection of a hyperbola* Let $X = \{xy - 1 = 0\} \subset \mathbb{A}^2$, and let $F\colon X \to \mathbb{A}^1$ be defined by

$$F(x, y) = x.$$

Note that in this example, the image set is $\mathbb{A}^1 \setminus \{0\}$ so the *image of a morphism does not need to be an affine subvariety*.

Examples of morphisms between affine varieties



More examples of morphisms between affine varieties

(E4) *A polynomial map defined on the affine line.* Let $F\colon \mathbb{A}^1 \to \mathbb{A}^3$ be defined by

$$F(t) = (t, t^2, t^3).$$

(E5) *The map to the cuspidal cubic.* Let $X = \mathbb{A}^1$, and

$$Y = \{x^3 - y^2\} \subset \mathbb{A}^2.$$

Let $F\colon X \to Y$ be defined by

$$f(t) = (t^2, t^3).$$

This is obviously a polynomial map $\mathbb{A}^1 \to \mathbb{A}^2$; to make it into a map to Y we just need to check that the image is contained in Y. This is easy.

Morphisms and coordinate rings

Given affine varieties $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, write $k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X)$, $k[Y] = k[y_1, \ldots, y_m]/\mathbb{I}(Y)$. Define

$$\mathrm{Hom}(X, Y) = \{\text{morphisms } F\colon X \to Y\}$$

and

$$\mathrm{Hom}_{k\text{-alg}}(k[Y], k[X]) = \{k\text{-algebra homs } k[Y] \to k[X]\}.$$

Theorem 4.6 (Fundamental theorem of affine algebraic geometry)

There is a one-to-one correspondence

$$\begin{array}{rcl} \mathrm{Hom}(X, Y) & \longleftrightarrow & \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X]) \\ F : X \to Y & \mapsto & F^* : k[Y] \to k[X] \\ \varphi^* : X \to Y & \leftarrow\!\shortmid & \varphi : k[Y] \to k[X] \end{array}$$

with a morphism F given in coordinates by $F(a) = (f_1(a), \ldots, f_m(a))$ and the maps given by

$$\begin{array}{rcl} F^*(y_j) & = & f_j(x_1, \ldots, x_n), \\ \varphi^*(a) & = & (\varphi(y_1)(a), \ldots, \varphi(y_m)(a)). \end{array}$$

More on morphisms and coordinate rings

The statement is almost a "tautology": it is easy to check that the maps in the statement are indeed mutual inverses (see Lecture Notes).
Note that in particular, we can think of

$$k[X] = \mathrm{Hom}(X, \mathbb{A}^1),$$

since an element $f \in k[X]$ is nothing but a function $a \mapsto f(a)$ from X to $\mathbb{A}^1$.
In this language, given a morphism $F\colon X \to Y$, the map

$$F^* : k[Y] \to k[X]$$

can be thought of as composition of F with a map to $\mathbb{A}^1$

$$F^* : \mathrm{Hom}(Y, \mathbb{A}^1) \to \mathrm{Hom}(X, \mathbb{A}^1),\ g \mapsto F^*g = g \circ F$$

and in particular, it indeed "goes backwards".

How the correspondence works in practice

It is easy to run the correspondence in practice!
Work with coordinates $\{x_1, \ldots, x_n\}$ on $\mathbb{A}^n$ and $\{y_1, \ldots, y_m\}$ on $\mathbb{A}^m$. A morphism $F: \mathbb{A}^n \to \mathbb{A}^m$ is given in these coordinates by a bunch of polynomial expressions

$$y_j = f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, m,$$

giving the map

$$F(x_1, \ldots, x_n) = (f_1(x_i), f_2(x_i), \ldots, f_m(x_i)).$$

You can read the same formulae as

$$y_j \mapsto f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, m$$

which immediately defines a ring morphism

$$k[y_1, \ldots, y_m] \longrightarrow k[x_1, \ldots, x_n],$$

which is exactly the dual map F^*!

How the correspondence works in practice

More generally, for affine varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, the story is exactly the same: have F defined by

$$F(x_1, \ldots, x_n) = (f_1(x_i), f_2(x_i), \ldots, f_m(x_i))$$

and dually F^* defined by

$$y_j \mapsto f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, m,$$

with the following conditions:

- The map F must have the property that $a \in X$ must be mapped to $F(a) \in Y$.
- Dually, the map F^* must descend to a ring map

$$k[Y] = k[y_1, \ldots, y_m]/\mathbb{I}_Y \longrightarrow k[X] = k[y_1, \ldots, x_n]/\mathbb{I}_X.$$

Morphisms and coordinate rings: Example 1

(E1) *Linear projection.* Let $n \geq m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1, \ldots, a_n) = (a_1, \ldots, a_m).$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{rccc} F^* & : & k[y_1, \ldots, y_m] & \to & k[x_1, \ldots, x_n] \\ & & y_i & \mapsto & x_i. \end{array}$$

Note that:

- The image of F is dense in $Y = \mathbb{A}^m$ (in fact it is equal to $\mathbb{A}^m$ in this case).
- F^* is injective.

Morphisms and coordinate rings: Example 2

(E2) *Inclusions of linear subspaces.* Let $n \le m$, and $F\colon \mathbb{A}^n \to \mathbb{A}^m$ be defined by

$$F(a_1,\ldots,a_n) = (a_1,\ldots,a_n,0,\ldots,0).$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{rcccl} F^* & : & k[y_1,\ldots,y_m] & \to & k[x_1,\ldots,x_n] \\ & & y_i & \mapsto & \begin{cases} x_i & \text{if } i \le n \\ 0 & \text{otherwise} \end{cases}. \end{array}$$

Note that:

- F is a closed inclusion.
- F^* is surjective.

C3.4 Algebraic Geometry

Lecture 5

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Recall: morphism between affine varieties

For affine varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, have F defined by

$$F(x_1, \ldots, x_n) = (f_1(x_i), f_2(x_i), \ldots, f_m(x_i))$$

and dually F^* defined by

$$y_j \mapsto f_j(x_1, \ldots, x_n), \quad j = 1, \ldots, m,$$

with the following conditions:

- The point $a \in X$ maps to $F(a) \in Y$.
- Dually, the map F^* must descend to a ring map

$$k[Y] = k[y_1, \ldots, y_m]/\mathbb{I}_Y \longrightarrow k[X] = k[y_1, \ldots, x_n]/\mathbb{I}_X.$$

Morphisms and coordinate rings: Example 3

(E3) **Projection of a hyperbola.** Let $X = \{xy - 1 = 0\} \subset \mathbb{A}^2$, and let $F\colon X \to \mathbb{A}^1$ be defined by

$$F(x,y) = x.$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{cccccc} F^* & : & k[\mathbb{A}^1] = k[t] & \to & k[X] = k[x,y]/\langle xy-1\rangle \\ & & t & \mapsto & x. \end{array}$$

Note that

- The image $\mathbb{A}^1 \setminus \{0\}$ of F is dense in $\mathbb{A}^1$.
- F^* is injective.

Morphisms and coordinate rings: Example 4

(E4) **A polynomial map defined on the affine line.** Let $F\colon \mathbb{A}^1 \to \mathbb{A}^3$ be defined by

$$F(t) = (t, t^2, t^3).$$

This corresponds to the k-algebra homomorphism

$$\begin{array}{rcccl} F^* & : & k[\mathbb{A}^3] = k[x,y,z] & \to & k[\mathbb{A}^1] = k[t] \\ & & x & \mapsto & t \\ & & y & \mapsto & t^2 \\ & & z & \mapsto & t^3 \end{array}$$

Note that, once again, this is a closed inclusion, and correspondingly the ring map is surjective.

Morphisms and coordinate rings: a variant of Example 4

(E4′) **A variant.** Let

$$Y = \{x^2 - y = 0, xy - z = 0\} \subset \mathbb{A}^3.$$

Let $F\colon \mathbb{A}^1 \to Y$ be defined by

$$F(t) = (t, t^2, t^3).$$

Note we are using the same formulae; to make sure this is a map to Y, we need to make sure that the image points are contained in Y, but that's easy to check. This corresponds to the k-algebra homomorphism

$$\begin{array}{rclcl} F^* & : & k[Y] = k[x,y,z]/\langle xy - z, y - x^2\rangle & \to & k[\mathbb{A}^1] = k[t] \\ & & x & \mapsto & t \\ & & y & \mapsto & t^2 \\ & & z & \mapsto & t^3 \end{array}$$

To make sure that this is well defined, we need to check that polynomials in the defining ideal of Y are mapped to zero, but that's easy.

Morphisms and coordinate rings: a variant of Example 4

(E4′) **A variant, continued.** On the other hand, let $G\colon Y \to \mathbb{A}^1$ be defined by $G(x,y,z)=x$, so

$$\begin{array}{rcrcl} G^* & : & k[\mathbb{A}^1]=k[t] & \to & k[Y]=k[x,y,z]/\langle xy-z, y-x^2\rangle \\ & & t & \mapsto & x \end{array}$$

Compute composites:

$$(F^*\circ G^*)(t)=F^*(G^*(t))=F^*(x)=t$$

whereas

$$\begin{array}{rclclll} (G^*\circ F^*)(x) & = & G^*(t) & = & x & & \\ (G^*\circ F^*)(y) & = & G^*(t^2) & = & x^2 & =y & \mod \mathbb{I}(Y) \\ (G^*\circ F^*)(z) & = & G^*(t^3) & = & x^3 & =z & \mod \mathbb{I}(Y). \end{array}$$

So F^*,G^* are mutual inverses, and so are F,G. In other words, $\mathbb{A}^1$ and Y are *isomorphic affine varieties*.

General properties of the correspondence

The correspondence between morphisms $F\colon X \to Y$ between affine varieties, and dual morphisms $F^*\colon k[Y] \to k[X]$ between coordinate rings, has the following basic properties.

- F is the *inclusion of a closed affine subvariety* if and only if F^* is *surjective*.
  We call such F a *closed embedding*.
- The image of F is *a Zariski dense subvariety of* Y if and only if F^* is *injective*.
  We call such F *dominant*.

For proofs, see Problem Sheet 1. For examples, see earlier!
Here ends (for now) our discussion of affine algebraic geometry.

Motivation for projective algebraic geometry

Affine algebraic geometry is a natural and beautiful subject. However, it lacks an essential "completeness" or "compactness" property.
This is *not* compactness in the Zariski topology, *nor* completeness in a metric sense, but rather the issue of "including points at infinity".

Bézout's theorem. Two non-overlapping plane curves of degrees d_1 and d_2 meet in *exactly* $d_1 \cdot d_2$ points.

- We need to work over an algebraically closed field.
- We need to count multiplicities.
- We need to *work in projective space*.

Simple special case. Two distinct lines in the plane meet in *exactly* one point.

- No need for an algebraically closed field; no issue with multiplicities.
- Still need to *work in projective space*!

Ways to discuss projective space

There are different ways to approach projective space and projective geometry, all having different merits.

(1) Axiomatic approach.
(2) Approach based on linear algebra.
(3) Coordinate-based approach.

Of these, (1) is beautiful but takes a long time to get to substantial results. We are going to use a combination of (2) and, for the most part, (3), following on from the Oxford Part A Projective Geometry course.

Definition of projective space via linear algebra

Let V be a finite-dimensional vector space over a field k. We let

$$\mathbb{P}(V) = \{\text{lines through the origin (1-dim linear subspaces) in } V\},$$

the *projective space of* V.
Let $k^* = k \setminus \{0\} =$ be the set of units (invertible elements) of k.
Since every line through the origin in V is determined by a direction (every 1-dim subspace has a generating vector, well defined up to scale), we can write

$$\mathbb{P}(V) = (V \setminus \{0\}) \,/ \sim$$

where $\sim$ is the equivalence relation on nonzero vectors in V given by

$$v \sim w \text{ if and only if } w = \lambda v \text{ for some } \lambda \in k^*.$$

Sometimes we simply write

$$\mathbb{P}(V) = (V \setminus \{0\}) \,/\, k^*$$

with the rescaling action of k^* understood.

Coordinate-based definition of projective space

It is easy to turn this description into coordinates. Choosing a basis of V, we have $V \cong k^{n+1}$, and every vector can be represented in coordinates as $v = (a_0, \ldots, a_n)$.

We get the definition of *projective n-space over k*:

$$\mathbb{P}^n_k = (k^{n+1} \setminus \{0\}) \,/\, k^* \text{ for all } \lambda \in k^*,$$

where the rescaling action is

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n) \text{ for all } \lambda \in k^*.$$

Write $[a_0 : a_1 : \ldots : a_n]$ for the equivalence class of $(a_0, a_1, \ldots, a_n) \in k^{n+1} \setminus \{0\}$; so in these *projective* or *homogeneous coordinates*,

$$[a_0 : \ldots : a_n] = [\lambda a_0 : \ldots : \lambda a_n] \qquad \text{for all } \lambda \in k^*.$$

Note not all a_i can be zero: $[0 : \ldots : 0]$ is not a valid choice.

Change of coordinates

Always remember: our coordinates $[x_0 : \ldots : x_n]$ on $\mathbb{P}^n$ depend on a *choice of basis* in the underlying k-vector space.

A change of basis corresponds to a matrix $A \in \mathrm{GL}(n+1, k)$, which gives a new coordinate system related to the old one by a linear transformation.

Change of basis might allow the simplification of a given polynomial (trick already used in Part A Projective Geometry course).

Example 5.1

For $\mathbb{P}^1$, these changes of basis correspond to the action of *Möbius transformations*:

$$[x_0 : x_1] \mapsto [ax_0 + bx_1 : cx_0 + dx_1]$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, k).$$

The projective line and the projective plane

Consider $n = 1$.
We have

$$[a_0 : a_1] = \begin{cases} [1 : 0] & \text{if } a_1 = 0 \\ [\frac{a_0}{a_1} : 1] & \text{otherwise.} \end{cases}$$

So we get

$$\mathbb{P}^1_k = \mathbb{A}^1_k \amalg \{\infty\}.$$

For $k = \mathbb{C}$, this is the *Riemann sphere* $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \amalg \{\infty\}$.
Now consider $n = 2$. We have

$$[a_0 : a_1 : a_2] = \begin{cases} [a_0 : a_1 : 0] & \text{if } a_2 = 0 \\ [\frac{a_0}{a_2} : \frac{a_1}{a_2} : 1] & \text{otherwise.} \end{cases}$$

So we get

$$\mathbb{P}^2_k = \mathbb{A}^2_k \amalg \mathbb{P}^1_k.$$

This is the description of the projective plane as the affine plane together with the *ideal line at infinity*.

Homogeneous polynomials

We continue to work with the ring of polynomials $R = k[x_0, \ldots, x_n]$, thought of as polynomials in our variables $x_0, \ldots, x_n$. We also continue to assume k algebraically closed.

Definition

A polynomial $F \in R$ is a *homogeneous polynomial of degree* d, if all the monomials $x_0^{i_0} \cdots x_n^{i_n}$ appearing in F have degree
$$d = i_0 + \cdots + i_n.$$
By convention, $0 \in R$ is homogeneous of every degree.

Lemma 5.2

$F \in R$ is homogeneous of degree d, if and only if for all $\lambda \in k^*$, we have
$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n).$$

Example 5.3

All monomials $\prod_i x_i^{d_i}$ are homogeneous (of some degree); $x_0^3 + x_1 x_2^2$ is homogeneous of degree 3; $x_0^3 + x_1 x_2 + x_3$ is not homogeneous.

Evaluating polynomials

How to evaluate polynomials on projective space?

- Given a polynomial $F \in R$, it does not make sense to "evaluate F at a point $p \in \mathbb{P}^n$", since the value of f is not well-defined because of the equivalence relation.
- Given a *homogeneous* polynomial $F \in R$, it *still* does not make sense to "evaluate F at a point $p \in \mathbb{P}^n$".
- Given a *homogeneous* polynomial $F \in R$, it *does* make sense to ask whether "F vanishes at a point $p \in \mathbb{P}^n$", as

$$F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n)$$

with $\lambda \in k^*$.

Homogeneous ideals

Definition

An ideal $I \lhd R$ is a *homogeneous ideal*, if it is generated by homogeneous polynomials.

Notes

- There is no requirement that the generators should all be of the same degree.
- Of course I will contain lots of non-homogeneous elements; the requirement is that it should be *generated* by such.

Example 5.4

- $I_1 = \langle x_0, \ldots, x_k \rangle \lhd k[x_0, \ldots x_n]$ is a homogeneous ideal generated by linear generators.
- $I_2 = \langle F \rangle \lhd k[x_0, \ldots x_n]$ for a homogeneous polynomial F is a principal, homogeneous ideal generated by a single homogeneous generator.
- $I_3 = \langle x_0^3 + x_1^3 + x_2^3, x_0^2 - x_1^2 \rangle \lhd k[x_0, x_1, x_2]$ is a homogeneous ideal generated by a quadric and a cubic.

Projective varieties

Definition

$X \subset \mathbb{P}^n$ is a *projective variety* if

$$X = \mathbb{V}(I) = \{[a] \in \mathbb{P}^n : F(a) = 0 \text{ for all homogeneous } F \in I\}$$

for some homogeneous ideal $I \lhd k[x_0 \ldots, x_n]$.

Example 5.5

- For $I_1 = \langle x_0, \ldots, x_k \rangle \lhd k[x_0, \ldots x_n]$,
$$\mathbb{V}(I_1) = \{x_0 = \ldots = x_k = 0\} \subset \mathbb{P}^n$$
is a *projective linear subspace*.
- For $I_2 = \langle F \rangle \lhd k[x_0, \ldots x_n]$ with F a homogeneous polynomial,
$$\mathbb{V}(I_2) = \{F = 0\} \subset \mathbb{P}^n$$
is a *projective hypersurface*.

More projective varieties

Some special cases:

- If $F \in k[x_0, x_1, x_2]$ is homogeneous of degree d,

$$\mathbb{V}(\langle F \rangle) = \{F = 0\} \subset \mathbb{P}^2$$

  is a *projective plane curve of degree* d.

- The *Fermat hypersurface* is

$$\mathbb{V}(\langle x_0^d + x_1^d + \ldots + x_n^d \rangle) = \left\{ x_0^d + x_1^d + \ldots + x_n^d = 0 \right\} \subset \mathbb{P}^n.$$

- Consider the vector space $V = M_n(k)$ of $n \times n$ matrices over k, with its standard basis. Then $\mathbb{P}V \cong \mathbb{P}^{n^2-1}$, with homogeneous coordinate variables being the matrix entries x_{ij}. Inside $\mathbb{P}^{n^2-1}$, we have the *determinantal hypersurface*

$$\{\det(A) = 0\} \subset \mathbb{P}^{n^2-1}.$$

  Indeed $\det$ is a homogeneous polynomial of degree n in the matrix entries.

Zariski topology on projective varieties

Definition

The *Zariski topology* on $\mathbb{P}^n$ has closed sets being projective varieties $\mathbb{V}(I)$.
The Zariski topology on a projective variety $X \subset \mathbb{P}^n$ is the subspace topology, so the closed subsets of X are $X \cap \mathbb{V}(J) = \mathbb{V}(I + J)$ for any homogeneous ideal J.
Equivalently, closed subsets in X are $\mathbb{V}(K)$ for homogeneous ideals $I \subset K \lhd R$.

Example 5.6

The Zariski closed subsets of $\mathbb{P}^1_k$ (over an algebraically closed field k) are $\mathbb{P}^1_k$ itself, $\emptyset$, and finite sets of points.

Proof.

This is basically a homogeneous version of the proof we had for $\mathbb{A}^1$. If F is a homogeneous polynomial of degree d, then

$$x_0^{-d} F(x_0, x_1) = f(x_1/x_0)$$

for an ordinary polynomial of the ratio x_1/x_0. The latter has a finite number of zeros in $\mathbb{A}^1_k \subset \mathbb{P}^1_k$. $\square$

Projective subvarieties

A *projective subvariety* $Y \subset X$ is a Zariski closed subset of $X = \mathbb{V}(I)$. So as before, $Y = \mathbb{V}(K)$ for a homogeneous ideal $I \subset K \lhd R$.

Example 5.7

- For $l \geq k$,

$$\{x_0 = \ldots = x_l = 0\} \subset \{x_0 = \ldots = x_k = 0\} \subset \mathbb{P}^n$$

  is a closed subvariety inside a projective linear subspace.

- In the projective space of matrices $\mathbb{P}M_n(k) \cong \mathbb{P}^{n^2-1}$,

$$\{A^2 = 0\} \subset \{\det(A) = 0\} \subset \mathbb{P}^{n^2-1}$$

  is a projective subvariety defined by quadratic equations. Indeed, for a matrix over a field k, if $A^2 = 0$, then $\det(A) = 0$.

C3.4 Algebraic Geometry

Lecture 6

Dominic Joyce, Oxford University, MT 2026

The affine cone

For a projective variety $X \subset \mathbb{P}^n$, the *affine cone* $\hat{X} \subset \mathbb{A}^{n+1}$ is the union of the straight lines in k^{n+1} corresponding to the points of X.

Lemma 6.1

If $\emptyset \neq X = \mathbb{V}(I) \subset \mathbb{P}^n$ for some homogeneous ideal $I \subset R$, then $\widehat{X}$ is the affine variety associated to the ideal $I \subset R$:

$$\widehat{X} = \mathbb{V}(I) \subset \mathbb{A}^{n+1}.$$

This follows basically from the definitions.
Note that the lemma does not hold if $X = \emptyset$. This will happen if $I \subset R$ does not vanish on any line in $\mathbb{A}^{n+1}$. By homogeneity of I, this forces $\mathbb{V}(I) \subset \mathbb{A}^{n+1}$ to be either $\emptyset$ or $\{0\}$, which by Nullstellensatz corresponds respectively to $I = R$ or $\sqrt{I} = \langle x_0, \ldots, x_n \rangle$.
The ideal $I = \langle x_0, \ldots, x_n \rangle$ is called the *irrelevant ideal* of the ring R.

The vanishing ideal

Definition

For any set $X \subset \mathbb{P}^n$, define the *vanishing ideal* $\mathbb{I}^h(X)$ to be the homogeneous ideal generated by the homogeneous polynomials vanishing on X:

$$\mathbb{I}^h(X) = \langle F \in R : F \text{ homogeneous}, F(X) = 0 \rangle.$$

Lemma 6.2

If I is homogeneous, then $\mathbb{V}(\mathbb{I}^h(\mathbb{V}(I))) = \mathbb{V}(I)$ and $I \subset \mathbb{I}^h(\mathbb{V}(I))$.

This follows, as in the affine case, essentially from definitions.

Lemma 6.3

For $X \subset \mathbb{P}^n$ a projective variety, we have an equality of ideals

$$\mathbb{I}^h(X) = \mathbb{I}(\widehat{X})$$

between the homogeneous vanishing ideal of X and the affine vanishing ideal of the cone over X.

Proof.

See notes. □

The projective Nullstellensatz

Let $R = k[x_0, \ldots, x_n]$, with k algebraically closed. Let $I_{irr} = \langle x_0, \ldots, x_n \rangle \lhd R$ be the irrelevant ideal.

Theorem 6.4 (Projective Nullstellensatz)

For any homogeneous ideal $I \lhd R$ with $\sqrt{I} \neq I_{irr}$, we have

$$I^h(\mathbb{V}(I)) = \sqrt{I}.$$

Proof.

We have $\mathbb{V}_{\text{affine}}(I) \neq \{0\}$ by the affine Nullstellensatz, Theorem 3.9, as $\sqrt{I} \neq I_{irr}$.
So $X = \mathbb{V}(I) \subset \mathbb{P}^n$ is non-empty, with affine cone $\widehat{X} = \mathbb{V}(I) \subset \mathbb{A}^{n+1}$.
Using Lemma 6.2 and the affine Nullstellensatz, we obtain:

$$\mathbb{I}^h(X) = \mathbb{I}(\widehat{X}) = \mathbb{I}(\mathbb{V}(I)) = \sqrt{I}. \qquad \square$$

Projective varieties and homogeneous ideals

The maps $X \mapsto \mathbb{I}^h(X)$ and $\mathbb{V}(I) \leftarrow I$ set up $1:1$ correspondences

$$\begin{aligned}
\{\text{proj. vars. } X \subset \mathbb{P}^n\} &\leftrightarrow \{\text{homogeneous radical ideals } I \neq I_{irr}\} \\
\{\text{irred. proj. vars. } X \subset \mathbb{P}^n\} &\leftrightarrow \{\text{homogeneous prime ideals } I \neq I_{irr}\} \\
\{\text{points of } \mathbb{P}^n\} &\leftrightarrow \{\text{``maximal'' homogeneous ideals } I \neq I_{irr}\} \\
\emptyset &\leftrightarrow \{\text{the homogeneous ideal } I_{irr} \text{ (or } R)\}.
\end{aligned}$$

Here a point $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n$ corresponds to the homogeneous ideal

$$\mathfrak{m}_p = \langle a_i x_j - a_j x_i : \text{all } i, j \rangle = \{\text{homogeneous polys vanishing at } a\}.$$

These are homogeneous ideals different from I_{irr}, maximal such with respect to inclusion.

Projective space covered by affine spaces

We work in projective space $\mathbb{P}^n$ over a field k. We have homogeneous coordinates $[x] = [x_0 : \ldots : x_n] \in \mathbb{P}^n$.
Define the Zariski closed sets

$$H_i = \{[x] \in \mathbb{P}^n : x_i = 0\} \subset \mathbb{P}^n$$

and the Zariski open sets

$$U_i = \mathbb{P}^n \setminus H_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\} \subset \mathbb{P}^n.$$

Easy Lemma 6.5

We have

(1) *There are bijections $U_i \leftrightarrow \mathbb{A}^n$ for each i.*

(2) *We have*

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i.$$

So $\mathbb{P}^n$ is covered by the Zariski open sets $U_0, \ldots, U_n$.

Projective space covered by affine spaces

Proof.

For (1), the bijection between U_i and $\mathbb{A}^n$ is given by

$$\phi_i : U_i \to \mathbb{A}^n$$
$$[x] = \left[\frac{x_0}{x_i} : \cdots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \cdots : \frac{x_n}{x_i}\right] \to \left(\frac{x_0}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right)$$

Next, for every point of $\mathbb{P}^n$, at least one of the homogeneous coordinates must be nonzero. So indeed $\mathbb{P}^n = \cup_{i=0}^n U_i$. □

The quantities $\frac{x_j}{x_i}$ for fixed i are called *affine coordinates* in the *affine chart* U_i.

In calculations, it is often advisable to introduce new variables for these affine coordinates:

$$y_j = \frac{x_j}{x_i}$$

are affine coordinates, identifying $U_i \subset \mathbb{P}^n$ with affine space $\mathbb{A}^n$ with coordinates $y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$.

Projective line covered by affine spaces

Example 6.6

For $n = 1$, we get

$$\mathbb{P}^1 = U_0 \cup U_1.$$

Points which have both coordinates nonzero are in both open sets. The complements of the open sets are $H_0 = \{[0:1]\}$ and $H_1 = \{[1:0]\}$. On $U_0 = \mathbb{A}^1$, we have the affine coordinate $y = \frac{x_1}{x_0}$. On $U_1 = \mathbb{A}^1$, we have the affine coordinate $z = \frac{x_0}{x_1}$.
For $k = \mathbb{C}$, this is the Riemann sphere as a union of two complex lines (planes), identified using stereographic projection:

$$\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}_0 \cup \mathbb{C}_\infty$$

with $\mathbb{C}_0$ being the chart around $0 = [0:1]$ and $\mathbb{C}_\infty$ the chart around $\infty = [1:0]$.
The special points are the north and south poles on the Riemann sphere.

Projective varieties covered by affine varieties

For any projective variety $X \subset \mathbb{P}^n$, the sets $X \cap H_i$ are Zariski closed in it, so the sets $X \cap U_i$ are Zariski open. We have

$$X = \bigcup_{i=0}^{n} (X \cap U_i).$$

Claim $X \cap U_i \subset U_i = \mathbb{A}^n$ are affine varieties.

Proof.

The set $X \cap U_i \subset \mathbb{A}^n$ is defined by the vanishing of all the equations of $X \subset \mathbb{P}^n$ after the substitution $x_i = 1$ (or, what is equivalent, re-expressing the equations in the affine coordinates on U_i). □

So indeed "projective varieties are locally affine": they are covered by open sets which are affine varieties. This is a very fruitful point of view. We will call $X \cap U_i \subset \mathbb{A}^n$ the *affine charts* of the projective variety $X \subset \mathbb{P}^n$.

Affine charts of a quadric

Example 6.7

Let us look at the quadric

$$Q = \{x_0^2 + x_1^2 - x_2^2 = 0\} \subset \mathbb{P}^2.$$

We have three affine charts

$$\mathbb{P}^2 = U_0 \cup U_1 \cup U_2.$$

(0) On U_0, we have $x_0 \neq 0$, and we have affine coordinates $y_1 = \frac{x_1}{x_0}, y_2 = \frac{x_2}{x_0}$. Dividing the equation by x_0^2, we get the affine quadric

$$Q \cap U_0 = \{1 + y_1^2 - y_2^2 = 0\} \subset U_0 = \mathbb{A}^2.$$

(1) On U_1, we have $x_1 \neq 0$, and we choose affine coordinates $z_0 = \frac{x_0}{x_1}, z_2 = \frac{x_2}{x_1}$. Dividing the equation by x_1^2, we get the affine quadric

$$Q \cap U_1 = \{z_0^2 + 1 - z_2^2 = 0\} \subset U_1 = \mathbb{A}^2.$$

(2) Finally on U_2 with variables t_0, t_1, we get

$$Q \cap U_2 = \{t_0^2 + t_1^2 - 1 = 0\} \subset U_2 = \mathbb{A}^2.$$

Affine charts and coordinate changes

Recall: our coordinates $[x_0 : \ldots : x_n]$ on $\mathbb{P}^n$ depended on a *choice of basis* in the underlying k-vector space.

A change of basis corresponds to a matrix $A \in \mathrm{GL}(n+1,k)$, which gives a new coordinate system related to the old one by a linear transformation.

With respect to a new coordinate system, we get a new system of hyperplanes, and new system of Zariski open sets identified with $\mathbb{A}^n$.

Example 6.7 (Continued)

Use the change of coordinates

$$X_0 = x_0, \quad X_1 = x_1 + x_2, \quad X_2 = x_1 - x_2$$

to write the equation of our quadric as

$$Q = \{X_0^2 + X_1 X_2 = 0\} \subset \mathbb{P}^2.$$

Then in this coordinate system, one of the open charts becomes

$$Q \cap U_2' = \{T_0^2 + T_1 = 0\} \subset U_2' = \mathbb{A}^2.$$

Projective closure

Given an affine variety $X \subset \mathbb{A}^n$, we can view $X \subset \mathbb{P}^n$ via:

$$X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

The *projective closure* $\overline{X} \subset \mathbb{P}^n$ of X is the Zariski closure (closure in the Zariski topology) of the set $X \subset \mathbb{P}^n$.
Note that, by definition,

$$\overline{X} \cap U_0 = X \subset U_0 = \mathbb{A}^n.$$

How to find equations for $\overline{X} \subset \mathbb{P}^n$? We need to *homogenize*.

Homogenizing polynomials

Given a polynomial $f \in k[x_1, \ldots, x_n]$, write $f = f_0 + f_1 + \cdots + f_d$ where f_i are the homogeneous parts of f of degree i. The *homogenization* of f is
$$\widetilde{f} = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + x_0 f_{d-1} + f_d.$$

Easy Lemma 6.8

$\widetilde{f} \in k[x_0, \ldots, x_n]$ is homogeneous of degree d, with $\widetilde{f}|_{x_0=1} = f$.

Example 6.9

- The quadric $x^2 + y^2 - 1 \in k[x, y]$ becomes the homogeneous quadric
$$x^2 + y^2 - z^2 \in k[x, y, z].$$
- The cubic $y^2 - x(x-1)(x-c) \in k[x, y]$ becomes the homogeneous cubic
$$y^2 z - x(x-z)(x-cz) \in k[x, y, z].$$

The ideal of the projective closure

Let $X = \mathbb{V}(I) \subset \mathbb{A}^n$ be an affine variety defined by an ideal $I \lhd k[x_1, \ldots, x_n]$. Define $\widetilde{I} = \langle \widetilde{f} : f \in I$ to be the homogeneous ideal in $k[x_0, \ldots, x_n]$ generated by the homogenizations of all elements of I (not just generators!).

Proposition 6.10

The projective closure $\overline{X} \subset \mathbb{P}^n$ of X of X is

$$\overline{X} = \mathbb{V}(\widetilde{I}) \subset \mathbb{P}^n.$$

Proof.

See Lecture Notes. □

Example 6.11

The Zariski closure of the affine cubic curve

$$\{y^2 = x(x-1)(x-c)\} \subset \mathbb{A}^2$$

is the projective cubic curve

$$\{y^2 z = x(x-z)(x-cz)\} \subset \mathbb{P}^2.$$

Morphisms of projective varieties

Fix projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. We work with $(m+1)$-tuples of homogeneous polynomials of the projective coordinates $x_0, \ldots, x_n$ on X. Let $R = k[x_0, \ldots, x_n]$.

Definition

A *morphism* $F : X \to Y$ of projective varieties is a function F such that for every $p \in X$, there is an open neighbourhood $p \in U \subset X$, and *homogeneous* polynomials $f_0, \ldots, f_m \in R$ *of the same degree*, with

$$F : U \to Y \text{ is given by } F([a]) = [f_0(a) : \cdots : f_m(a)].$$

(1) The fact that the degrees of the f_i are equal ensures that the map is well-defined:

$$\begin{aligned} F([\lambda a]) &= [f_0(\lambda a) : \cdots : f_m(\lambda a)] \\ &= [\lambda^d f_0(a) : \cdots : \lambda^d f_m(a)] \\ &= [f_0(a) : \cdots : f_m(a)] \\ &= F([a]). \end{aligned}$$

Morphisms of projective varieties

A morphism $F : X \to Y$ of projective varieties is defined on open sets $\in U \subset X$ by *homogeneous* polynomials $f_0, \ldots, f_m \in R$ of the same degree:

$$F : U \to Y \text{ is given by } F([a]) = [f_0(a) : \cdots : f_m(a)].$$

(2) When constructing F, we have to make sure the f_i do not vanish simultaneously at any a.
(3) Need to make sure the image point always lands in Y, i.e. the values

$$[f_0(a) : \cdots : f_m(a)]$$

have to satisfy all the defining equations of Y.
(4) Often a single set of polynomials $f_0, \cdots, f_m$ suffices, but already in relatively simple cases more than one set of polynomials may be needed.

Definition

An *isomorphism* of projective varieties is a morphism $F : X \to Y$ that has a (two-sided) inverse $G : Y \to X$.

An example: a Veronese embedding of $\mathbb{P}^1$

Consider the morphism (Veronese embedding)

$$F_1 : \mathbb{P}^1 \to \mathbb{P}^2$$

given by

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is a morphism:

(1) It is defined on the whole of $\mathbb{P}^1$ by degree 2 polynomials.

(2) The polynomials do not vanish simultaneously.

(3) There are no equations to check in the image.

The example revisited

Let

$$Y = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2.$$

Consider the morphism (Veronese embedding)

$$F_2 : \mathbb{P}^1 \to Y \subset \mathbb{P}^2$$

given by the same formula

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is a morphism from $\mathbb{P}^1$ to Y:

(1) It is defined on the whole of $\mathbb{P}^1$ by degree 2 polynomials.

(2) The polynomials do not vanish simultaneously.

(3) The image values satisfy the defining polynomial of Y:

$$(s^2)(t^2) = (st)^2.$$

C3.4 Algebraic Geometry

Lecture 7

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Return to our example (end of Lecture 6)

Start with

$$Y = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$$

and the morphism

$$F_2 : \mathbb{P}^1 \to Y \subset \mathbb{P}^2$$

given by

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is a morphism from $\mathbb{P}^1$ to Y:

(1) It is defined on the whole of $\mathbb{P}^1$ by degree 2 polynomials.

(2) The polynomials do not vanish simultaneously.

(3) The image values satisfy the defining polynomial of Y:

$$(s^2)(t^2) = (st)^2.$$

The example continued

We want to build an inverse morphism to F_2. Define

$$G : Y \to \mathbb{P}^1$$

by $[x : y : z] \mapsto [x : y]$ if $x \neq 0$, and $[x : y : z] \mapsto [y : z]$ if $z \neq 0$.
Note the Zariski open sets $\{x \neq 0\}$ and $\{z \neq 0\}$ cover Y.
We get a well-defined map, since on the overlap $x \neq 0, z \neq 0$ we have

$$[x : y] = [xz : yz] = [y^2 : yz] = [y : z].$$

It is also easy to check that $F_2 \circ G = \mathrm{id}$, $G \circ F_2 = \mathrm{id}$. So F_2, G are inverse isomorphisms of projective varieties.

Another example: projection

Consider a projective variety $X \subset \mathbb{P}^n$. Assume $[1:0:\ldots:0] \notin X$. Define

$$\pi : X \to \mathbb{P}^{n-1}$$

by the formula

$$[x_0 : \ldots : x_n] \mapsto [x_1 : \ldots : x_n].$$

This is a morphism:

(1) It is defined on X by degree 1 polynomials.

(2) The polynomials do not vanish simultaneously as $[1:0:\ldots:0] \notin X$.

(3) There are no equations to check in the image.

Geometric interpretation: Projection from the point $p = [1:0:\ldots:0]$, see Lecture Notes.

Last example: projective equivalences

An isomorphism $X \cong Y$ of projective varieties $X, Y \subset \mathbb{P}^n$ is a *projective equivalence*, if it is the restriction of a linear isomorphism

$$\mathbb{P}^n \to \mathbb{P}^n,\ [x] \mapsto [Ax]$$

given by an invertible $(n+1) \times (n+1)$ matrix A over k.
This morphism is induced by a linear isomorphism $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$, $x \mapsto Ax$, where $A \in GL(n+1, k)$.
Since $[Ax] = [\lambda Ax]$, we only care about A modulo scalar matrices $\lambda \cdot I$.
Thus we only need to consider

$$A \in PGL(n+1, k) = GL(n+1, k)/k^*.$$

In particular, the group $PGL(n+1, k)$ acts on projective space $\mathbb{P}^n$ by projective equivalences.

Graded rings

Definition

Let A be a commutative ring. An $\mathbb{N}$-*grading of* A means a direct sum decomposition

$$A = \bigoplus_{d \geq 0} A_d$$

as an abelian group under addition, so that the grading is compatible with multiplication:

$$A_d \cdot A_e \subset A_{d+e}.$$

The elements in A_d are called the *homogeneous elements of degree* d.
Note every $f \in A$ is uniquely a finite sum $\sum f_d$ of homogeneous elements $f_d \in A_d$.
A *homomorphism of graded rings* $f : A \to B$ is a homomorphism of rings which respects the grading so that $f|_{A_d} : A_d \to B_d$.
In all our examples, A will be a f.g. commutative unital k-algebra with $A_0 \cong k$.

Homogeneous ideals

Suppose A is a graded ring as above. For an ideal $I \lhd A$, let
$$I_d = I \cap A_d.$$

Definition

$I \lhd A$ is a *homogeneous ideal*, if
$$I = \bigoplus_{d \geq 0} I_d.$$

Easy Lemma 7.1

1. *$I \lhd A$ is homogeneous if and only if I generated by homogeneous elements.*
2. *I is homogeneous if and only for every $f \in I$, also all homogeneous parts $f_d \in I$.*
3. *If is I homogeneous,*
$$I \textit{ a prime ideal } \Leftrightarrow \forall \textit{ hgs } f, g \in A, fg \in I \textit{ implies } f \in I \textit{ or } g \in I.$$
4. *Sums, products, intersections, radicals of homogeneous ideals are homogeneous.*

Examples

Example 7.2

This is the key example. The ring

$$R = k[x_0, \ldots, x_n]$$

is graded, by declaring that $\deg x_i = 1$. The graded pieces are

$$R_d = k[x_0, \ldots, x_n]_d,$$

the spaces of homogeneous polynomials of degree d.

Example 7.3

We could generalize this! Declare $\deg x_i = w_i$, some positive integer. We get a *different* $\mathbb{N}$-grading on the same ring $R = k[x_0, \ldots, x_n]$. We will not use this construction in this course, but it would be an interesting direction to go in.

Quotients of graded rings

The following is still easy, but it is a key statement.

Proposition 7.4

Let $A = \oplus A_d$ be a graded ring, $I \lhd A$ a homogeneous ideal. Then the quotient ring $S = R/I$ is also graded, with

$$S_d = R_d/I_d.$$

Example 7.5

Let $f \in k[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree d. Then the ring

$$S = R/\langle f \rangle$$

is graded, with

$$S_e = \begin{cases} k[x_0, \ldots, x_n]_e & \text{if } e < d; \\ k[x_0, \ldots, x_n]_e / f \cdot k[x_0, \ldots, x_n]_{e-d} & \text{otherwise.} \end{cases}$$

The homogeneous coordinate ring of a projective variety

We work with $R = k[x_0, \ldots, x_n]$, with the usual grading in which $x_0, \ldots, x_n$ all have degree 1.

Consider a projective variety $X \subset \mathbb{P}^n$. The *homogeneous coordinate ring* $S(X)$ is the graded ring

$$S(X) = R/\mathbb{I}^h(X),$$

where $\mathbb{I}^h(X) \lhd R$ is the homogeneous ideal of X.

Example 7.6

We have

$$S(\mathbb{P}^n) = R = k[x_0, \ldots, x_n].$$

Example 7.7

For $X = \mathbb{V}(yz - x^2) \subset \mathbb{P}^2$, we have

$$S(X) = k[x, y, z]/(yz - x^2).$$

Reconstructing a projective variety from a graded ring

There is also a converse process. Suppose that A is a reduced, graded k-algebra with $A_0 \cong k$. Suppose also

Key assumption: A is generated by finitely many elements, all in degree 1.

Let $g_0, \ldots, g_n$ be a set of generators in degree 1 of A as a k-algebra. Consider the homomorphism of graded k-algebras

$$\varphi\colon R = k[x_0, \ldots, x_n] \to A$$

given by $\varphi(x_i) = g_i$. This is surjective, since the elements g_i generate A. Let

$$I = \ker \varphi \lhd R.$$

Then I is necessarily a graded ideal, since φ respects the gradings (as g_i is of degree 1!). I is also a radical ideal, since A is reduced.
Let $X = \mathbb{V}(I) \subset \mathbb{P}^N$. Then by the Nullstellensatz $\mathbb{I}^h(X) = I$. Hence

$$S(X) \cong R/I \cong A.$$

So the projective variety $X \subset \mathbb{P}^n$ has homogeneous coordinate ring A.

Veronese subrings of a graded ring

Let $A = \oplus_{e\geq 0} A_e$ be a graded ring, d a fixed integer. Consider

$$A^{(d)} = \bigoplus_{e\geq 0} A_{d\cdot e},$$

the d-th Veronese subring of A. This is a graded ring, with *new grading* $A^{(d)}_e = A_{d\cdot e}$.

Example 7.8

Let $A = k[x, y]$ in the standard grading. Then for $d = 2$,

$$A^{(2)} = k \oplus \langle x^2, xy, y^2\rangle \oplus \langle x^4, x^3y, x^2y^2, xy^3, y^4\rangle \oplus \ldots.$$

More General Example

Let $R = k[x_0, \ldots, x_n]$ in the standard grading. Then

$$R^{(d)} = k \oplus k[x_0, \ldots, x_n]_d \oplus k[x_0, \ldots, x_n]_{2d} \oplus \cdots .$$

Towards the Veronese embedding

We have the graded ring

$$R^{(d)} = \bigoplus_{e \geq 0} k[x_0, \ldots, x_n]_{ed}.$$

Easy Lemma 7.9

$R^{(d)}$ is reduced, and generated by its degree 1 piece $R^{(d)}_1 = R_d$.

Proof.

$R^{(d)}$ is a subring of the reduced ring R, so it is reduced itself. Also every monomial of degree $e \cdot d$ in $x_0, \ldots, x_n$ is a product of d monomials of degree e (usually in many ways). □

Natural question What projective variety do we get if we consider this graded ring $R^{(d)}$ and perform the construction of a projective variety from it?

The first example of a Veronese embedding

Example 7.10

Consider the simplest case $n = 1$, $d = 2$. We have, as before,

$$R^{(2)} = k \oplus \langle x_0^2, x_0x_1, x_1^2\rangle \oplus \langle x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4\rangle \oplus \ldots.$$

The degree one piece $R_1^{(2)}$ is generated by three polynomials x_0^2, x_0x_1, x_1^2. So we consider the surjection

$$\varphi\colon k[y_0, y_1, y_2] \to R^{(2)}$$

given by $y_0 \mapsto x_0^2, y_1 \mapsto x_0x_1, y_2 \mapsto x_1^2$. The kernel is the ideal

$$\ker\varphi = \langle y_0y_2 - y_1^2\rangle.$$

We get

$$Y = \{y_0y_2 - y_1^2 = 0\} \subset \mathbb{P}^2,$$

the image of the Veronese embedding $F_2\colon \mathbb{P}^1 \to Y \subset \mathbb{P}^2$ we considered in Lecture 6, and early in this lecture!

Counting and listing monomials of a given degree

We will count the number of monomials of $x_0, \dots, x_n$ of degree d.

Proposition 7.11

We have

$$\dim_k k[x_0, \dots, x_n]_d = \binom{n+d}{d}.$$

Proof.

Several proofs are possible. A combinatorial proof is explained in the Lecture Notes. A proof by double induction on (n, d) is left as an exercise. $\square$

In concrete calculations, it is also useful to be able to list these monomials as a linear list. The most common way to do so is the lexicographic ordering. Instead of a detailed explanation, an example should suffice.

Example 7.12

For $n = 2$ and $d = 3$, we have $\dim_k k[x_0, x_1, x_2]_3 = \binom{5}{3} = 10$.
A lexicographically ordered basis of $k[x_0, x_1, x_2]_3$ is

$$\{x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_1x_2, x_0x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}.$$

Definition of the Veronese embedding

Fix d, n as before. Let $N = \binom{n+d}{d} - 1$.
Define *the d-th Veronese map on $\mathbb{P}^n$* to be the projective morphism

$$\begin{array}{rcccc} \nu_d & : & \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ & & [x_0 : \ldots : x_n] & \mapsto & [\ldots : x^I : \ldots] \end{array}$$

where the index set runs over all monomials $x^I = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$ of degree $d = i_0 + \cdots + i_n$, in other words a basis of the space $k[x_0, \ldots, x_n]_d$.
The image of ν_d inside $\mathbb{P}^N$ is called a *Veronese variety*.
Note: indeed ν_d is a projective morphism, as it is defined by homogeneous polynomials of the same degree d, not all zero as the monomials include $x_0^d, \ldots, x_n^d$.

Equations satisfied by the image of the Veronese map

We now want to write down equations that the image of ν_d satisfies. The morphism is defined by considering all degree d monomials x^I of $x_0, \ldots, x_n$.

Consider multi-indices of type $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$.

Note that if for multi-indices I, J, K, L, we have $I + J = K + L$, then

$$x^I x^J = x^K x^L.$$

This means that, considering variables z_I on $\mathbb{P}^N$ for all possible multi-indices I, we get

$$\mathrm{image}(\nu_d) \subset \mathbb{V}(z_I z_J - z_K z_L) \subset \mathbb{P}^N$$

for all multi-indices I, J, K, L with $I + J = K + L$.

Example 7.13

For the familiar case $n = 1$, $d = 2$, we have

$$(x_0 x_1)^2 = (x_0^2)(x_1^2)$$

which corresponds to the multi-index identity

$$(1, 1) + (1, 1) = (2, 0) + (0, 2).$$

The main result

Theorem 7.14

The Veronese map is a closed embedding $\nu_d\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^N$. It defines an isomorphism between $\mathbb{P}^n$ and its image

$$\begin{aligned} \mathrm{Image}(\nu_d) &= \mathbb{V}(\langle z_I z_J - z_K z_L : I+J=K+L\rangle) \\ &= \bigcap_{I+J=K+L} \mathbb{V}(z_I z_J - z_K z_L) \subset \mathbb{P}^N, \end{aligned}$$

where we run over all multi-indices I, J, K, L of type $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$.

C3.4 Algebraic Geometry

Lecture 8

Dominic Joyce, Oxford University, MT 2026

The main result about the Veronese map

Theorem 8.1

The Veronese map is a closed embedding $\nu_d \colon \mathbb{P}^n \hookrightarrow \mathbb{P}^N$. It defines an isomorphism between $\mathbb{P}^n$ and its image

$$\begin{aligned} \mathrm{Image}(\nu_d) &= \mathbb{V}(\langle z_I z_J - z_K z_L : I + J = K + L \rangle) \\ &= \bigcap_{I+J=K+L} \mathbb{V}(z_I z_J - z_K z_L) \subset \mathbb{P}^N \end{aligned}$$

where we run over all multi-indices I, J, K, L of type $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$.

Proof, Step 1. Let $Y = \bigcap_{I+J=K+L} \mathbb{V}(z_I z_J - z_K z_L) \subset \mathbb{P}^N$. The discussion in the previous lecture showed that $\mathrm{Image}(\nu_d) \subset Y$. So indeed we can think of ν_d as a morphism $\nu_d \colon \mathbb{P}^n \to Y$.

To finish the argument, we will write down an explicit inverse morphism for ν_d on Y.

The main result: construction of the inverse

Proof, Step 2. Fix $J = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $d - 1 = i_0 + \cdots + i_n$, and denote $J_\ell = (j_0, \ldots, j_\ell + 1, \ldots, j_n)$; these indices now sum to d. Let

$$\varphi_J : Y \setminus Z_J \to \mathbb{P}^n \text{ defined by } [\ldots : z_I : \ldots] \mapsto [z_{J_0} : z_{J_1} : \ldots : z_{J_n}]$$

which is a well-defined morphism away from the closed set Z_J where all $z_{J_\ell} = 0$.
These morphisms φ_J fit together to a morphism

$$\varphi \colon Y \to \mathbb{P}^n.$$

Indeed for two such J, J', notice $J_\ell + J'_{\ell'} = J_{\ell'} + J'_\ell$ (this equals $J + J'$ plus add 1 in the two slots ℓ, ℓ').
Hence $z_{J_\ell} z_{J'_{\ell'}} = z_{J_{\ell'}} z_{J'_\ell}$, and thus $[z_{J_\ell} : z_{J_{\ell'}}] = [z_{J'_\ell} : z_{J'_{\ell'}}]$ and so finally

$$\varphi_J([z]) = \varphi_{J'}([z]).$$

The main result: checking the inverse properties

Proof, Step 3. We show that ν_d, φ are inverse morphisms, finally proving

$$\mathbb{P}^n \cong Y \subset \mathbb{P}^N.$$

Notice

$$\varphi_J \circ \nu_d([x]) = [x^{J_0} : \ldots : x^{J_n}] = [x_0 : \ldots : x_n],$$

as we can just rescale by $1/x^J$.
Conversely, one can also check

$$\nu_d \circ \varphi_J([z_I]) = [z_I].$$

For the somewhat intricate combinatorial argument to prove this, see Lecture Notes. $\square$

Rational normal curves

The d-th Veronese embedding of $\mathbb{P}^1$. For $n = 1, d$ arbitrary, we get a closed inclusion

$$\nu_d \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$

defined by $[x_0 : x_1] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d]$.
The image is traditionally called the *rational normal curve of degree d*.
In this case, the equations can be written in the following attractive form.

$$\nu_d(\mathbb{P}^1) = \left\{ \mathrm{rank} \begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix} \leq 1 \right\} \subset \mathbb{P}_d.$$

Indeed, the condition that the rank of the given matrix is at most 1 is captured by the vanishing of 2×2 determinants

$$\det \begin{pmatrix} y_i & y_j \\ y_{i+1} & y_{j+1} \end{pmatrix} = y_i y_{j+1} - y_j y_{i+1}.$$

These are precisely the quadrics from the theorem above.

The Veronese surface

The second Veronese embedding of $\mathbb{P}^2$. For $n = 2, d = 2$, we get a closed inclusion

$$\nu_2 \colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

defined by $[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]$.
The image $S = \mathrm{im}(\nu_2) \subset \mathbb{P}^5$ is traditionally called the *Veronese surface*.

Products in algebraic geometry

Given two algebraic varieties X and Y (say affine or projective), we would like to define what it means to form the product $X \times Y$ of X and Y.
Reasonable requests from any kind of product:

(1) $X \times Y$ should be the same kind of object as X and Y are (affine or projective variety).

(2) There should be projection maps $X \times Y \to X$ and $X \times Y \to Y$ which are morphisms of the appropriate structures.

(3) The set of points of $X \times Y$ should be in one-to-one correspondence with pairs (x, y) with $x \in X$ and $y \in Y$.

A brief reflection shows that for $X = \mathbb{A}^n$ and $Y = \mathbb{A}^m$, the natural choice is $X \times Y = \mathbb{A}^{n+m}$, with affine projection maps. However, this also shows

(4) We cannot require the Zariski topology on $X \times Y$ to be the product topology; that would have too few open sets.

Ways of forming products in algebraic geometry

There are different points of view on how to form products in algebraic geometry.

(1) Algebraic: tensor product

(2) Categorical: using the *universal property* of the product

(3) Geometric: via defining equations and the Segre embedding

(1) and (2) are briefly discussed in the Lecture Notes (non-examinable). We will follow approach (3).

The product of two affine varieties

Here we take as our starting point the idea that $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ which is certainly very natural.
Start with general affine varieties given by equations as follows:

$$X = \mathbb{V}(f_1, \ldots, f_N) \subset \mathbb{A}^n, \quad f_j = f_j(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$$

and

$$Y = \mathbb{V}(g_1, \ldots, g_M) \subset \mathbb{A}^m, \quad g_i = g_i(y_1, \ldots, y_m) \in k[y_1, \ldots, y_m].$$

Definition

Define the product $X \times Y$ to be

$$X \times Y = \mathbb{V}(f_1, \ldots, f_N, g_1, \ldots, g_M) \subset \mathbb{A}^{n+m}$$

using the coordinate ring $k[\mathbb{A}^{n+m}] = k[x_1, \ldots, x_n, y_1, \ldots, y_m]$.

Checking properties of the product

For $X = \mathbb{V}(f_1, \ldots, f_N) \subset \mathbb{A}^n$ and $Y = \mathbb{V}(g_1, \ldots, g_M) \subset \mathbb{A}^m$, we define

$$X \times Y = \mathbb{V}(f_1, \ldots, f_N, g_1, \ldots, g_M) \subset \mathbb{A}^{n+m}.$$

Properties:

(1) $X \times Y$ thus defined is indeed an affine variety: it is the zero-set of a finite set of polynomials in $\mathbb{A}^{n+m}$.

(2) There is a projection morphism

$$p_1 \colon X \times Y \to \mathbb{A}^n$$

given by $p_1(x_i, y_j) = (x_i)$. If $(x, y) \in X \times Y$, then the x coordinates satisfy the polynomials f_j and so we can consider p_1 as a morphism of affine varieties

$$p_1 \colon X \times Y \to X.$$

The argument for the other projection is the same.

Checking properties of the product

For $X = \mathbb{V}(f_1, \ldots, f_N) \subset \mathbb{A}^n$ and $Y = \mathbb{V}(g_1, \ldots, g_M) \subset \mathbb{A}^m$, we define

$$X \times Y = \mathbb{V}(f_1, \ldots, f_N, g_1, \ldots, g_M) \subset \mathbb{A}^{n+m}.$$

(3) The argument used to show (2) also shows that indeed the set of points of $X \times Y$ is in one-to-one correspondence with pairs (x, y) with $x \in X$ and $y \in Y$.

Example 8.2

Let

$$X = \mathbb{V}(x^2 - 1) \subset \mathbb{A}^1$$

and

$$Y = \mathbb{V}(y^2 - 4) \subset \mathbb{A}^1.$$

Then both X and Y consist of two points $x = \pm 1$, respectively $y = \pm 2$. Their product

$$X \times Y = \mathbb{V}(x^2 - 1, y^2 - 4) \subset \mathbb{A}^2$$

consists of four points $(\pm 1, \pm 2)$.

The algebra of products

A different point of view: assume $X = \mathbb{V}(I) \subset \mathbb{A}^n$ and $Y = \mathbb{V}(J) \subset \mathbb{A}^m$ for $I \lhd k[x_1, \ldots, x_n]$ and $J \lhd k[y_1, \ldots, y_m]$.
Consider

$$K = I \cdot k[x_i, y_j] + J \cdot k[x_i, y_j] \lhd k[x_1, \ldots, x_n, y_1 \ldots, , y_m].$$

Then it is immediate from the definitions that

$$X \times Y = \mathbb{V}(K).$$

For those who know tensor products, this leads to the following statement.

Proposition 8.3

The coordinate rings of X, Y and their product are related by

$$k[X \times Y] = k[X] \otimes_k k[Y].$$

Proof.

See notes. □

How to form the product of projective varieties?

What should we do when $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$?
In the affine case, we used $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.
However, in the projective case $\mathbb{P}^n \times \mathbb{P}^m$ does not have an obvious projective variety structure. It certainly is *not* $\mathbb{P}^{n+m}$.
We need to start by realizing the set $\mathbb{P}^n \times \mathbb{P}^m$ as a projective variety.

The Segre map

The *Segre map* is the function

$$\begin{array}{rcl}\sigma_{n,m}:\mathbb{P}^n\times\mathbb{P}^m & \to & \mathbb{P}^{(n+1)(m+1)-1}=\mathbb{P}^{nm+n+m}\\ ([x_0:\cdots:x_n],[y_0:\cdots:y_m]) & \mapsto & [x_0y_0:x_0y_1:\cdots:x_iy_j:\cdots:x_ny_m]\end{array}$$

The *Segre variety* is

$$\Sigma_{n,m}=\sigma_{n,m}(\mathbb{P}^n\times\mathbb{P}^m)\subset\mathbb{P}^{nm+n+m}$$

At the moment, it is best to regard $\sigma_{n,m}$ as a map of sets, and $\Sigma_{n,m}$ as a subset of $\mathbb{P}^{nm+n+m}$.

The Segre map in matrix language

One way to understand the Segre map: in the language of matrices.
Its target $\mathbb{P}^{nm+n+m}$ is the projective space of $(n+1) \times (m+1)$ matrices.
It maps a pair of vectors $[x_i]$, $[y_j]$ to the matrix whose (i,j) entry is $z_{ij} = x_i y_j$. So

- all the columns are multiples of $[x]$, with proportionality constants y_j;
- all the rows are multiples of $[y]$, with proportionality constants x_i.

Its image is exactly the locus of rank-1 matrices (up to scale).

The Segre variety in $\mathbb{P}^3$

Example 8.4

Look at the first interesting case. We have

$$\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

mapping $([x : y], [u : v]) \mapsto [xu : xv : yu : yv]$ or perhaps

$$([x : y], [u : v]) \mapsto \left[\begin{pmatrix} xu & xv \\ yu & yv \end{pmatrix}\right].$$

From an image point $[z_{00} : z_{01} : z_{10} : z_{11}]$, we can recover

$$[x : y] = [ux : uy] = [z_{00} : z_{10}]$$

if $u \neq 0$, or

$$[x : y] = [vx : vy] = [z_{01} : z_{11}]$$

if $v \neq 0$. So in all cases, we can recover $[x : y]$.
Similarly, we can also recover $[u : v]$.
The image of $\sigma_{1,1}$ is defined by the equation $z_{00}z_{11} - z_{10}z_{01} = 0$ inside $\mathbb{P}^3$.

Properties of the Segre map

Theorem 8.5

The Segre map is an injection

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^{nm+n+m}.$$

Its image is exactly the subvariety

$$\mathbb{V}(z_{ij}z_{k\ell} - z_{kj}z_{i\ell} : 0 \leq i < k \leq n, 0 \leq j < \ell \leq m) \subset \mathbb{P}^{nm+n+m}.$$

Proof.

Everything follows from the matrix interpretation. From an image point we can recover both $[x]$ and $[y]$ as the one-dimensional column space, respectively row space.

The stated equations simply say that a matrix is of rank 1 if and only if all its 2×2 minors

$$\begin{vmatrix} z_{ij} & z_{i\ell} \\ z_{kj} & z_{k\ell} \end{vmatrix}$$

vanish. □

Products of projective varieties

We can now *define* the projective variety $\mathbb{P}^n \times \mathbb{P}^m$ to be the projective subvariety of $\mathbb{P}^{nm+n+m}$ given by the above equations. The Theorem above shows that its points are indeed in bijection with points of the set $\mathbb{P}^n \times \mathbb{P}^m$.
Given $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ projective varieties, we can now proceed as before: consider

$$\sigma_{n,m}(X \times Y) \subset \mathbb{P}^{nm+n+m}.$$

Proposition 8.6

The above set is a projective subvariety of $\mathbb{P}^{nm+n+m}$, whose points are in bijection with $X \times Y$.

Proof.

Say $X = \mathbb{V}(F_1, \ldots, F_N)$, and $Y = \mathbb{V}(G_1, \ldots, G_M)$.
The set $\sigma_{n,m}(X \times Y)$ can then be written as

$$\Sigma_{n,m} \cap \mathbb{V}(F_k(z_{0j}, \ldots, z_{nj}), G_\ell(z_{i0}, \ldots, z_{im}) : \text{all } k, \ell, i, j) \subset \mathbb{P}^{nm+n+m}.$$

As $\sigma_{n,m}$ is an injection, the rest follows. □

Projective varieties from matrices

Using the language of matrices, we can begin to write down some interesting chains of projective varieties.

Example 8.7

Let $\mathbb{P}\mathrm{M}_k(3) \cong \mathbb{P}^8$ be the projective space of 3×3 matrices over k. Then there is a chain of subvarieties

$$\Sigma_{2,2} \subset \Delta \subset \mathbb{P}^8.$$

Here

$$\Delta = \{[A] \colon \det A = 0\} \subset \mathbb{P}^8$$

is a projective cubic hypersurface, defined by the determinant polynomial. Also $\Sigma_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2$ is the Segre variety in $\mathbb{P}^8$.
Note that $\Delta \subset \mathbb{P}^8$ is the (projective) locus of singular matrices: 3×3 matrices of rank at most 2. This locus contains the locus of matrices of rank 1, which is exactly $\Sigma_{2,2}$.

The definition of the Grassmannian

We first define the Grassmannian as a set.

Definition

The *Grassmannian* of d-planes in k^n is

$$\mathrm{Gr}(d,n) = \{d\text{-dimensional vector subspaces } U \subset k^n\}.$$

Example 8.8

- $\mathrm{Gr}(1,n) = \mathbb{P}(k^n) \cong \mathbb{P}^{n-1}$.
- $\mathrm{Gr}(n-1,n) = \mathbb{P}((k^n)^*) \cong \mathbb{P}^{n-1}$.
- Projective duality more generally says

$$\mathrm{Gr}(d,n) \cong \mathrm{Gr}(n-d,n).$$

As the initial examples suggest, $\mathrm{Gr}(d,n)$ may always have the structure of a projective variety. We will show this, and work out some equations, at least in a simple case.

Describing the Grassmannian by linear algebra

The vector space k^n comes with a fixed basis $\{e_1, \ldots, e_n\}$.
Suppose that $U \subset k^n$ is a d-dimensional linear subspace. Choose a basis $u_1, \ldots, u_d$ for it. We can arrange these as rows of a $d \times n$ matrix

$$A_U \in M_k(d, n)$$

which has maximal rank d (as $u_1, \ldots, u_d$ form a basis).
If we change the basis in U using a change-of-basis matrix $P \in \mathrm{GL}(d)$, we get a new matrix

$$B_U = PA_U \in M_k(d, n),$$

also of rank d, that corresponds to the same subspace U.

Proposition 8.9

There is a one-to-one correspondence

$$\mathrm{Gr}(d, n) \leftrightarrow \mathrm{Max}(d, n)/\mathrm{GL}(d)$$

where $\mathrm{Max}(d, n) \subset M_k(d, n)$ *is the subset of matrices of maximal rank* d.

Describing the Grassmannian by linear algebra

Proposition 8.10

There is a one-to-one correspondence

$$\mathrm{Gr}(d,n) \leftrightarrow \mathrm{Max}(d,n)/\mathrm{GL}(d)$$

where $\mathrm{Max}(d,n) \subset M_k(d,n)$ *is the subset of matrices of maximal rank* d.

Example 8.11

When $d = 1$, this says that

$$\mathrm{Gr}(1,n) \leftrightarrow \mathrm{Max}(1,n)/\mathrm{GL}(1).$$

Here $\mathrm{Max}(1,n)$ is row vectors except the zero vector, so

$$\mathrm{Max}(1,n) = k^n \setminus 0.$$

Also

$$\mathrm{GL}(1) = k^*.$$

We recover

$$\mathbb{P}^{n-1} \leftrightarrow (k^n \setminus 0)/k^*.$$

C3.4 Algebraic Geometry

Lecture 9

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The Grassmannian

Definition

The *Grassmannian* of d-planes in k^n is

$$\mathrm{Gr}(d,n) = \{d\text{-dimensional vector subspaces } U \subset k^n\}.$$

Proposition 9.1

There is a one-to-one correspondence

$$\mathrm{Gr}(d,n) \leftrightarrow \mathrm{Max}(d,n)/\mathrm{GL}(d)$$

where $\mathrm{Max}(d,n) \subset M_k(d,n)$ *is the subset of matrices of maximal rank* d.

Coordinates on the Grassmannian

Suppose we want to define a map

$$\mathrm{Gr}(d,n) \to \mathbb{P}^N$$

to some projective space. What should our coordinate functions be?
We need to define functions on $\mathrm{Max}(d,n)$ which are independent of the particular vector representative chosen, i.e. which are *invariant under the action of the change of basis group* $GL(d)$.
Answer: consider $d \times d$ *minors (determinants of* $d \times d$ *submatrices) of the matrix* A_U *attached to the subspace* U. We need to choose d columns among the n columns in total, so there are $\binom{n}{d}$ such minors.
The collection of minors is invariant under the group action: if we left-multiply A_U by a $d \times d$ matrix P, all minors change by $\det P$, so the projective point they describe does not change.
Also not all minors are zero by the maximal rank assumption.

The Plücker embedding

Theorem 9.2

Associating to a subspace U the collection of $d \times d$ minors of its representing matrix A_U gives a closed embedding

$$\mathrm{Gr}(d,n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1},$$

whose image is a projective subvariety of $\mathbb{P}^{\binom{n}{d}-1}$. In particular, $\mathrm{Gr}(d,n)$ is projective.

We will not discuss the general proof. Instead, we will look at the simplest nontrivial example.
If $n \leq 3$, all Grassmannians are either points or projective spaces. So the first interesting case is when $n = 4$ and $d = 2$.

Plücker relation(s)

Example 9.3

Let $d=2, n=4$. The Plücker map is $\mathrm{Gr}(2,4)\to\mathbb{P}^5$ given in matrix form by

$$\left[\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}\right]$$
$$\longmapsto [af-be : ag-ce : ah-de : bg-cf : bh-df : ch-dg].$$

Proposition 9.4

The Plücker map is an embedding

$$\mathrm{Gr}(2,4)\hookrightarrow\mathbb{P}^5$$

with image

$$\mathbb{V}(y_0y_5-y_1y_4+y_2y_3)\subset\mathbb{P}^5.$$

Proof. It is easy to check that the image of the Plücker map satisfies this quadratic relation, the *Plücker relation*. To complete the proof, we consider affine charts.

An affine open set in the Plücker embedding

The Plücker map of $\mathrm{Gr}(2,4)$ has image contained in

$$\mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \subset \mathbb{P}^5.$$

Let us consider the affine open set

$$\mathrm{Gr}(2,4)_0 \subset \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \cap \{y_0 \neq 0\} \subset \{y_0 \neq 0\} = \mathbb{A}^5 \subset \mathbb{P}^5.$$

In the matrix coordinates used before, this means $af - be \neq 0$. So for the corresponding 2-dimensional subspace $U \subset k^4$, the first two columns of the matrix A_U are linearly independent.
This means that we can pre-multiply the matrix A_U by a unique change-of-basis matrix P so that the first two columns become the standard basis vectors of a 2-dimensional vector space.
We get an equivalence of matrices

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & C & D \\ 0 & 1 & G & H \end{pmatrix}.$$

An affine open set in the Plücker embedding

For $U \in \mathrm{Gr}(2,4)_0$, we have the representing matrix

$$\begin{pmatrix} 1 & 0 & C & D \\ 0 & 1 & G & H \end{pmatrix}.$$

We can then read off the affine Plücker coordinates of this subspace U as

$$U \mapsto (G, H, -C, -D, CH - DG).$$

We deduce the following:

- The affine Plücker relation

$$y_5 - y_1 y_4 + y_2 y_3 = 0$$

  indeed holds.
- There are no further equations involving the Plücker coordinates.
- In this open set, we can recover the subspace U uniquely from its Plücker image.

An affine open set in the Plücker embedding

Considering all such affine charts, we deduce that over the whole Grassmannian $\mathrm{Gr}(2,4)$, the Plücker map is an embedding, and its image equals

$$\mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \subset \mathbb{P}^5.$$

This proves Proposition 9.4. $\square$

From the preceding argument, we also deduce

Corollary 9.5

The affine open set $\mathrm{Gr}(2,4)_0$ is isomorphic to affine four-space $\mathbb{A}^4$.

Proof.

The projection $\mathbb{A}^5 \to \mathbb{A}^4$ to the first four coordinates, and the inclusion $\mathbb{A}^4 \to \mathbb{A}^5$ defined by

$$(x_1, x_2, x_3, x_4) \to (x_1, x_2, -x_3, -x_4, x_2x_3 - x_1x_4),$$

give inverse isomorphisms between $\mathrm{Gr}(2,4)_0 \subset \mathbb{A}^5$ and $\mathbb{A}^4$. $\square$

Irreducibility

Theorem 9.6

The Grassmannian $\mathrm{Gr}(d, n)$ *is an irreducible variety.*

We need a lemma.

Lemma 9.7

Let $\mathrm{GL}_n(k) \subset \mathbb{A}^{n^2}$ *be the space of invertible linear matrices inside the affine space of all* $n \times n$ *matrices over* k*. Then* $\mathrm{GL}_n(k)$ *is an irreducible affine variety.*

Proof.

Let $\Delta \in k[\mathbb{A}^{n^2}]$ be the determinant polynomial on the space of matrices. Then $\mathrm{GL}_n(k) = D_\Delta$, the principal open subset defined by the non-vanishing of Δ. The first statement is then a general instance of the phenomenon that a basic open set in an affine variety is affine. See Lecture 12 later!

Also $\mathrm{GL}_n(k) = D_\Delta \subset \mathbb{A}^{n^2}$ is dense, as affine space $\mathbb{A}^{n^2}$ is irreducible. A dense subset of an irreducible variety must itself be irreducible. This concludes the proof. □

Irreducibility: the proof

Proof of Theorem 9.6.

We define a surjective polynomial map

$$\varphi\colon \mathrm{GL}_n(k) \to \mathrm{Gr}(d, n).$$

A surjective image of an irreducible variety is irreducible, so the existence of the map φ proves the irreducibility of $\mathrm{Gr}(d, n)$.
Inside the vector space k^n with fixed basis $\{e_1, \ldots, e_n\}$, let $W = \langle e_1, \ldots, e_d \rangle$ be a reference d-dimensional subspace.
Suppose that $V \subset k^n$ is an arbitrary d-dimensional linear subspace.
Choose a basis $v_1, \ldots, v_d$ for it, and complete to a basis $v_1, \ldots, v_n$ of k^n.
Then $A \in \mathrm{GL}_n(k)$ with columns v_i will map W to V. This defines the surjective polynomial map φ, thus concluding the proof. □

Irreducibility: a comment

You may have noticed that this proof was a little bit cheating: the morphism

$$\varphi\colon \mathrm{GL}_n(k) \to \mathrm{Gr}(d,n)$$

is not a morphism of affine varieties, nor a morphism of projective varieties. We will fill in the gap in Lecture 11, where we will discuss morphisms of quasi-projective varieties.

Chains in varieties and dimension

Let X be a variety (affine or projective). A *chain of length* m means a strict chain of inclusions

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_m \subset X,$$

where each $X_i \subset X$ is an *irreducible* subvariety.
For example, one can start with $X_0 = \{p\}$ a point of X.
If X is irreducible, then one can end with $X_m = X$.

Definition

- The *local dimension* $\dim_p X$ of X at a point $p \in X$ is the maximum over all lengths of chains starting with $X_0 = \{p\}$.
- The *dimension* of X is the maximum of the lengths of all chains,

$$\begin{aligned} \dim X &= \max_m \left(\exists \text{ chain } X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_m\right) \\ &= \max_{p \in X} \dim_p X. \end{aligned}$$

Say X has *pure dimension* if the $\dim_p X$ are equal for all $p \in X$.

Some elementary examples

Example 9.8

(a) The chain $\mathbb{A}^0 = \{0\} = \mathbb{V}(x_1, \ldots, x_n) \subset \mathbb{A}^1 = \mathbb{V}(x_2, \ldots, x_n) \subset \cdots \subset \mathbb{A}^{n-1} = \mathbb{V}(x_n) \subset \mathbb{A}^n$ shows $\dim \mathbb{A}^n \geq n$.
(b) Similarly, $\dim \mathbb{P}^n \geq n$.
Fact. (to be discussed later) $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$, both varieties being of pure dimension.
(c) Consider $X = \mathbb{V}(xy, xz) \subset \mathbb{A}^3$.
As we saw earlier, X is a union of the (y, z)-plane and the x-axis. We have $\dim_p X = 2$ at points p in the plane, and $\dim_p X = 1$ at other points.
(d) An (affine or projective) variety X of dimension $\dim X = 0$ is a finite set of points.

Proof of Example 9.8(d).

Let $p \in X$ and X_0 an irreducible component of X containing p. Suppose $X_0 \neq \{p\}$. Then we have the chain $\{p\} \subsetneq X_0$ in X, so we would have to have $\dim X \geq \dim_p X \geq 1$. So each of the finitely many irreducible components of X must be a point. $\square$

Chains in commutative algebra

Let A be a ring (commutative with unit). A *chain of length* m means a strict chain of inclusions

$$\wp_0 \supsetneqq \wp_1 \supsetneqq \cdots \supsetneqq \wp_{m-1} \supsetneqq \wp_m$$

where each $\wp_i \lhd A$ is a prime ideal.
One can start with a maximal ideal $\wp_0 = \mathfrak{m} \subset A$. If A is an integral domain, one can end with $\wp_m = \{0\}$.
Fact For A Noetherian, the descending chain condition holds for prime ideals, i.e. there are no chains of infinite length.

Definition

The *height* $\mathrm{ht}(\wp)$ of a prime ideal $\wp \lhd A$ is the maximal length of a chain with $\wp_0 = \wp$:

$$\mathrm{ht}(\wp) = \max_m (\exists \text{ chain } \wp \supsetneqq \wp_1 \supsetneqq \cdots \supsetneqq \wp_{m-1} \supsetneqq \wp_m).$$

For an arbitrary ideal $I \lhd A$, the height of I is $\mathrm{ht}(I) = \min_\wp \mathrm{ht}(\wp)$ over all prime ideals $\wp$ containing I.

Dimension in commutative algebra

Definition

The *Krull dimension* of A is the maximum height of all maximal ideals $\mathfrak{m} \lhd A$ (equivalently, prime ideals):

$$\dim A = \max \mathrm{ht}(\mathfrak{m} \colon \mathfrak{m} \lhd A \text{ maximal}).$$

Example 9.9

1. A field $A = k$ has $\dim A = 0$.
2. If A is a PID but not a field, then $\dim A = 1$.
3. The chain $(x_1, \ldots, x_n) \supset (x_1, \ldots, x_{n-1}) \supset \cdots \supset (x_1) \supset \{0\}$ shows $\dim k[x_1, \ldots, x_n] \geq n$.

Some key results from commutative algebra

The following results are proved in commutative algebra courses.

Theorem 9.10 (Krull's principal ideal theorem, Hauptidealsatz)

For any Noetherian ring A, if $f \in A$ is neither a zero-divisor nor a unit, then

$$\mathrm{ht}((f)) = 1.$$

Theorem 9.11 (Krull's height theorem)

For any Noetherian ring A, and $\langle f_1, \ldots, f_m \rangle \neq A$,

$$\mathrm{ht}(\langle f_1, \ldots, f_m \rangle) \leq m.$$

So the height $\mathrm{ht}(\wp)$ is at most the number of generators of $\wp$.

Dimension and transcendence degree

Definition

Consider a field extension K/k. Then the *transcendence degree* $\mathrm{trdeg}_k K$ of K/k is the maximum number of elements of K which are algebraically independent over k (i.e. they satisfy no polynomial relations with coefficients in k).

A key result in field theory says that for a *finitely generated* field extension K/k, $\mathrm{trdeg}_k K = m$ if and only if there are m algebraically independent elements $\alpha_1, \ldots, \alpha_m \in K$ with $K/k(\alpha_1, \ldots, \alpha_m)$ a finite extension.

Theorem 9.12

Let A be a finitely generated k-algebra which is an integral domain. Let $\mathrm{Frac}(A)$ be the field of fractions of A. Then

$$\dim A = \mathrm{trdeg}_k \mathrm{Frac}(A).$$

Notice that in this case, a finite set of generators of A as a k-algebra provides a finite generating set for the field extension $\mathrm{Frac}(A)/k$.

Additivity for prime ideals

Our last result from commutative algebra is the following.

Theorem 9.13 (additivity for prime ideals)

Let A be a finitely generated k-algebra which is an integral domain. Then for every prime ideal $\wp \lhd A$, we have

$$\mathrm{ht}(\wp) + \dim A/\wp = \dim A.$$

Compare this with the following, much easier result.

Proposition 9.14

For any Noetherian ring A, let $f \in A$, neither a zero-divisor nor a unit. Then

$$\dim A/(f) \leq \dim A - 1,$$

but equality frequently fails.

Proof.

Lift a chain of prime ideals from $A/(f)$ to A, and use that $\mathrm{ht}((f)) = 1$ by the Hauptidealsatz, Theorem 9.10. □

A key deduction

Corollary 9.15

We have $\dim k[x_1, \ldots, x_n] = n$.

Proof 1.

We know the maximal ideals are $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$, so they have height at most n by Krull's height theorem.

So $\dim k[x_1, \ldots, x_n] \leq n$.

On the other hand, we noted the easy direction $\dim k[x_1, \ldots, x_n] \geq n$ above. □

Proof 2.

We have

$$\dim k[x_1, \ldots, x_n] = \operatorname{trdeg}_k k(x_1, \ldots, x_n).$$

But this latter quantity is clearly n, using the key result in field theory mentioned before. □

C3.4 Algebraic Geometry

Lecture 10

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Algebraic and geometric dimension coincide

Theorem 10.1

If $X \subset \mathbb{A}^n$ is an affine variety, then the (geometric) dimension of X and the (Krull) dimension of its coordinate ring agree:

$$\dim X = \dim k[X].$$

Proof.

By Hilbert's Nullstellensatz, there is an inclusion-reversing bijection between irreducible subvarieties $X_j \subset X$ and prime ideals $\wp_j \lhd k[X]$, given by $\wp_j = \mathbb{I}(X_j)$ and $X_j = \mathbb{V}(\wp_j)$. This gives a bijection between maximal chains. □

Similarly,

Theorem 10.2

For a projective variety $X \subset \mathbb{P}^n$, $\dim X$ equals the maximal length of chains of homogeneous prime ideals of its projective coordinate ring $S(X)$ which do not contain the irrelevant ideal $(x_0, \ldots, x_n)$.

In particular, $\dim X = \dim \hat{X} - 1$, where $\hat{X} \subset \mathbb{A}^{n+1}$ is the affine cone of X.

Basic properties of dimension

Proposition 10.3

If X, Y are isomorphic affine, respectively projective varieties, then they have the same dimension.

Proof.

This is clear from the geometric definition: isomorphisms map maximal chains of irreducible subvarieties to each other. □

Caveat. Note if $X \cong Y$ are *affine* varieties, then $k[X] \cong k[Y]$ and so we can also see immediately that Krull dimensions coincide.
However, this is *not* true in the projective case; an isomorphism of projective varieties does not induce an isomorphism between homogeneous coordinate rings!

Proposition 10.4

If X, Y are affine, respectively projective varieties, then
$$\dim(X \times Y) = \dim X + \dim Y.$$

Proof is sketched in the Lecture Notes.

Basic properties of dimension

Proposition 10.5

If $X \subset \mathbb{A}^n$ is an irreducible affine variety, and $\overline{X} \subset \mathbb{P}^n$ its projective closure, then

$$\dim X = \dim \overline{X}.$$

Proof.

One direction $\dim X \leq \dim \overline{X}$ is clear: given a chain of irreducible subvarieties in X, we can take their projective closure to get a chain of irreducible subvarieties of $\overline{X}$. The proof of the full result is omitted. □

Corollary 10.6

If $X \subset \mathbb{P}^n$ is an irreducible projective variety, and $U \subset \mathbb{A}^n$ an affine open subset of X, then

$$\dim X = \dim U.$$

Linear subspaces

A *linear subspace* of $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ is a projectivization $L = \mathbb{P}(U)$ of a vector subspace $U \subset k^{n+1}$.

If $\dim_k U = m+1$ in the sense of linear algebra, then

$$U = \langle v_0, \ldots, v_m \rangle \subset V.$$

Changing basis in k^{n+1} so that these vectors belong to the basis, we can write

$$L = \mathbb{P}U = \{x_{m+1} = x_{m+2} = \ldots = x_n = 0\} \subset \mathbb{P}^n.$$

So the homogeneous ideal defining L is

$$\mathbb{I}_L^h = \langle x_{m+1}, x_{m+2}, \ldots, x_n \rangle \lhd k[x_0, \ldots, x_n].$$

So its homogeneous coordinate ring is

$$S(L) \cong k[x_0, \ldots, x_m]$$

and so

$$\dim L = \dim k[L] - 1 = m$$

the projective (linear) dimension of L.

Hypersurfaces

Theorem 10.7

For an irreducible affine variety $X \subset \mathbb{A}^n$, *we have* $\dim X = n - 1$ *if and only if* $X = \mathbb{V}(f) \subset \mathbb{A}^n$ *for an irreducible* $f \in R = k[x_1, \ldots, x_n]$. *The analogous result also holds for* $X \subset \mathbb{P}^n$ *an irreducible projective variety and* f *homogeneous in* $k[x_0, \ldots, x_n]$.

Proof.

$(\Rightarrow)$: $\dim X = n - 1 \Rightarrow \mathbb{I}(X) \neq (0) \Rightarrow \exists f \neq 0 \in \mathbb{I}(X)$.
Since $\mathbb{I}(X)$ is prime, it must contain an irreducible factor of the factorization of f. So we can assume that f is irreducible, hence prime, in the UFD R.
Then $X \subset \mathbb{V}(f) \subsetneq \mathbb{A}^n$ is a chain of irreducibles, $\dim X = n - 1$ and $\dim \mathbb{A}^n = n$, thus we must have $X = \mathbb{V}(f)$.
$(\Leftarrow)$: As f is irreducible, $\wp = \langle f \rangle$ is a prime ideal. By the Hauptidealsatz, Theorem 9.10, $\mathrm{ht}(\wp) = 1$. Now use Theorem 9.13 on additivity for prime ideals. □

Such (affine or projective) varieties are called *hypersurfaces*.

Examples

Example 10.8

Let $f \in k[x_0, x_1, x_2]$ be a non-constant homogeneous polynomial of degree d. Let

$$C = \mathbb{V}(f) \subset \mathbb{P}^2.$$

Then we have $\dim C = 1$ by the Hauptidealsatz, Theorem 9.10
Indeed, $C \subset \mathbb{P}^2$ is a *plane curve*, a hypersurface in $\mathbb{P}^2$.

Example 10.9

Consider the Segre embedding $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Its image is given by

$$\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{V}(x_0x_3 - x_1x_2) \subset \mathbb{P}^3.$$

This is a hypersurface of dimension $2 = 1 + 1 = \dim \mathbb{P}^1 + \dim \mathbb{P}^1$.

An example from the Veronese embedding

Example 10.10

Consider the Veronese embedding $\nu_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$.
The image is given by

$$X = \nu_3(\mathbb{P}^1) \cong \mathbb{V}(x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2) \subset \mathbb{P}^3.$$

Consider also

$$X \subset Y = \mathbb{V}(x_0x_3 - x_1x_2) \subset \mathbb{P}^3.$$

Then we have

- $\dim \mathbb{P}^1 = 1$.
- $\dim \nu_3(\mathbb{P}^1) = 1$, as ν_3 is an isomorphism onto its image.
- $\dim \mathbb{P}^3 = 3$.
- $\dim Y = 2$, by the Hauptidealsatz, Theorem 9.10. Y is an irreducible hypersurface.
- $\dim Y - \dim X = 1$, even though $X \subset Y$ is given by *two further equations*.

Grassmannians

Recall that $\mathrm{Gr}(2,4) \cong \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \hookrightarrow \mathbb{P}^5$. This being a hypersurface, we get $\dim \mathrm{Gr}(2,4) = 4$.
Note that this is compatible with the fact that we found an affine open set $U_0 \cong \mathbb{A}^4 \subset \mathrm{Gr}(2,4)$.

Proposition 10.11

In general, we have

$$\dim \mathrm{Gr}(d,n) = d(n-d).$$

Sketch proof.

The argument given in Lecture 8 generalizes to show that for arbitrary (d,n), there is an affine open set

$$U_0 \cong \mathrm{Mat}_k(d,n-d) \cong \mathbb{A}^{d(n-d)} \subset \mathrm{Gr}(d,n).$$

As $\mathrm{Gr}(d,n)$ is irreducible, its dimension agrees with any of its affine open sets, so we deduce the statement. □

Definition of degree

Recall that a *linear subspace* of $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ is a projectivization $L = \mathbb{P}(U)$ of a vector subspace $U \subset k^{n+1}$.
Consider a projective variety $X \subset \mathbb{P}^n$ of dimension d. Denote $m = n - d$, the *complementary dimension*.

Definition

We define the *degree* of $X \subset \mathbb{P}^n$ to be

$$\deg(X) = \#(X \cap L) \colon L \subset \mathbb{P}^n \text{ a general linear subspace, } \dim L = m.$$

This needs an explanation. Consider $\mathrm{Gr}(m+1, n+1)$, the Grassmannian of $(m+1)$-dimensional subspaces of k^{n+1}.
Fact. There is an open subset $U \subset \mathrm{Gr}(m+1, n+1)$ such that for $L \in U$, there is a finite number of intersection points $(X \cap L)$, and the number $\#(X \cap L)$ is maximal. We define this maximal value to be the degree $\deg X$.
Note: As we know, $\mathrm{Gr}(m+1, n+1)$ is irreducible, so every open set is automatically dense in it.

Linear subspaces

Example 10.12

Let $X = \mathbb{P}(V)$ a linear subvariety itself, of dimension d, with $V \subset k^{n+1}$ a vector subspace of dimension $d+1$.

Then the dimension of intersection formula in projective geometry says that for any other projective linear subspace $L = \mathbb{P}U$,

$$\dim(X \cap L) = \dim X + \dim L - \dim\langle X, L\rangle.$$

So if $\dim L = n - \dim X$, the complementary dimension, we get that $\dim(X \cap L) = 0$ as long as the linear span $\langle X, L\rangle = \mathbb{P}^n$.

The latter will be the case as long as among the $m+1$ vectors giving a basis of U and the $d+1$ vectors giving a basis of V, we can find a basis of k^{n+1}. This will happen "most of the time" (in an open set of the corresponding Grassmannian).

The number of intersection points $\#(X \cap L) = 1$ in this case, so $\deg X = 1$.

Hypersurfaces

Example 10.13

Let $X = \mathbb{V}(F) \subset \mathbb{P}^n$ a hypersurface defined by a homogeneous polynomial of degree d without multiple factors. Then $\dim X = n - 1$, the complementary dimension is thus $m = 1$.
Let $L \subset \mathbb{P}^n$ be a line. We know $L \cong \mathbb{P}^1 \subset \mathbb{P}^n$. For example, as before, we can choose a basis of $\mathbb{P}^n$ so that

$$L = \{x_2 = \ldots = x_n = 0\} \subset \mathbb{P}^n.$$

Then

$$X \cap L = \{[x_0 \colon \ldots : x_n] \colon F(x_0, x_1) = 0, x_2 = \ldots = x_n = 0\}.$$

Here $F(x_0, x_1)$ is a homogeneous polynomial of degree d in two variables. The Fact referred to in the definition of degree translates here to the fact that if we choose a general L, then this polynomial will have d distinct roots.

Corollary 10.14

For $X = \mathbb{V}(F) \subset \mathbb{P}^n$ *as above,* $\deg X = d$.

Extreme cases

At one extreme, let $p_1, \ldots, p_d \in \mathbb{P}^n$ be a finite set of points and

$$X = \{p_1, \ldots, p_d\} \subset \mathbb{P}^n.$$

Note that this has the structure of a projective variety, with $\dim X = 0$, as discussed in the last lecture.
The complementary dimension is $m = n$.
So we have

$$\deg X = \#(X \cap \mathbb{P}^n) = d.$$

At the other extreme, for $X = \mathbb{P}^n \subset \mathbb{P}^n$, the complementary dimension is $m = 0$, so $L = \{p\}$ a point. So

$$\deg \mathbb{P}^n = \#(\mathbb{P}^n \cap L) = 1.$$

This is of course also covered by the case of general linear subspaces.

Weak Bézout theorem

We state the following result without proof.

Theorem 10.15 (Weak Bézout Theorem)

Let $X, Y \subset \mathbb{P}^n$ be projective varieties of pure dimension with

$$\dim X \cap Y = \dim X + \dim Y - n.$$

Then

$$\deg X \cap Y \leq \deg X \cdot \deg Y.$$

Note: the dimension assumption says that $X \cap Y$ is "of the expected dimension", given by the dimension of intersection formula for linear subspaces spanning $\mathbb{P}^n$.

Weak Bézout for plane curves

Example 10.16 (Weak Bézout for plane curves)

Let $C_1, C_2 \subset \mathbb{P}^2$ be plane curves, given by homogeneous polynomials $F_i \in k[x_0, x_1, x_2]$ of degrees d_i. Then as discussed before, $\dim C_i = 1$ and $\deg C_i = d_i$.

The dimension assumption above translates to the fact that

$$\dim C_1 \cap C_2 = \dim C_1 + \dim C_2 - 2 = 0,$$

so $C_1 \cap C_2$ is a finite set of points. In other words, it means that C_1, C_2 should have no common components.

Corollary 10.17 (Weak Bézout for curves)

With these assumptions, we have

$$\#(C_1 \cap C_2) \leq d_1 \cdot d_2.$$

Proof.

Use the general Weak Bézout Theorem 10.15, together with

$$\deg(C_1 \cap C_2) = \#(C_1 \cap C_2)$$

for the finite set $(C_1 \cap C_2)$. □

Grassmannians

Recall once again

$$\mathrm{Gr}(2,4) \cong \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \hookrightarrow \mathbb{P}^5.$$

This being a hypersurface, we get $\deg \mathrm{Gr}(2,4) = 2$.

Theorem 10.18

In general, let

$$\mathrm{Gr}(d,n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$$

be the Plücker embedding. Then in this embedding, we have

$$\deg \mathrm{Gr}(d,n) = (d(n-d))! \frac{1! \cdot 2! \cdot \ldots \cdot (d-1)!}{(n-d)! \cdot (n-d+1) \cdot \ldots \cdot (n-1)!}.$$

The proof, as you can guess from the formula, is nontrivial. Note that even the fact that the right hand side is an integer needs a moment's reflection.

The Hilbert function of a projective variety

Let $X \subset \mathbb{P}^n$ be a projective variety. Its homogeneous ideal

$$\mathbb{I}^h(X) \lhd R = k[x_0, \dots, x_n]$$

is graded:

$$\mathbb{I}^h(X) = \bigoplus \mathbb{I}^h(X)_m.$$

We also have the graded homogeneous coordinate ring

$$S(X) = R/\mathbb{I}^h(X) \cong \bigoplus S(X)_m,$$

with

$$S(X)_m = k[x_0, \dots, x_n]_m / \mathbb{I}^h(X)_m.$$

Definition

The *Hilbert function* of the graded ring $S(X)$ is the function

$$h_X : \mathbb{N} \to \mathbb{N}$$

defined by

$$h_X(m) = \dim_k S(X)_m.$$

The Hilbert function of a projective variety: first example

Easy Lemma 10.19

We have the formula

$$h_X(m) = \binom{m+n}{m} - \dim_k \mathbb{I}^h(X)_m.$$

Example 10.20

Let $X = \mathbb{P}^n$. Then $\mathbb{I}^h(X) = (0)$, so

$$\begin{aligned} h_{\mathbb{P}^n}(m) &= \binom{m+n}{m} \\ &= \frac{(m+n)\cdots(m+1)}{n!} \\ &= \frac{1}{n!}m^n + \text{lower order terms in } m. \end{aligned}$$

The Hilbert function of a plane curve

Example 10.21

Let $C = \mathbb{V}(F) \subset \mathbb{P}^2$, for F irreducible homogeneous of degree d. Then $\mathbb{I}^h(X) = \langle F \rangle$, so for $m \geq d$,

$$\mathbb{I}^h(X)_m \cong k[x_0, x_1, x_2]_{m-d}.$$

We get, for $m \geq d$,

$$\begin{aligned} h_C(m) &= \binom{m+2}{m} - \binom{m-d+2}{m-d} \\ &= \frac{(m+2)(m+1)}{2} - \frac{(m-d+2)(m-d+1)}{2} \\ &= \frac{1}{2}\big(m^2+3m+2-(m^2-2md+3m)-(d-2)(d-1)\big) \\ &= dm - \frac{(d-1)(d-2)}{2} + 1. \end{aligned}$$

The Hilbert function of a plane curve

Introduce the quantity

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Corollary 10.22

The Hilbert function of a plane curve $C = \mathbb{V}(F) \subset \mathbb{P}^2$ of degree d is given by

$$h_C(m) = d \cdot m - g(C) + 1$$

for $m \geq d$.

Note. The quantity $g(C)$ is the *genus* of C, an invariant of fundamental importance in the study of (plane) algebraic curves. There are many other definitions.

Example 10.23

- For $d \leq 2$, we have $g(C) = 0$. Indeed, for $d = 1$ we have $C = L = \mathbb{P}^1$. For $d = 2$, we have a quadric which (if nonsingular) is isomorphic to $\mathbb{P}^1$.
- For $d = 3$, we have $g(C) = 1$. This is the case of cubic elliptic curves.

C3.4 Algebraic Geometry

Lecture 11

Dominic Joyce, Oxford University, MT 2026

The Hilbert polynomial

We state the following fundamental results without proof.

Theorem 11.1

For any projective variety $X \subset \mathbb{P}^n$, there exists a polynomial $p_X \in \mathbb{Q}[x]$, such that for sufficiently large m,

$$h_X(m) = p_X(m).$$

p_X *is called the* Hilbert polynomial *of $X \subset \mathbb{P}^n$. The leading term of p_X is*

$$\frac{\deg X}{(\dim X)!} \cdot m^{\dim X}.$$

In other words, the Hilbert function is "eventually polynomial", of degree $\dim X$.

The Hilbert polynomial: examples

Example 10.20, revisited. We computed

$$h_{\mathbb{P}^n}(m) = \frac{1}{n!} m^n + \text{lower order terms in } m$$

which agrees with the statement of the Theorem: $\dim \mathbb{P}^n = n$ and $\deg \mathbb{P}^n = 1$.

Example 10.21, revisited. For $C = \mathbb{V}(F) \subset \mathbb{P}^2$ a plane curve of degree d, we computed

$$h_C(m) = d \cdot m - g(C) + 1$$

for $m \geq d$. This also agrees with the Theorem, as $\dim C = 1$ and $\deg C = d$.

An important caveat

Note that both the degree $\deg X$ and the Hilbert polynomial p_X of a projective variety $X \subset \mathbb{P}^n$ were defined *in terms of the embedding*. They are *not* invariants of X up to isomorphism! An honest, though clumsy, notation would be $\deg(X \subset \mathbb{P}^n)$ and $p_{X \subset \mathbb{P}^n}$.

Example 11.2

Recall the Veronese embedding $\nu_2 \colon \mathbb{P}^1 \to \mathbb{P}^2$. Its image is a quadric plane curve

$$C = \{x_0 x_2 - x_1^2 = 0\} \subset \mathbb{P}^2.$$

The Veronese map and its inverse on C give an isomorphism of projective varieties $C \cong \mathbb{P}^1$.
Yet we have $\deg \mathbb{P}^1 = 1$ and $\deg C = 2$, and the Hilbert polynomials are also different.

Contrast this with the quantity $\dim X$ which, as we discussed before, is an invariant of X up to isomorphism. Indeed, $\dim C = \dim \mathbb{P}^1 = 1$.

Regular functions at points and on open sets

Let $X \subset \mathbb{A}^n$ be an affine variety. Then its coordinate ring

$$k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X)$$

can be thought of as the ring of regular or polynomial functions on X. Every element $f \in k[X]$ can be interpreted as a function

$$f \colon X \to k$$

by evaluating a representing polynomial in $k[x_1, \ldots, x_n]$.
We would like to extend this definition to study functions which are regular near a point, or on some open set in X.
Let $p \in X$. A function $f : U \to k$ defined on a neighbourhood of p is called *regular at* p, if on some open set $p \in W \subset U$,

$$f|_W = \frac{g}{h} \qquad \text{for some } g, h \in k[X] \text{ with } h(w) \neq 0 \text{ for all } w \in W.$$

For an open subset $U \subset X$, write $\mathcal{O}_X(U)$ for the *k-algebra of regular functions on U*, functions $f : U \to k$ that are regular at all points $p \in U$.

Examples

1. If $f \in k[X]$ then f is regular at all points of X: use $f = \frac{f}{1}$. Thus $f \in \mathcal{O}_X(X)$.
   We will prove later that in fact this inclusion
   $$k[X] \hookrightarrow \mathcal{O}_X(X)$$
   is an isomorphism; note that this is not at all obvious!
2. If $X = \mathbb{A}^1$, then $1/x^n \in \mathcal{O}_X(U)$ for any open set U not containing the origin.
3. More generally, for any X and $f \in k[X]$,
   $$1/f^n \in \mathcal{O}_X(D_f),$$
   where
   $$D_f = X \setminus \mathbb{V}(f)$$
   is the basic open set of X defined by the non-vanishing of f.

Germs of regular functions

Let $p \in X$. We want a notion of "ring of regular functions near p on X". We want to regard two functions as the same, if they agree "sufficiently close to p".
Here is how to formalize this idea.

Definition

A *function germ at* p is an equivalence class of pairs (U, f), with $p \in U \subset X$ open, and $f : U \to k$ a regular function, where we identify

$$(U, f) \sim (V, g)$$

if $f|_W = g|_W$ on an open $p \in W \subset U \cap V$.
Denote by $\mathcal{O}_{X,p}$ the set of germs of regular functions at p. This is a k-algebra in an obvious way.

Function germs: an example

Function germs can behave in unexpected ways, especially on reducible varieties.

Example 11.3

Let
$$X = \mathbb{V}(xy) \subset \mathbb{A}^2.$$
Then X is a reducible variety: it is the union of the x-axis and the y-axis.
Let
$$U = X \setminus \mathbb{V}(x)$$
the open subset of X consisting of the x-axis without the origin.
Consider $f : U \to k$ given by $f = \frac{y}{x}$. This is a regular function on U:
$$f \in \mathcal{O}_X(U).$$
But along the x-axis away from the origin,
$$(U, f) \sim (U, 0)$$
as $y = 0 : U \to k$, so $[(U, f)] = 0$.

Multiplicative sets in rings

To relate the ring of germs $\mathcal{O}_{X,p}$ to other rings of functions algebraically, we need to recall a notion from Commutative Algebra. Let A be a ring (commutative with 1).

Definition

A subset $S \subset A$ is a *multiplicative set* if $1 \in S$ and $S \cdot S \subset S$.

Example 11.4

1. $S = A \setminus \{0\}$ for any integral domain A.
2. $S = A \setminus \wp$ for any prime ideal $\wp \lhd A$.
3. $S = A \setminus \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \lhd A$.
4. $S = \{1, f, f^2, \ldots\}$ for any $f \in A$.

Localizing rings

Let $S \subset A$ be a multiplicative subset.

Definition

The *localization* of A at (or with respect to) S is the ring

$$S^{-1}A = (A \times S)/\sim$$

where we abbreviate the pairs (r, s) by $\frac{r}{s}$, and the equivalence relation is:

$$\frac{r}{s} \sim \frac{r'}{s'} \iff t(rs' - r's) = 0 \text{ for some } t \in S.$$

Remark

If A is an integral domain, then t in the definition of the equivalence relation is unnecessary. Indeed, for integral domains the theory of localization substantially simplifies.

Notation. For $f \in A$ and $S = \{1, f, f^2, \ldots\}$, denote

$$A_f = S^{-1}(A).$$

For a prime ideal $\wp \lhd A$ and $S = A \setminus \wp$, denote

$$A_\wp = S^{-1}(A).$$

Examples of localization

Example 11.5

1. If $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus 0$, then $S^{-1}A \cong \mathbb{Q}$.
2. More generally, if A is an integral domain and $S = A \setminus \{0\}$, then
$$\mathrm{Frac}(A) = S^{-1}A$$
is the (definition of the) field of fractions of A.
3. If A is an integral domain, $f \in A \setminus \{0\}$ and $S = \{1, f, f^2, \ldots\}$, then we have
$$A_f = A[f^{-1}] = \{a/f^r : a \in A\}.$$
If f is a zero-divisor, A_f can be more complicated to describe.

A further example

Let us see an example where A is not a domain.

Example 11.6

Let $A = k[x, y]/(xy)$, and consider the multiplicative subset $S = \{1, x, x^2, \ldots\}$ in A.
In the the localization $S^{-1}A$, the element $y = \frac{y}{1}$ is zero, since y is annihilated by $x \in S$.
Thus

$$k[x]_x = S^{-1}A \cong k[x, x^{-1}] \subset \mathrm{Frac}(k[x]) = k(x).$$

Some basic algebraic properties

Theorem 11.7 (Algebraic Properties of Localisation)

(1) *If A is an integral domain, then for any multiplicative set S in A, there are injections*

$$A \hookrightarrow S^{-1}(A) \hookrightarrow \mathrm{Frac}(A).$$

(2) *For any prime ideal $\wp \lhd A$ and $S = A \setminus \wp$, the localization $A_\wp$ has a unique maximal ideal*

$$\wp A_\wp = \left\{ \tfrac{r}{s} : r \in \wp, s \notin \wp \right\} / \sim .$$

(3) *For A an integral domain,*

$$A = \bigcap_{max\ \mathfrak{m} \lhd A} A_\mathfrak{m} = \bigcap_{prime\ \wp \lhd A} A_\wp \subset \mathrm{Frac}(A).$$

(4) *For any ideal $I \lhd A$, and for any prime ideal $\wp \lhd A$ containing I, the localization $A_\wp$ has an induced prime ideal $IA_\wp$, and we have an isomorphism*

$$(A/I)_\wp \cong A_\wp / IA_\wp.$$

The ring of germs of functions is a localization

Proposition 11.8

Let $X \subset \mathbb{A}^n$ be an affine variety. At a point $p \in X$, the ring of germs of functions at p is the localization

$$\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p}$$

of the coordinate ring $k[X]$ at its maximal ideal

$$\mathfrak{m}_p = \mathbb{I}(p) = \{f \in k[X] : f(p) = 0\}$$

corresponding to p.

Proof. Consider the map

$$\mathcal{O}_{X,p} \to k[X]_{\mathfrak{m}_p}$$

defined by

$$(U, f) \mapsto \frac{g}{h},$$

where $f|_U = \frac{g}{h}$ for $g, h \in k[X]$, $h(p) \neq 0$.

The ring of germs of functions is a localization

The map is well-defined: $h(p) \neq 0 \Rightarrow h \notin \mathfrak{m}_p \Rightarrow \frac{g}{h} \in k[X]_{\mathfrak{m}_p}$.
Moreover, if $(U, f) \sim (U', f')$, so $\frac{g}{h} = \frac{g'}{h'}$ on a basic open $p \in D_s \subset U \cap U'$, where $s \in k[X]$, then $gh' - g'h = 0$ on D_s.
Since $s(p) \neq 0$, we have $s \notin \mathfrak{m}_p$. Thus $s \cdot (gh' - g'h) = 0$ everywhere on X, so $s \cdot (gh' - g'h) = 0$ in $k[X]$.
Thus $\frac{g}{h} = \frac{g'}{h'}$ in $k[X]_{\mathfrak{m}_p}$.
We build the inverse map: for $h \notin \mathfrak{m}_p$, let $U = D_h$, then send $\frac{g}{h} \mapsto (U, \frac{g}{h})$.
If $\frac{g}{h} = \frac{g'}{h'}$ in $k[X]_{\mathfrak{m}_p}$, then $s \cdot (gh' - g'h) = 0$ for some $s \in k[X] \setminus \mathfrak{m}_p$.
Then $s(p) \neq 0$ so $p \in D_s$, and $gh' - g'h = 0$ on D_s.
Thus $\frac{g}{h} = \frac{g'}{h'}$ as functions $D_s \to k$, as required.
By construction, the two maps are inverse to each other, so they both define isomorphisms. $\square$

An example

Let us return to $A = k[x,y]/(xy)$, corresponding to $X \subset \mathbb{A}^2$ being the union of coordinate axes.
Consider the point $p = (1,0)$. Its maximal ideal is

$$\mathfrak{m}_p = \langle x-1, y\rangle \lhd k[x,y]/(xy) = k[X].$$

Naively, we would like to invert all functions that do not vanish at p, so allow denominators from the set

$$S = \{f \in A : f \notin \langle x-1, y\rangle\} = A \setminus \mathfrak{m}_p.$$

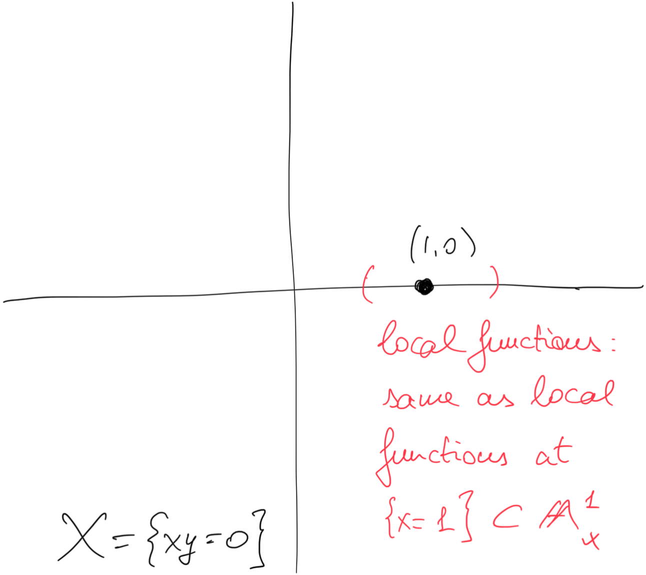
However, while the global functions 0 and y are different in $k[X]$, as we saw before, they become equal near p, and also in the localization $S^{-1}A = k[X]_{\mathfrak{m}_p}$. Indeed, in this example we get

$$k[x]_{\mathfrak{m}_p} = k[x][\tfrac{1}{h} : h(p) \neq 0] \subset \mathrm{Frac}(k[x]) = k(x).$$

This is the same as what we would have obtained for

$$p = (1,0) \subset Y = \{y = 0\} \subset \mathbb{A}^2.$$

An example



local functions: same as local functions at $\{x=1\} \subset \mathbb{A}^1_x$

$X = \{xy=0\}$

An example, continued

Continue with $A = k[x, y]/(xy)$. Consider the point $q = (0, 0) \in X$, the intersection point of the two components. Its maximal ideal is

$$\mathfrak{m}_q = \langle x, y\rangle \lhd k[x, y]/(xy) = k[X].$$

Now we have $y \neq 0$ near this point, so in the local ring $k[X]_{\mathfrak{m}_q}$.
Indeed there are surjective restriction maps, obtained by setting $y = 0$, respectively $x = 0$:

$$k[X]_{\mathfrak{m}_q} \to k[x]_{\langle x\rangle}$$

and

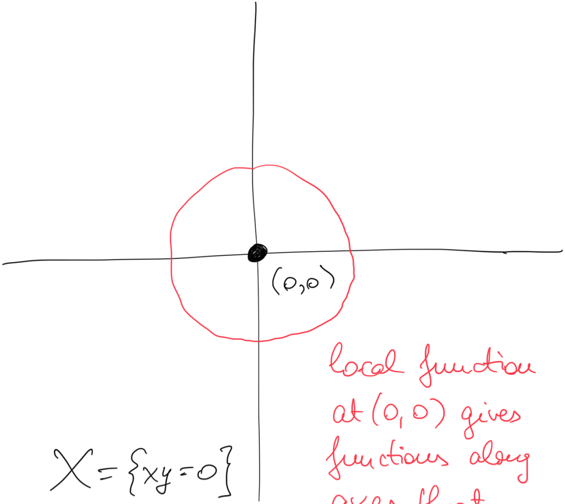
$$k[X]_{\mathfrak{m}_q} \to k[y]_{\langle y\rangle},$$

the local rings at the origin in the two copies of the affine line.
Using these two surjective maps, it is an easy exercise to show that

$$k[X]_{\mathfrak{m}_q} \subset k[x]_{\langle x\rangle} \times k[y]_{\langle y\rangle}$$

is the subring of elements of the product ring consisting of pairs of functions with the same value at the origin.

An example, continued



$X = \{xy = 0\}$

local function at (0,0) gives functions along axes that agree at 0

Localization: final general properties on irreducible varieties

For $X \subset \mathbb{A}^n$, an *irreducible* variety corresponding to a coordinate ring $k[X]$ which is a domain, we obtain from Theorem 11.7(1) above,

$$k[X] \subset \mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p} = k[X][\tfrac{1}{h} : h(p) \neq 0] \subset \mathrm{Frac}(k[X]) = k(X).$$

We also have, from Theorem 11.7(3) above,

$$k[X] = \bigcap_{p \in X} \mathcal{O}_{X,p} \subset k(X).$$

In other words, on an irreducible affine variety, a rational function is regular if and only if it defines a regular germ at each point.

Localization on reducible varieties

For $X \subset \mathbb{A}^n$ *reducible*, write

$$X = \bigcup_{i=1}^{N} X_i$$

an irreducible decomposition.
Consider a point

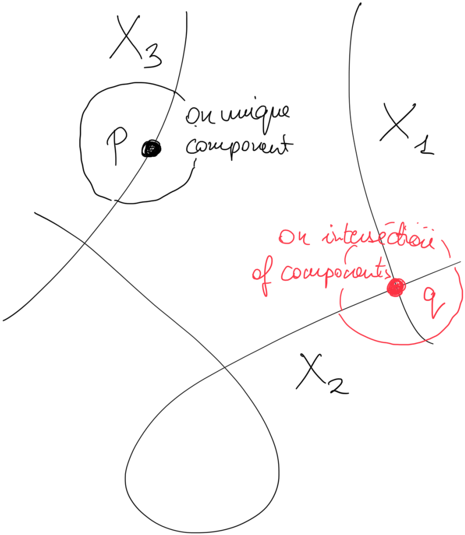
$$p \in X_i \setminus \bigcup_{j \neq i} X_j$$

lying on a unique irreducible component.
Then essentially the same argument that was used in the example before proves that the ring of germs at p is

$$\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p} \cong k[X_i]_{\mathfrak{m}_p}.$$

In other words, for such points the ring of germs "sees" only the (unique) irreducible component p lies on.

Localization on reducible varieties



C3.4 Algebraic Geometry

Lecture 12

Dominic Joyce, Oxford University, MT 2026

Intersection numbers via localization

This is "off-topic" but too nice not to mention.

Let $f_1, f_2 \in k[x,y]$ be two polynomials without common factors. Let

$$C_i = \{f_i = 0\} \subset \mathbb{A}^2$$

be the two affine plane curves defined by the f_i.

Let $p \in C_1 \cap C_2$ be an intersection point. Let $\mathfrak{m}_p \lhd k[x,y]$ be the maximal ideal of this point on the affine plane.

Definition

The *intersection multiplicity* of C_1 and C_2 at p is the (finite) quantity

$$\dim_k k[x,y]_{\mathfrak{m}_p}/\langle f_1, f_2 \rangle.$$

This is the (local) quantity that goes into the full version of Bézout's theorem.

Why quasiprojective varieties?

We would like to consider a class (category, if you wish) of "algebraic varieties" that includes affine and projective varieties, but also (closed and) open subsets of those.

We would also like to define (to get a category), what morphisms of these are.

Example 12.1

The open set $U = \mathbb{A}^1 \setminus \{0\}$ of $\mathbb{A}^1$ can be thought of as an affine variety: the first projection of the affine variety

$$Y = \mathbb{V}(xy - 1) \subset \mathbb{A}^2$$

is a bijection (*isomorphism*) between Y and U.

Example 12.2

The open set $V = \mathbb{A}^2 \setminus \{0\}$ can *not* be thought of as an affine variety: it is not (*isomorphic to*) an affine variety $Y \subset \mathbb{A}^n$.

We will see that $\mathbb{A}^2 \setminus \{0\}$ is an example of a quasiprojective variety which is not affine, nor projective.

Quasiprojective varieties

Let $R = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of projective space $\mathbb{P}^n$.

Definition

A *quasi-projective variety* $X \subset \mathbb{P}^n$ is any open subset of a projective variety, so there exists ideals $I, J \lhd R$ with

$$X = U_J \cap \mathbb{V}(I)$$

where

$$U_J = \mathbb{P}^n \setminus \mathbb{V}(J).$$

Notice we can also write X as the difference of two closed sets:

$$X = \mathbb{V}(I) \setminus \mathbb{V}(I + J).$$

A *quasi-projective subvariety* Y of X is a subset of X which is also a quasi-projective variety.

Examples

The following are all examples of quasi-projective varieties.

1. Any projective variety $X \subset \mathbb{P}^n$ is a quasi-projective variety as
$$X = \mathbb{P}^n \cap X.$$
2. Any affine variety $X \subset \mathbb{A}^n$ is a quasi-projective variety as
$$X = U_0 \cap \overline{X},$$
where $\overline{X} \subset \mathbb{P}^n$ is the projective closure of X, and $U_0 = \mathbb{A}^n$ is one of the covering open affines of $\mathbb{P}^n$.
3. We have
$$\mathbb{A}^2 \setminus \{0\} = (U_0 \cap (U_1 \cup U_2)) \cap \mathbb{P}^2,$$
where $U_i = \{x_i \neq 0\}$ are the covering affine open subsets of $\mathbb{P}^2$.
4. Any open subset of a quasi-projective variety is also a quasi-projective variety, since
$$U_{J'} \cap (U_J \cap \mathbb{V}(I)) = (U_{J'} \cap U_J) \cap \mathbb{V}(I).$$

Morphisms

Morphisms are defined in the same way as for projective varieties. "Morphisms in projective geometry are defined locally." Fix quasi-projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. We work with $(m+1)$-tuples of homogeneous polynomials of the projective coordinates $x_0, \ldots, x_n$ on X. Let $R = k[x_0, \ldots, x_n]$.

Definition

A *morphism* $F : X \to Y$ of quasi-projective projective varieties is a function F such that for every $p \in X$, there is an open neighbourhood $p \in U \subset X$, and *homogeneous* polynomials $f_0, \ldots, f_m \in R$ *of the same degree*, with

$$F : U \to Y \text{ given by } F([a_0 : \cdots : a_n]) = [f_0(a) : \cdots : f_m(a)].$$

Morphisms of affine varieties

We need to check that for $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ affine, this definition agrees with the definition of a morphism of affine varieties.
Suppose a morphism $f\colon X \to Y$ between affine varieties is given by an m-tuple $f_1, \ldots, f_m$ of polynomials of the variables $x_1, \ldots, x_n$ of X:

$$\begin{array}{rccc} f: & X & \to & Y \\ & (x_1, \ldots, x_n) & \mapsto & (f_1(x_i), \ldots, f_m(x_i)). \end{array}$$

Let $d = \max \deg f_i$, $F_0(x) = X_0^d$, $F_i(X_i) = X_0^d f_i(x_i)$, where the affine variables $(x_1, \ldots, x_n)$ and the projective variables $[X_0 : \ldots : X_n]$ are related as usual by $x_i = X_i/X_0$ for $i \neq 0$.
Then we can use the quasi-projective representation

$$[F_0(X_i) : \cdots : F_m(X_i)] = [X_0^d : X_0^d f_1(x_i) : \cdots : X_0^d f_m(x_i)].$$

On the open set $\mathbb{A}^n$ of $\mathbb{P}^n$ where we can set $X_0 = 1$, we indeed get

$$[1 : x_1 : \cdots : x_n] \mapsto [1 : f_1(x_i) : \cdots : f_m(x_i)].$$

Affine quasi-projective varieties

As we have a notion of morphism, we automatically have a notion of isomorphism for quasi-projective varieties.
The following terminology is unfortunate, but completely standard in the subject.

Definition

A quasi-projective variety X is called *affine*, if it is isomorphic (as a quasi-projective variety) to a Zariski closed subset $Y \subset \mathbb{A}^m$.
We could call these "affine quasi-projective varieties" but everybody calls these "affine varieties".

As we will see later, the ring $k[Y] = k[\mathbb{A}^m]/\mathbb{I}(Y)$ has an intrinsic definition in terms of X.

Example 12.3

The open set $V = \mathbb{A}^2 \setminus \{0\}$ is not an affine (quasi-projective) variety: it is not isomorphic to a closed subset $Y \subset \mathbb{A}^n$.

An example of an affine quasi-projective variety

Example 12.4

Consider the quasi-projective variety

$$U = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}.$$

Let

$$Y = \mathbb{V}(xy - 1) \subset \mathbb{A}^2,$$

which can be written as a quasi-projective variety as

$$Y = \mathbb{V}(x_1 x_2 - x_0^2) \cap U_0 \subset \mathbb{P}^2.$$

Then the formula

$$[x_0 : x_1 : x_2] \to [x_0 : x_2]$$

defines a well-defined map of quasi-projective varieties $F\colon Y \to U$. Its inverse $G\colon U \to Y$ can be written in the coordinates z_0, z_1 of $\mathbb{P}^1$ as

$$G\colon [z_0 : z_1] \mapsto [z_0 z_1 : z_0^2 : z_1^2].$$

Check that these indeed define mutual inverses and behave as expected!

Basic open sets in affine varieties are affine

Lemma 12.5

Let $X \subset \mathbb{A}^n$ be an affine variety, $f \in k[X]$. Then the basic open set

$$D_f = X \setminus \mathbb{V}(f)$$

is an affine (quasi-projective) variety with

$$k[D_f] \cong k[X]_f.$$

The proof is just a more general version of the previous example.
Proof. Let $\mathbb{I}(X) \lhd k[x_1, \dots, x_n]$ be the ideal of X. Let

$$\widetilde{I} = \langle \mathbb{I}(X), x_{n+1}f - 1 \rangle \subset k[x_1, \dots, x_n, x_{n+1}].$$

Let

$$Y = \mathbb{V}(\widetilde{I}) \subset \mathbb{A}^{n+1}$$

be the affine variety corresponding to the ideal $\widetilde{I}$. The proof will be concluded by the following two Claims.

Basic open sets in affine varieties are affine

Claim 1. We have $k[Y] \cong k[X]_f$.

Proof of Claim 1.

There is a natural map from left to right given by x_{n+1} mapping to $\frac{1}{f} \in k[X]_f$ and $g \in k[X]$ mapping to $\frac{g}{1} \in k[X]_f$.
There is also a natural map from right to left given by $\frac{g}{f^a} \mapsto g x_{n+1}^a$.
It can be checked easily that these maps are mutual inverses. □

Claim 2. We have $Y \cong D_f$ as quasi-projective varieties.

Proof of Claim 2.

The first projection $(a_1, \ldots, a_n, a_{n+1}) \mapsto (a_1, \ldots, a_n)$ restricts to a map $Y \to \mathbb{A}^n$ whose image is D_f.
It has an inverse which is the map

$$(a_1, \ldots, a_n) \mapsto \left(a_1, \ldots, a_n, \frac{1}{f(a)}\right)$$

from D_f to Y.
Introducing homogeneous coordinates, it is easy to write these as genuine maps of quasi-projective varieties. For details, see Lecture Notes. The proof of Lemma 12.5 is complete. □

Quasi-projective varieties are locally affine

Theorem 12.6

Every quasi-projective variety X has a finite open cover by affine (quasi-projective) subvarieties. In particular, affine open subsets form a basis for the Zariski topology on X.

Proof.

We have, picking generators for ideals,

$$\mathbb{P}^n \supset X = \mathbb{V}(F_1, \ldots, F_N) \setminus \mathbb{V}(G_1, \ldots, G_M).$$

Because $\mathbb{P}^n$ has an affine open cover by its standard open sets U_i, it suffices to check the claim on the open subset $U_0 \cap X$ of X.
Then $U_0 \cap X$ is

$$\mathbb{V}(f_1, \ldots, f_N) \setminus \mathbb{V}(g_1, \ldots, g_M) = \bigcup_j \left(\mathbb{V}(f_1, \ldots, f_N) \setminus \mathbb{V}(g_j)\right) = \bigcup_j D_{g_j},$$

where

$$D_{g_j} = \{g_j \neq 0\} \subset \mathbb{V}(f_1, \ldots, f_N)$$

is a basic open subset in an affine variety.
Now apply Lemma 12.5. □

Quasi-projective varieties as glued from affine varieties

The Theorem shows that quasi-projective varieties are basically affine varieties "glued together". We could take this as a starting point!

$$X = \bigcup_i U_i, \quad \text{with} \quad U_i \subset \mathbb{A}^{n_i} \text{ affine.}$$

This point of view takes algebraic geometry closer to other subjects like differential geometry: we specify a set of "local models" and build global objects from local ones.
This point of view expands the list of possible objects further, allowing so-called "abstract varieties" and eventually "schemes":

$$\{\text{affine varieties}\} \subset \{\text{quasi-proj. varieties}\} \subset \{\text{abstract varieties}\} \subset \{\text{schemes}\}.$$

Objects in algebraic geometry

We have as possible objects

$$\{\text{affine varieties}\} \subset \{\text{quasi-proj. varieties}\} \subset \{\text{abstract varieties}\} \subset \{\text{schemes}\}.$$

Here

- affine varieties (corresponding to finitely generated reduced k-algebras) are basic building blocks;
- quasi-projective varieties are what we can glue inside a convenient global model (projective space);
- abstract varieties are what we can glue abstractly, without a global model;
- schemes are what we can glue when we do not insist that our algebras should be reduced.

This list could go on!

Morphisms in algebraic geometry

Morphisms for our objects

$$\{\text{affine varieties}\} \subset \{\text{quasi-proj. varieties}\} \subset \{\text{abstract varieties}\} \subset \{\text{schemes}\}$$

are then defined as follows:

- morphisms of affine varieties are algebra homomorphisms of the corresponding to finitely generated reduced k-algebras (backwards);
- for all other types of objects, morphisms are collections of local morphisms compatible with gluing.

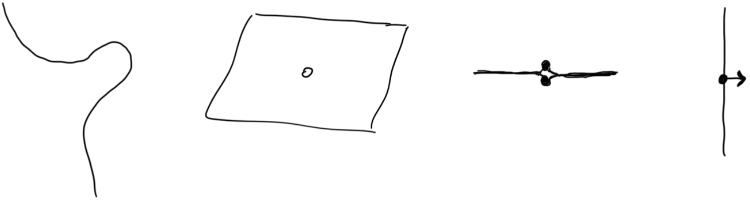
This point of view is explained further in the Lecture Notes (non-examinable), and in second courses in Algebraic Geometry.

Examples of different types of objects in algebraic geometry

Examples:

$$\{\text{affine varieties}\} \subset \{\text{quasi-proj. varieties}\} \subset \{\text{abstract varieties}\} \subset \{\text{schemes}\}$$

$\{x^3+y^3+1=0\} \subset \mathbb{A}^2$ $\qquad$ $\mathbb{A}^2 \setminus \{0\}$ $\qquad$ $\mathbb{A}^1 \cup_{\mathbb{A}^1\setminus\{0\}} \mathbb{A}^1$ $\qquad$ $\operatorname{Spec} k[x,y]/(x^2,xy)$



Regular functions in the affine case

Start with an affine variety $X \subset \mathbb{A}^n$. For $U \subset X$ open, recall

$$\begin{aligned} \mathcal{O}_X(U) &= \{\text{regular functions } f : U \to k\} \\ &= \{f : U \to k : f \text{ is regular at each } p \in U\}. \end{aligned}$$

Here f regular at p means: on some open $p \in W \subset U$, the following functions $W \to k$ are equal:

$$f = \frac{g}{h} \qquad \text{some } g, h \in k[X] \text{ and } h(w) \neq 0 \text{ for all } w \in W.$$

Example 12.7

1. Let $U = D_x = \mathbb{A}^2 \setminus \mathbb{V}(x) \subset \mathbb{A}^2$, then $f : D_x \to k$ defined by $f(x, y) = \frac{y}{x}$ is a regular function on $U = D_x$:
$$f \in \mathcal{O}_X(D_x).$$
2. More generally, for any $g, h \in k[X]$, with $h \neq 0$, we have $\frac{g}{h} \in \mathcal{O}_X(D_h)$.

Regular functions on quasi-projective varieties

This definition generalizes readily to quasi-projective varieties, remembering the slogan "quasi-projective varieties are locally affine". Let $X \subset \mathbb{P}^n$ be a quasi-projective variety.

Definition

For any open subset $U \subset X$, let

$$\mathcal{O}_X(U) = \{F : U \to k : F \text{ is regular at each } p \in U\}$$

be the *ring of regular functions on* U.
Here F *regular at* p means: on some *affine* open $p \in W \subset U$, $F|_W$ is regular at p as previously defined.

Rings of germs of functions on quasi-projective varieties

Let $X \subset \mathbb{P}^n$ be a quasi-projective variety, and $p \in X$. The following definition is now immediate; note that all terms we use here are now defined for general quasi-projective varieties.

Definition

A *function germ at* p is an equivalence class of pairs (U, f), with $p \in U \subset X$ open, and $f : U \to k$ a regular function, where we identify

$$(U, f) \sim (V, g)$$

if $f|_W = g|_W$ on an open $p \in W \subset U \cap V$.
Denote by $\mathcal{O}_{X,p}$ the set of germs of regular functions at p. This is a k-algebra in an obvious way.

Rings of germs of functions on quasi-projective varieties

In practice, it is easiest to approach $\mathcal{O}_{X,p}$ on an *affine* open neighbourhood $p \in V \subset X$.

Theorem 12.6 shows that affines cover a quasi-projective variety.

Proposition 12.8

Let $p \in V \subset X$ be an affine *neighbourhood of p in the quasi-projective variety X. Then*

$$\mathcal{O}_{X,p} \cong k[V]_{\mathfrak{m}_p},$$

where $\mathfrak{m}_p \lhd k[V]$ is the maximal ideal of $p \in V$ in the coordinate ring of the affine variety V.

Proof.

Use Proposition 11.8. □

One way to view this is to say that that the localization on the right hand side is *independent* of the affine neighbourhood of p chosen.

C3.4 Algebraic Geometry

Lecture 13

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Gluing regular functions

Consider open sets U_1, U_2 in a quasi-projective variety X, and regular functions $f_1 \in \mathcal{O}_X(U_1)$ and $f_2 \in \mathcal{O}_X(U_2)$.

Claim A necessary and sufficient condition to be able to find a regular function $f \in \mathcal{O}_X(U_1 \cup U_2)$ restricting to f_i on U_i is that

$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}.$$

Proof.

Necessity is obvious. To see sufficiency, define $f = f_i$ on U_i. Then $f : U_1 \cup U_2 \to k$ is well-defined. Regularity follows because regularity is a local condition and we already know it is satisfied by f_1, f_2 on U_1, U_2. □

This result is almost a "tautology". But it would play an important role in a different development of the subject; it says that the collection of rings $\mathcal{O}_X(U)$ attached to open sets $U \subset X$ forms a *sheaf*. We will not use this language, but see the Lecture Notes for some more details (non-examinable).

Regular functions are not necessarily global quotients

For $f \in \mathcal{O}_X(U)$, it may not be possible to find a fraction $f = \frac{g}{h}$ that works on all of U.

Example 13.1

Consider the affine variety
$$X = \mathbb{V}(xw - yz) \subset \mathbb{A}^4.$$
Let $U = D_y \cup D_w \subset X$, a quasi-projective variety.
Consider
$$f = \frac{x}{y} = \frac{z}{w} \in k(X) = \mathrm{Frac}\; k[X].$$
This is a *regular* function
$$f \in \mathcal{O}_X(U),$$
glued from regular functions on D_y, D_w that agree on the intersection. It can be proved that there is no a global expression $f = \frac{g}{h}$ defined on all of U that defines f.

For algebra aficionados: this is a reflection of the fact that $k[X]$ is not a UFD.

Regular functions on affine varieties

We noted before that for $X \subset \mathbb{A}^n$ an affine variety, elements of its coordinate ring define regular functions: there is an inclusion

$$k[X] \hookrightarrow \mathcal{O}_X(X)$$

defined by mapping $f \mapsto \frac{f}{1}$.
We now have the means to prove the converse: the locally everywhere regular functions on an affine variety are exactly elements of its coordinate ring.

Theorem 13.2

Let $X \subset \mathbb{A}^n$ be an affine variety. Then

$$O_X(X) \cong k[X].$$

This is important, as it gives a "local" way to characterize the coordinate ring of an affine (quasi-projective) variety as the ring of everywhere regular functions.

Regular functions on affine varieties: an application

Before we give the proof, let us see an application.

Proposition 13.3

The quasi-projective variety $X = \mathbb{A}^2 \setminus \{0\}$ is not *affine.*

The proof consists of two parts.

Claim 1 . We have

$$\mathcal{O}_X(X) \cong k[x, y].$$

Claim 2. This isomorphism implies that X is not affine.

Colloquial summary

- If X were affine, its coordinate ring would have to be its ring of regular functions.
- But then X would have to be $\mathbb{A}^2$, which it isn't.

Proof of Claim 1

Claim 1. We have
$$\mathcal{O}_X(X) \cong k[x, y].$$

Proof.

We have $\mathbb{A}^2 \setminus \{0\} = D_x \cup D_y$, so $f \in \mathcal{O}_X(X)$ defines regular functions $f_1 = f|_{D_x} \in k[D_x]$, $f_2 = f|_{D_y} \in k[D_y]$ which agree on the intersection: $f_1|_{D_x \cap D_y} = f|_{D_x \cap D_y} = f_2|_{D_x \cap D_y} \in k[D_x \cap D_y]$.
We know
$$\mathcal{O}_X(D_x) = k[x, y]_x \subset k(x, y)$$
and similarly $\mathcal{O}_X(D_y) = k[x, y]_y$ (Lemma from Lecture 12).
We get the following intersection in $k(x, y)$:
$$\begin{aligned} \mathcal{O}_X(X) &= k[D_x] \cap k[D_y] \\ &= k[x, y]_x \cap k[x, y]_y \\ &= k[x, x^{-1}, y] \cap k[x, y, y^{-1}] \\ &= k[x, y]. \end{aligned}$$
□

Proof of Claim 2

Claim 2. The isomorphism $\mathcal{O}_X(X) \cong k[x,y]$ implies that X is not affine.

Proof.

Suppose that X is isomorphic to an affine variety

$$X \cong Y \subset \mathbb{A}^n.$$

Then this isomorphism would give us isomorphisms

$$k[Y] \cong \mathcal{O}_Y(Y) \cong \mathcal{O}_X(X) \cong k[x,y].$$

So $Y \cong \mathbb{A}^2$ as affine varieties can be recognized from their coordinate rings. Assume that we have an isomorphism $\varphi : k[\mathbb{A}^2] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_X(X)$. The preimage of the prime ideal $I = \langle x,y \rangle \subset \mathcal{O}_X(X)$ yields a prime ideal $J = \varphi^{-1}(I) \lhd k[\mathbb{A}^2]$.
But $\mathbb{V}(I) = \emptyset \subset X$, so $\mathbb{V}(J) = \varphi^*(\mathbb{V}(I)) = \emptyset \subset \mathbb{A}^2$.
So $J = k[x,y]$ by the affine Nullstellensatz.
But φ is an isomorphism, so $I = \varphi(J) = k[x,y]$, contradiction. $\square$

Regular functions on affine varieties: the proof

Theorem 13.2 (Repeated)

Let $X \subset \mathbb{A}^n$ be an affine variety. Then
$$O_X(X) \cong k[X].$$

Proof. We will prove the theorem under the additional assumption that X is irreducible. The general case is treated in the Lecture Notes.
We need to show that any element of $O_X(X)$ comes from the coordinate ring. So let $f \in O_X(X)$.
For all $p \in X$, there exists an open neighbourhood $p \in U_p \subset X$ such that

$$f = \frac{g_p}{h_p} \text{ as maps } U_p \to k,$$

where $g_p, h_p \in k[X]$, and $h_p \neq 0$ at all points of U_p.
Note that at this point, we have an infinite collection of (g_p, h_p) representing f, one for each point of X.

Regular functions on affine varieties: the proof

Consider the ideal of denominators

$$J = \langle h_p : p \in X \rangle \subset k[X].$$

We have

$$\mathbb{V}(J) = \emptyset,$$

since $h_p(p) \neq 0$.
By the Nullstellensatz then, $J = \langle 1 \rangle$ the whole coordinate ring. So there exists a finite decomposition

$$1 = \sum_{i=1}^{N} \alpha_i h_{p_i} \in k[X] \tag{13.1}$$

for some *finite* collection of $p_i \in X$, and $\alpha_i \in k[X]$.
Abbreviate $h_i = h_{p_i}$, $g_i = g_{p_i}$, $D_i = D_{h_{p_i}}$. Note that (13.1) implies that the D_i are an open cover of X: at each point of X, at least one of the h_i must be nonzero.

Regular functions on affine varieties: end of the proof

We have now built a finite open cover $\{D_i\}$ of X, such that on each of these open sets, f is represented by g_i/h_i, a ratio of elements of $k[X]$. On the overlap $D_i \cap D_j$, we have $g_i/h_i = g_j/h_j$, so on this intersection,

$$g_i h_j - h_i g_j = 0 \in \mathcal{O}_X(D_i \cap D_j).$$

However, since X is irreducible, this vanishing must be true over the whole of X, so for all i, j

$$g_i h_j - h_i g_j = 0 \in k[X].$$

We then deduce, on D_j,

$$f = \frac{g_j}{h_j} = 1 \cdot \frac{g_j}{h_j} = \sum_{i=1}^{N} \alpha_i h_i \cdot \frac{g_j}{h_j} = \sum_{i=1}^{N} \alpha_i \frac{h_i g_j}{h_j} = \sum_{i=1}^{N} \alpha_i \frac{h_j g_i}{h_j} = \sum_{i=1}^{N} \alpha_i g_i.$$

Once again, $D_j \subset X$ is dense, so we deduce over the whole of X that

$$f = \sum_{i=1}^{N} \alpha_i g_i \in k[X]. \qquad \square$$

Regular functions on projective varieties

Affine varieties admit many global regular functions: elements of their coordinate rings.
Let us conclude this lecture by discussing the opposite case: projective varieties.

Theorem 13.4

Let $X \subset \mathbb{P}^n$ be an irreducible projective *variety. Then the only global regular functions on X are the constants:*

$$\mathcal{O}_X(X) \cong k.$$

The proof, while not difficult, is somewhat lengthy, so we omit it. We discuss one example, and then sketch the proof of one more general case.

Corollary 13.5

The variety $X = \mathbb{A}^2 \setminus \{0\}$ is not projective.

Regular functions on the projective line

Example 13.6

Consider $X = \mathbb{P}^1 = U_0 \cup U_1$, with projective coordinates $(x_0 : x_1)$. The open sets $U_i = \{x_i \neq 0\} \cong \mathbb{A}^1$ intersect in $V = U_i \cap U_j \cong \mathbb{A}^1 \setminus \{0\}$. Then U_0, U_1 are affine varieties, with

$$\mathcal{O}_X(U_0) \cong k[U_0] = k[x], \quad \mathcal{O}_X(U_1) \cong k[U_1] = k[y].$$

The affine coordinates $x = x_1/x_0$, $y = x_0/x_1$ are related by $y = 1/x$, so we can write

$$\mathcal{O}_X(U_0) = k[x], \quad \mathcal{O}_X(U_1) = k[x^{-1}].$$

We also have

$$\mathcal{O}_X(V) \cong k[V] = k[x]_x = k[x, x^{-1}].$$

We thus deduce

$$\mathcal{O}_X(X) = k[x] \cap k[x^{-1}] \subset k[x, x^{-1}].$$

This intersection is clearly

$$\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k.$$

Regular functions on projective space

Theorem 13.7

The only global regular functions on $\mathbb{P}^n$ are the constants:

$$\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \cong k.$$

Sketch Proof.

Write $\mathbb{P}^n = \cup_{i=0}^n U_i$ for the standard open cover with $U_i \cong \mathbb{A}^n$.
We have

$$\mathcal{O}_{\mathbb{P}^n}(U_i) \cong k[U_i] \cong k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

A non-constant f would have to restrict as a non-constant polynomial to each of the U_i. That means that its expression as a rational function in the homogeneous coordinates x_i will have a denominator. But this denominator has to vanish somewhere on $\mathbb{P}^n$, and f cannot be regular there. □

The function field of an irreducible variety

All varieties in this lecture will be irreducible.
Let $X \subset \mathbb{A}^n$ be an irreducible affine variety.
Recall that this means that the coordinate ring

$$k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X)$$

is an integral domain.

Definition

The *function field* of X is

$$k(X) = \mathrm{Frac}(k[X]) = \{\tfrac{g}{h} : g, h \in k[X], h \neq 0\},$$

the field of fractions of the integral domain $k[X]$.

Examples

Example 13.8

- For $X = \mathbb{A}^1$, we have
$$k[\mathbb{A}^1] \cong k[x],$$
and so
$$k(\mathbb{A}^1) \cong k(x),$$
the field of rational functions in the variable x.
- For $X = \mathbb{A}^1 \setminus 0$, we have
$$k[X] \cong k[x]_x = k[x, x^{-1}],$$
and so
$$k(X) \cong k(x) \cong k(\mathbb{A}^1).$$
- For $X = \mathbb{A}^n$, we have $k[\mathbb{A}^n] \cong k[x_1, \ldots, x_n]$. So
$$k(\mathbb{A}^n) \cong k(x_1, \ldots, x_n).$$

A basic lemma

Let X be an irreducible affine variety.

Lemma 13.9

For $h \in k[X]$, let

$$D_h = X \setminus \mathbb{V}(h) \subset X$$

be the corresponding basic open subset. Then

$$k(D_h) \cong k(X).$$

Proof.

By Lemma 12.5, we have

$$k[D_h] \cong k[X]_h = k[X][h^{-1}].$$

Thus

$$k(D_h) = \mathrm{Frac}(k[X]_h) \cong \mathrm{Frac}(k[X]) \cong k(X).$$ □

Important corollaries

Corollary 13.10

If $U \subset X$ is any affine open subset in an irreducible affine variety, then

$$k(U) \cong k(X).$$

Proof.

Basic open sets form a basis of the topology of X.
Let $D_h \subset U$ be a basic open of X for $h \in k[X]$. Consider

$$g = h|_U \in \mathcal{O}_X(U) \cong O_U(U) \cong k[U],$$

the latter isomorphism following from U being affine.
So $D_h = D_g$ is also a basic open of U.
Now we obtain

$$k(U) \cong k(D_g) = k(D_h) \cong k(X).$$

□

Important corollaries

Corollary 13.11

If $U_1, U_2 \subset X$ are affine open subsets in an irreducible quasi-projective variety $X \subset \mathbb{P}^n$, then

$$k(U_1) \cong k(U_2).$$

Proof.

As affine opens form a basis of the topology of X, the intersection $U_1 \cap U_2$ contains an affine subvariety V. Then by Corollary 13.10,

$$k(U_1) \cong k(V) \cong k(U_2).$$

□

The function field of an irreducible quasi-projective variety

Let $X \subset \mathbb{P}^n$ be an irreducible quasi-projective variety.

Definition

The *function field* of X is

$$k(X) = k(U) = \mathrm{Frac}(k[U]),$$

where $U \subset X$ is an affine open subset.

Note that by Corollary 13.11, the answer is independent of the affine open chosen. We also deduce

Corollary 13.12

If $U \subset X$ is any open subset of an irreducible quasi-projective variety $X \subset \mathbb{P}^n$, then

$$k(U) \cong k(X).$$

Example 13.13

Let

$$U = \mathbb{A}^2 \setminus \mathbb{V}(x) \subset X = \mathbb{A}^2 \setminus \{(0,0)\} \subset \mathbb{A}^2.$$

Then $U \subset \mathbb{A}^2$ are both affine, X is not. Their function fields agree:

$$k(U) \cong k(X) \cong k(\mathbb{A}^2) \cong k(x,y).$$

C3.4 Algebraic Geometry

Lecture 14

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The structure of function fields

Theorem 14.1

(1) *Let X be an irreducible quasi-projective variety. Then $k(X)$ is a finitely generated field extension of k.*

(2) *Conversely, given any finitely generated field extension K/k, there exists an irreducible quasi-projective variety X with $k(X) \cong K$.*

Proof of (1). We may assume that $X \subset \mathbb{A}^n$ is irreducible and affine. Then

$$k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X).$$

So

$$k(X) = \mathrm{Frac}(k[X])$$

is generated as a field over k by the images of $x_1, \ldots, x_n$. In particular, $k(X)$ is finitely generated over k.

The structure of function fields

Proof of (2). Conversely, let K be a field generated over k by some elements $\alpha_1, \ldots, \alpha_n$. Define a ring homomorphism

$$\phi\colon R = k[x_1, \ldots, x_n] \to K$$

by mapping x_i to α_i.
Let $J = \ker \varphi$. Then by the Isomorphism Theorem $R/J \hookrightarrow K$, so J is a prime ideal as K is an integral domain.
Let

$$X = \mathbb{V}(J) \subset \mathbb{A}^n$$

be the corresponding irreducible affine variety. Then

$$k(X) = \operatorname{Frac} R/J \hookrightarrow K$$

contains the generators α_i in the image. So $k(X) \cong K$. $\square$

The function field of a hypersurface

Example 14.2

Let $V_f \subset \mathbb{A}^n$ be an irreducible hypersurface defined by an irreducible polynomial $f \in k[x_1, \ldots, x_n]$.
Let us assume that f involves the variable x_n and is of degree d in that variable:

$$f(x_1, \ldots, x_n) = c_0(x_1, \ldots, x_{n-1})x_n^d + c_1(x_1, \ldots, x_{n-1})x_n^{d-1} + \ldots$$

We have

$$k[V_f] = k[x_1, \ldots, x_n]/\langle f \rangle.$$

From Lecture 9, we also know

$$\dim V_f = n - 1 = \mathrm{trdeg}_k\, k(V_f).$$

There is a map of rings

$$k[x_1, \ldots, x_{n-1}] \to k[V_f]$$

which is injective, as there is no algebraic relation between $x_1, \ldots, x_{n-1}$.
So we get an injection

$$k(x_1, \ldots, x_{n-1}) \hookrightarrow k(V_f).$$

The function field of a hypersurface

We have an injection

$$k(x_1, \ldots, x_{n-1}) \hookrightarrow k(V_f).$$

We get a realization of the function field of V_f as

$$k(V_f) =$$
$$k(x_1, \ldots, x_{n-1})[x_n]/\langle c_0(x_1, \ldots, x_{n-1})x_n^d + c_1(x_1, \ldots, x_{n-1})x_n^{d-1} + \cdots \rangle,$$

a finite extension of degree d of the purely transcendental field

$$k(\mathbb{A}^{n-1}) = k(x_1, \ldots, x_{n-1}).$$

The same argument also computes the function field of an irreducible projective hypersurface; recall the function field only depends on a dense open subset, so it is enough to consider any affine chart.
For example, the function field of a plane curve of degree d is a degree d extension of $k(x)$.

Rational maps

Let X be an irreducible quasi-projective variety.

Definition

A *rational map* $f : X \dashrightarrow Y$ to another quasi-projective variety Y is an equivalence class of pairs (f, U), where $f : U \to Y$ is a regular map defined on a non-empty open subset of X, and we identify pairs of maps which agree on a non-empty open subset.

Example 14.3

We have a rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

given by

$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_{n-1}).$$

This is defined on $U = \mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$.

Rational functions

Definition

A *rational function* on an irreducible quasi-projective variety X is a rational map $f : X \dashrightarrow \mathbb{A}^1$.

Lemma 14.4

There is a natural bijection

$$k(X) \leftrightarrow \{\textit{rational functions } f : X \dashrightarrow \mathbb{A}^1\}.$$

Remark

Compare this with what we had before: for any quasi-projective variety X, there is a bijection

$$\mathcal{O}_X(X) \leftrightarrow \{\text{morphisms } f : X \to \mathbb{A}^1\}.$$

In particular, if X is affine, we have a bijection

$$k[X] \leftrightarrow \{\text{morphisms } f : X \to \mathbb{A}^1\}.$$

Rational functions

Lemma 14.4 (Repeated)

There is a natural bijection

$$k(X) \leftrightarrow \{\textit{rational functions } f : X \dashrightarrow \mathbb{A}^1\}.$$

Proof.

Both sides are invariant under passing to an open subset, so we may assume that X is affine. In this case,

$$k(X) = \mathrm{Frac}(k[X]).$$

A natural map is given for $f, g \in k[X]$ by

$$\frac{f}{g} \mapsto \left[\left(D_g, \frac{f}{g}\right)\right].$$

We need to check that we get every rational function $X \dashrightarrow \mathbb{A}^1$ this way. By the above Remark, any such corresponds to an element $h \in \mathcal{O}_X(U)$ for some open $U \subset X$. Taking a basic open $D_g \subset U$, we have an element $h|_{D_g} \in \mathcal{O}_X(D_g) = k[X]_g$. So we can write $h = f/g^N$ for $f \in k[X]$. □

Rational maps to affine space

Corollary 14.5

On an irreducible quasi-projective variety X, the data of a rational map $f : X \dashrightarrow \mathbb{A}^n$ is equivalent to n elements $f_i \in k(X)$:

$$f(x) = (f_1(x), \ldots, f_n(x)),$$

valid in some open set $x \in U \subset X$.

Proof.

A rational map $f : X \dashrightarrow \mathbb{A}^n$ determines, and is determined by, n rational functions $f_i : X \dashrightarrow \mathbb{A}^1$, the coordinates of f. By Lemma 14.4, each of these can be thought of as an element $f_i \in k(X)$.

Each $f_i \in k(X)$ is a regular function on some open set $U_i \subset X$.

The formula

$$f(x) = (f_1(x), \ldots, f_n(x))$$

is valid on the intersection

$$U = \bigcap_i U_i \subset X$$

of these open sets U_i. □

Composition of rational maps

Rational maps are not functions in the classical sense (they are only "partially defined").
In general, composition of rational maps may not be possible, even in the simplest cases.

Example 14.6

Let $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ be defined by $a \mapsto 0$. This is a regular, hence rational map.
Let $g\colon \mathbb{A}^1 \to \mathbb{A}^1$ be $a \mapsto a^{-1}$. This is a rational map.
The image of f is $\{0\} \subset \mathbb{A}^1$. The domain of g is $\mathbb{A}^1 \setminus \{0\}$.
So the composite $g \circ f$ is not defined.

Dominant rational maps

Definition

A rational map

$$f = [(U, F)] : X \dashrightarrow Y$$

is *dominant*, if the image $F(U) \subset Y$ is dense.

The point of the definition is that dominant rational maps can always be composed.

Consider $f = [(F, U)] : X \dashrightarrow Y$ and $g = [(G, V)] : Y \dashrightarrow Z$ dominant rational.

Let

$$W = U \cap F^{-1}(V) \subset X,$$

an open subset of X.

Then F is defined on W, and maps to V.

Now we can compose with G to get a representative $[W, G \circ F]$ for $g \circ f$.

Birational equivalence

Definition

A *birational equivalence* $f : X \dashrightarrow Y$ is a dominant rational map between irreducible quasiprojective varieties, which has a dominant rational inverse: a rational map $g : Y \dashrightarrow X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ (equalities of rational maps).
We say $X \simeq Y$ are *birational*.

Example 14.7

$\mathbb{A}^n \simeq \mathbb{P}^n$ are birational, via the rational map $\mathbb{A}^n \dashrightarrow \mathbb{P}^n$ defined by the inclusion $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$. (This is in fact a morphism.)
It has the rational inverse $\mathbb{P}^n \dashrightarrow \mathbb{A}^n$ defined by

$$[x_0 : \cdots : x_n] \to \left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right),$$

defined on U_0.

Example 14.8

Similarly, for an irreducible affine variety $X \subset \mathbb{A}^n$, let $\bar{X} \subset \mathbb{P}^n$ be the projective closure. Then the inclusion $X \to \bar{X}$ gives a birational equivalence $X \simeq \bar{X}$.

A famous birational self-equivalence

Consider the *Cremona transformation*

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

given by $[x : y : z] \mapsto [yz : xz : xy]$.
f is defined on the open set $U \subset \mathbb{P}^2$, where at least two coordinates are non-zero.
This rational map is equivalent to $[x : y : z] \mapsto [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}]$, defined on the open $V \subset \mathbb{P}^2$ where all coordinates are non-zero.
This form shows that

$$f \circ f = \mathrm{id}_{\mathbb{P}^2},$$

so the map f is its own inverse, and in particular is a birational self-equivalence.
Note that f does *not* have the form $[x] \mapsto [Ax]$ for $A \in \mathrm{GL}(3, k)$, which are the regular self-morphisms (automorphisms) of $\mathbb{P}^2$.

The Cremona group

Let

$$\mathrm{Aut}(\mathbb{P}^n) = \{\varphi\colon \mathbb{P}^n \to \mathbb{P}^n \mid \phi \text{ is an automorphism}\}.$$

It is not too hard to show that

$$\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n+1, k),$$

the group of projective transformations.
Define also the *Cremona group*

$$\mathrm{Bir}(\mathbb{P}^n) = \{\varphi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n \mid \phi \text{ is a birational automorphism}\}.$$

Then clearly

$$\mathrm{Aut}(\mathbb{P}^n) \subset \mathrm{Bir}(\mathbb{P}^n).$$

For $n = 1$,

$$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{Bir}(\mathbb{P}^1) \cong \mathrm{PGL}(2, k)$$

is the group of Möbius transformations.

The Cremona group

For $n \geq 2$, the inclusion

$$\mathrm{Aut}(\mathbb{P}^n) \subset \mathrm{Bir}(\mathbb{P}^n)$$

is no longer an equality, as the classical Cremona transformation above shows for $n = 2$.

The study of $\mathrm{Bir}(\mathbb{P}^n)$ is an active area of research. For example, it was only proved recently that $\mathrm{Bir}(\mathbb{P}^2)$ has nontrivial normal subgroups (i.e. it is not simple).

Dominant rational maps and field homomorphisms

Theorem 14.9

Consider quasi-projective varieties X, Y. There is a one-to-one correspondence between dominant rational maps $f\colon X \dashrightarrow Y$ and field homomorphisms $\phi\colon k(Y) \to k(X)$.

Proof, forward construction. We may restrict to open affines, so we may as well assume X, Y are affine. First assume that we have a dominant rational map $f\colon X \dashrightarrow Y$.

Consider the k-algebra homomorphism

$$f^* : k[Y] \to k(X)$$

defined by

$$(y : Y \to \mathbb{A}^1) \mapsto (f^* y = y \circ f : X \dashrightarrow \mathbb{A}^1).$$

We claim that as f is dominant, f^* is injective.

If so, we can define $f^* : k(Y) \to k(X)$ by

$$\frac{g}{h} \mapsto \frac{f^* g}{f^* h}.$$

Dominant rational maps and field homomorphisms

Proof of claim. The k-algebra homomorphism

$$f^* : k[Y] \to k(X)$$

is defined by

$$(y : Y \to \mathbb{A}^1) \mapsto (f^* y = y \circ f : X \dashrightarrow \mathbb{A}^1).$$

Suppose that $y \in k[Y]$ is in the kernel of Y. Write $f = [(U, F)]$. Then $F^* y = 0$ implies that for all $u \in U$,

$$y(F(u)) = 0.$$

Thus for all $u \in U$,

$$F(u) \subset \mathbb{V}(y)$$

and so

$$F(U) \subset \mathbb{V}(y) \subset Y.$$

As f was assumed dominant, $y = 0$.

Dominant rational maps and field homomorphisms

Proof, reverse construction. Suppose we have a field homomorphism $\phi\colon k(Y) \to k(X)$.
As fields have no proper ideals, ϕ is injective.
Let $y_1, \ldots, y_n$ be generators of $k[Y]$. Then

$$\varphi(y_j) = \frac{g_j}{h_j} \in k(X).$$

Let $U = \cap D_{h_j} = D_{h_1 \cdot \ldots \cdot h_n}$, an affine subvariety of the affine variety X.
We have $\varphi(y_j) \in \mathcal{O}_X(U) = k[U]$.
Since $k[Y]$ is generated by the y_j, we get an inclusion

$$\varphi\colon k[Y] \hookrightarrow k[U].$$

This corresponds to a dominant morphism $\varphi^* : U \to Y$ of affine varieties.
The pair $[(U, \phi^*)]$ represents a dominant rational map $\varphi^* : X \dashrightarrow Y$. $\square$

Birational varieties

Corollary 14.10

Two quasi-projective varieties X, Y are birational if and only if their function fields are isomorphic.

Example 14.11

As we saw before, we have a birational equivalence $\mathbb{A}^2 \simeq \mathbb{P}^2$. We also have $\mathbb{A}^1 \simeq \mathbb{P}^1$ and it is not hard to argue from this that $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We deduce a birational equivalence

$$\mathbb{P}^2 \simeq \mathbb{A}^2 \cong \mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

This is the first example of two *projective* varieties $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ that are birational to each other. In fact they are not isomorphic!

Sketch proof.

Suppose there is an isomorphism $\phi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$. Consider $C_0 = \{0\} \times \mathbb{P}^1$ and $C_1 = \{1\} \times \mathbb{P}^1$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. These are two curves (one-dimensional varieties) in $\mathbb{P}^1 \times \mathbb{P}^1$, with $C_0 \cap C_1 = \emptyset$.
Let $D_i = \phi(C_i)$. As ϕ is an isomorphism, D_1, D_2 are two curves in $\mathbb{P}^2$ with $D_1 \cap D_2 = \emptyset$. This contradicts Bézout's Theorem. □

C3.4 Algebraic Geometry

Lecture 15

Dominic Joyce, Oxford University, MT 2026

Tangent spaces of an affine variety

Consider $f \in k[x_1, \ldots, x_n]$, and $p = (p_1, \ldots, p_n) \in \mathbb{A}^n$.
The linear polynomial $d_p f \in k[x_1, \ldots, x_n]$ is defined by

$$d_p f = df|_{x=p} \cdot (x - p) = \sum \frac{\partial f}{\partial x_j}(p) \cdot (x_j - p_j).$$

Consider an affine variety $X \subset \mathbb{A}^n$ with $\mathbb{I}(X) = \langle f_1, \ldots, f_N \rangle$.

Definition

The *tangent space* to X at the point p is the space

$$T_p X = \mathbb{V}(d_p f_1, \ldots, d_p f_N) = \cap \ker df_i \subset \mathbb{A}^n.$$

This is an intersection of hyperplanes, so a linear subspace of some dimension.

Trivial Example

For $X = \mathbb{A}^n$,

$$T_p X = \mathbb{A}^n$$

at every point $p \in \mathbb{A}^n$.

Tangent spaces of a hypersurface

Example 15.1

Consider a hypersurface $X = \mathbb{V}(f)$ defined by a polynomial f. We have

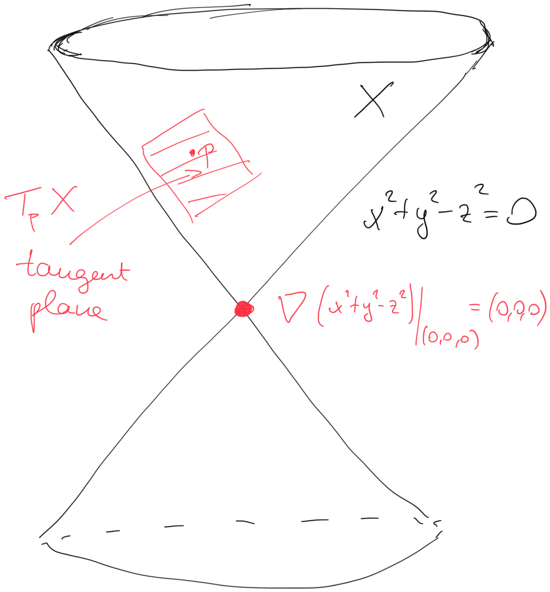
$$T_pX = \mathbb{V}(d_pf) = \ker(\nabla f_p) \subset \mathbb{A}^n.$$

- If the gradient $\nabla f_p \neq 0$, then this is a codimension-one linear subspace of $\mathbb{A}^n$, naturally thought of as the tangent space at p to $X = \mathbb{V}(f)$.
- If the gradient $\nabla f_p = 0$, then

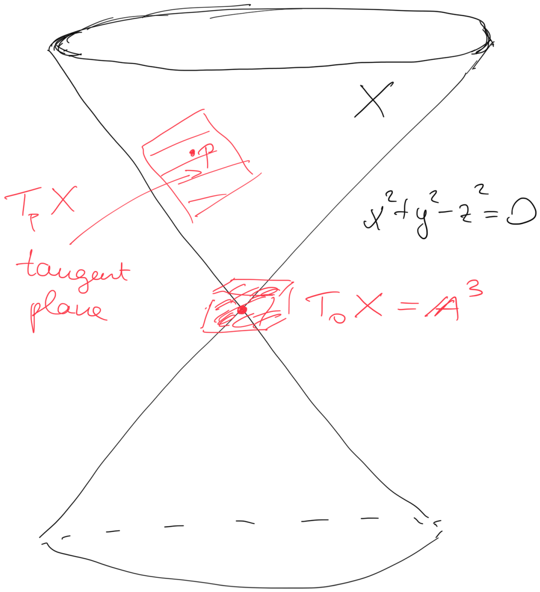
$$T_pX = \mathbb{A}^n.$$

Algebraic geometry does not say *there are no tangent vectors at such a point* $p \in X$.
It says *all vectors are tangent vectors at such a point* $p \in X$.

Tangent spaces of a hypersurface



Tangent spaces of a hypersurface



Smooth and singular points of an affine variety

Key fact For any point $p \in X$,

$$\dim_k T_pX \geq \dim_p X.$$

Definition

A point $p \in X$ is a *smooth point* if

$$\dim_k T_pX = \dim_p X.$$

A point $p \in X$ is a *singular point* if

$$\dim_k T_pX > \dim_p X.$$

Let

$$\mathrm{Sing}(X) = \{p \in X : p \text{ is a singular point of } X\} \subset X.$$

The variety X is *nonsingular* if

$$\mathrm{Sing}(X) = \emptyset.$$

Smooth and singular points of an affine hypersurface

Example 15.2

For an irreducible hypersurface $X = \mathbb{V}(f) \subset \mathbb{A}^n$, we have

$$\dim_p X = n - 1$$

at every point $p \in X$.

- If the gradient $\nabla f_p \neq 0$, then
$$\dim_k T_p X = n - 1 = \dim_p X,$$
so such a point is smooth.
- If the gradient $\nabla f_p = 0$, then
$$\dim_k T_p X = n > \dim_p X$$
so such a point is singular.

Remark

For $k = \mathbb{R}$, near a smooth point p, $X = \mathbb{V}(f)$ is a *codimension-one submanifold* of $\mathbb{R}^n$.

The singular set is Zariski closed

Let X be an irreducible affine variety of dimension d with

$$\mathbb{I}(X) = \langle f_1, \ldots, f_N \rangle.$$

Theorem 15.3

The set $\mathrm{Sing}(X) \subset X \subset \mathbb{A}^n$ *is a closed subvariety, given by the vanishing in* X *of all* $(n-d) \times (n-d)$ *minors of the Jacobian matrix*

$$\mathrm{Jac} = \left(\frac{\partial f_i}{\partial x_j} \right).$$

Proof. By definition, T_pX is the zero set of

$$\varphi_p : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial F_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial F_1}{\partial x_n}\Big|_p \\ \vdots & & \\ \frac{\partial F_N}{\partial x_1}\Big|_p & \cdots & \frac{\partial F_N}{\partial x_n}\Big|_p \end{pmatrix} \cdot \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix}.$$

The singular set is Zariski closed

We have that T_pX is the zero set of

$$\varphi_p : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial F_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial F_1}{\partial x_n}\Big|_p \\ \vdots & & \\ \frac{\partial F_N}{\partial x_1}\Big|_p & \cdots & \frac{\partial F_N}{\partial x_n}\Big|_p \end{pmatrix} \cdot \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix}.$$

Hence

$$p \in \operatorname{Sing} X \Leftrightarrow \dim \varphi_p^{-1}(0) > d \Leftrightarrow \dim \ker \operatorname{Jac}_p > d.$$

Claim. The last condition is equivalent to the vanishing of all $(n-d) \times (n-d)$ minors of Jac_p.

For otherwise, we could find $n-d$ linearly independent columns in Jac_p, the columns involved in that minor.

Hence the rank of Jac_p would have dimension at least $n-d$.

So the kernel of Jac_p would have dimension at most d. $\square$

The tangent space is an intrinsic notion

Let $X \subset \mathbb{A}^n$ be an affine variety, and let $p \in X$.
Recall the ring of germs of functions $\mathcal{O}_{X,p}$ and its maximal ideal

$$\mathfrak{m}_p = \left\{ \frac{f}{g} \in \mathcal{O}_{X,p} : f(p) = 0 \right\} \subset \mathcal{O}_{X,p}.$$

Theorem 15.4

There is a canonical isomorphism

$$T_pX \cong \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*.$$

The vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ is called the cotangent space.

Remark

What this result shows is that the tangent space can equivalently be defined in a purely local way, only using information about the ring of local germs of functions.

The tangent space is an intrinsic notion

Theorem 15.5

There is a canonical isomorphism

$$T_pX \cong \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*.$$

Proof. By a translation that does not change anything, we can assume

$$p = 0 \in X \subset \mathbb{A}^n.$$

First, we will look at the case $X = \mathbb{A}^n$ itself. For $F \in k[x_1, \ldots, x_n]$, we can consider the linear functional

$$d_0F = \left(\frac{\partial F}{\partial x_i}\right)\Big|_{x_i=0} \colon \mathbb{A}^n \equiv T_0\mathbb{A}^n \to k;$$

this is basically taking inner product of a vector with $\nabla_0 F$. So we get an element

$$d_0F \in (T_0\mathbb{A}^n)^*,$$

giving us a linear map

$$d_0 \colon k[x_1, \ldots, x_n] \to (T_0\mathbb{A}^n)^*.$$

The tangent space is an intrinsic notion

Have a linear map

$$d_0\colon k[x_1,\ldots,x_n]\to (T_0\mathbb{A}^n)^*.$$

Restricting to the maximal ideal $\mathfrak{m}$ at the origin on the left, we get a linear map

$$d_0|_{\mathfrak{m}}:\mathfrak{m}\to (T_0\mathbb{A}^n)^*.$$

Claim 1. The linear map $d_0|_{\mathfrak{m}}$ is surjective, and its kernel is $\mathfrak{m}^2$.
Claim 1 is easy to check — see Lecture Notes for details.
By the Isomorphism Theorem, we then obtain

$$(T_0\mathbb{A}^n)^*\cong \mathfrak{m}/\mathfrak{m}^2,$$

which is the dual of the isomorphism claimed by the Theorem in this case.

The tangent space is an intrinsic notion

Let us consider the general case $X \subset \mathbb{A}^n$. Let us continue to denote by $\mathfrak{m}$ the maximal ideal of the origin in $\mathbb{A}^n$.
The inclusion $j : T_0 X \hookrightarrow T_0 \mathbb{A}^n$ is injective, so the dual map is surjective, hence we get a surjection

$$j^* : \mathfrak{m}/\mathfrak{m}^2 \cong (T_0 \mathbb{A}^n)^* \to (T_0 X)^*.$$

and thus a surjection

$$j^* \circ d_0 \colon \mathfrak{m} \to (T_0 X)^*.$$

Claim 2. We have

$$\ker(j^* \circ d_0) = \mathfrak{m}^2 + \mathbb{I}(X).$$

For standard details of the proof of Claim 2, see Lecture Notes for details.
We deduce, once again by the isomorphism theorem,

$$\mathfrak{m}/(\mathfrak{m}^2 + \mathbb{I}(X)) \cong (T_0 X)^*.$$

The tangent space is an intrinsic notion

Write $\overline{\mathfrak{m}} \lhd k[X]$ for the image of $\mathfrak{m}$ in the quotient $k[X] = R/\mathbb{I}(X)$; this is the maximal ideal of $p = 0$ in the coordinate ring $k[X]$.
The last isomorphism then implies

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \cong (T_0 X)^*.$$

Finally, we need to relate this quotient to the corresponding quotient in the localized ring.
Claim 3. The standard inclusion $k[X] \hookrightarrow \mathcal{O}_{X,P}$ gives an isomorphism

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \cong \mathfrak{m}_p/\mathfrak{m}_p^2,$$

where recall $\mathfrak{m}_p \lhd \mathcal{O}_{X,P}$ is the maximal ideal of non-vanishing germs of regular functions at p.
For standard details of the proof of Claim 3, see Lecture Notes for details.
The proof of Theorem 15.5 is complete. $\square$

The tangent space is an intrinsic notion

Remark

The argument presented above also proves

$$T_pX \cong (\mathcal{I}_p/\mathcal{I}_p^2)^*,$$

where

$$\mathcal{I}_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle \subset k[X]$$

is the maximal ideal of an arbitrary point $p \in X$ inside the coordinate ring.

Corollary 15.6

The tangent space T_pX only depends on an open neighbourhood of $p \in X$.

Proof.

By the Theorem, tangent space only depends on the local ring $\mathcal{O}_{X,p}$ and its unique maximal ideal $\mathfrak{m}_p$. □

Tangent spaces and nonsingularity, quasi-projective case

Definition

For X a quasi-projective variety, we define the *tangent space* at $p \in X$ by

$$T_pX = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^* .$$

A point $p \in X$ is *smooth* if $dim_k T_pX = \dim_p X$.. The quasi-projective variety X is *nonsingular* if $\mathrm{Sing}(X) = \emptyset$.

Proposition 15.7

The set of non-smooth (singular) points $p \in X$ forms a Zariski closed subvariety $\mathrm{Sing}(X)$ of X.

Proof.

This is true in the (affine) neighbourhood of any point. □

Example 15.8

For an irreducible projective hypersurface $X = \mathbb{V}(f) \subset \mathbb{P}^n$, singular points $p \in X$ are located where

$$\nabla f_p = 0.$$

A further example

Recall from Lecture 7 the chain of subvarieties

$$\Sigma_{2,2} \subset \Delta \subset \mathbb{P}^8,$$

inside the projective space $\mathbb{P}\mathrm{M}_k(3) \cong \mathbb{P}^8$ of 3×3 matrices over k.
Here $\Delta = \{[A] : \det A = 0\} \subset \mathbb{P}^8$ is the projective cubic hypersurface defined by the determinant polynomial.
This contains $\Sigma_{2,2} \cong \mathbb{P}^2 \times \mathbb{P}^2$, the Segre variety in $\mathbb{P}^8$.
Fact. The hypersurface Δ is singular, with singular locus

$$\mathrm{Sing}(\Delta) = \Sigma_{2,2}.$$

C3.4 Algebraic Geometry

Lecture 16

Dominic Joyce, Oxford University, MT 2026

The blowup of affine space at the origin

The blow-up of $\mathbb{A}^n$ at the origin is the set

$$B_0\mathbb{A}^n = \{(x,\ell) : \mathbb{A}^n \times \mathbb{P}^{n-1} : x \in \ell\}.$$

Use coordinates $(x_1,\ldots,x_n)$ on $\mathbb{A}^n$ and $[y_1 : \cdots : y_n]$ on $\mathbb{P}^{n-1}$.
$x \in \ell$ if and only if $(x_1,\ldots,x_n)$ and $(y_1,\cdots,y_n)$ are proportional.
Equivalently the matrix

$$\begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{pmatrix}$$

has rank 1. This happens if and only if its 2×2 minors vanish.
So we can write the equations

$$B_0\mathbb{A}^n = \mathbb{V}(x_iy_j - x_jy_i) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

This shows in particular that $B_0\mathbb{A}^n$ is a quasi-projective variety.

The blowup of affine space at the origin

The morphism

$$\pi : B_0\mathbb{A}^n \to \mathbb{A}^n, \, \pi(x, [y]) = x$$

is birational with inverse

$$\mathbb{A}^n \dashrightarrow B_0\mathbb{A}^n, \quad x \mapsto (x, [x])$$

defined on $x \neq 0$: the point is that for $x \neq 0$, x lies in a unique line $[x]$. The fibre $\pi^{-1}(x)$ is a point for $x \neq 0$. Also

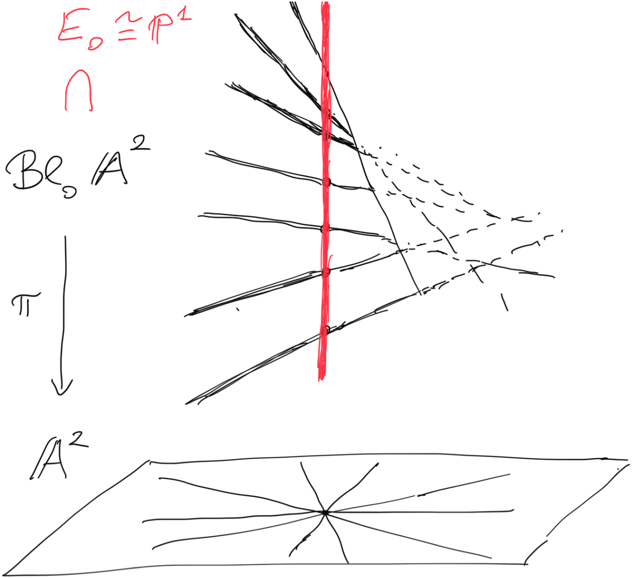
$$E_0 = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1},$$

called the *exceptional set* of π.
Thus

$$\pi : B_0\mathbb{A}^n \setminus E_0 \to \mathbb{A}^n \setminus 0$$

is an isomorphism, and π collapses E_0 to the point 0.

The blowup of affine space at the origin



The blowup of an affine variety in one of its points

Let $X \subset \mathbb{A}^n$ be an affine variety with $0 \in X$.

Definition

The *blow-up* of X at 0 is the Zariski closure

$$B_0X = \overline{\pi^{-1}(X \setminus \{0\})} \subset B_0\mathbb{A}^n.$$

The map $\pi|_{B_0X} : B_0X \to X$ is birational, and we have the *exceptional set*

$$E = \pi^{-1}(0) \cap B_0X.$$

Interpretation

The blowup B_0X keeps track of those directions $E \subset E_0$ along which X approaches 0.

Remark

The blowup B_0X, the Zariski closure of $\pi^{-1}(X \setminus \{0\})$, is different from

$$\pi^{-1}(X) = B_0X \cup_E E_0.$$

An example

Consider

$$X = \mathbb{V}(xy) \subset \mathbb{A}^2,$$

the union of the coordinate axes. Then $0 \in X$, and we can consider its blowup.

First,

$$\pi^{-1}(X \setminus 0) =$$
$$\{((x,y),[a:b]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xb - ya = 0, xy = 0, (x,y) \neq (0,0)\}.$$

We get the following solutions: either

$$((x,0),[1:0]) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

for $x \neq 0$, or

$$((0,y),[0:1]) \subset \mathbb{A}^2 \times \mathbb{P}^1$$

for $y \neq 0$.

Then we get the closure

$$B_0X = (\mathbb{A}^1 \times 0, [1:0]) \sqcup (0 \times \mathbb{A}^1, [0:1]) \subset B_0\mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1,$$

a disjoint union of two lines.

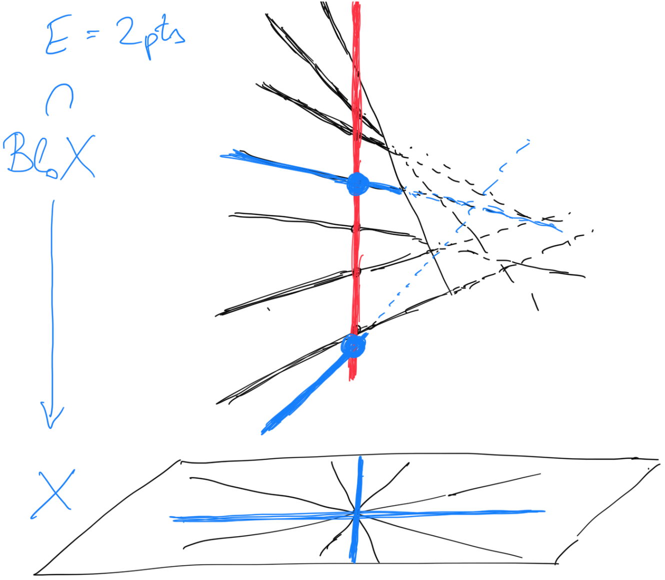
An example

We have

$$B_0X = (\mathbb{A}^1 \times 0, [1:0]) \amalg (0 \times \mathbb{A}^1, [0:1]) \subset B_0\mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1,$$

a disjoint union of two lines.
The exceptional set E consists of two points: $((0,0),[1:0])$, $((0,0),[0:1])$, the 2 directions of the lines in X.
Note also that X is *singular* at the origin, while B_0X is *nonsingular*.
Indeed, it is often (though not always) the case that if $0 \in X$ is a singular point, B_0X is 'less singular'.

An example



Another example: the cuspidal curve

Consider the cuspidal curve

$$Y = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2.$$

It is singular at $0 \in Y$. Let us describe the variety

$$B_0 Y \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Use coordinates $((x, y), [a : b])$; we have the equation $xb - ya = 0$ as one of the equations of the blowup. We want to compute the Zariski closure of

$$\pi^{-1}(Y \setminus \{0\}) \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Instead of this mixture of affine and projective coordinates, it is a lot easier to work in affine coordinates (remember quasi-projective varieties are locally affine!).

Another example: the cuspidal curve

Use the usual open cover of $\mathbb{P}^1$. We have the open set

$$(B_0Y)_a = (B_0Y) \cap (a \neq 0) \subset \mathbb{A}^2 \times \mathbb{A}^1,$$

with coordinates $(x, y, u = b/a)$ and equation $y = ux$. Substitute into the equation of the curve Y, to get the equation

$$0 = y^2 - x^3 = x^2u^2 - x^3.$$

The proper transform (Zariski closure) is obtained by dropping the x^2 factor:

$$u^2 - x = 0.$$

We obtain a description of our open set

$$(B_0Y)_a = \{u^2 - x = 0, y = ux\} \subset \mathbb{A}^3_{x,y,u}.$$

This is a nonsingular curve, isomorphic (using projection to the last coordinate u) to $\mathbb{A}^1$.

Another example: the cuspidal curve

The second open set is

$$(B_0 Y)_b = (B_0 Y) \cap (b \neq 0) \subset \mathbb{A}^2 \times \mathbb{A}^1,$$

with coordinates $(x, y, v = a/b)$ and equation $x = vy$. Substitute into the equation of the curve Y, to get the equation

$$0 = y^2 - x^3 = y^2 - y^3 v^3.$$

The proper transform (Zariski closure) is now obtained by dropping the y^2 factor:

$$1 - yv^3 = 0.$$

We get

$$(B_0 Y)_b = \{1 - yv^3 = 0, x = vy\} \subset \mathbb{A}^3_{x,y,v}.$$

Again, this is a nonsingular curve.

Another example: the cuspidal curve

The sets $(a \neq 0)$ and $(b \neq 0)$ cover $\mathbb{P}^1$. So the two open sets we looked at cover the blowup:

$$B_0 Y = (B_0 Y)_a \cup (B_0 Y)_b.$$

These open sets are described as explicit affine varieties

$$(B_0 Y)_a = \{u^2 - x = 0, y = ux\} \subset \mathbb{A}^3_{x,y,u}$$

and

$$(B_0 Y)_b = \{1 - yv^3 = 0, x = vy\} \subset \mathbb{A}^3_{x,y,v}.$$

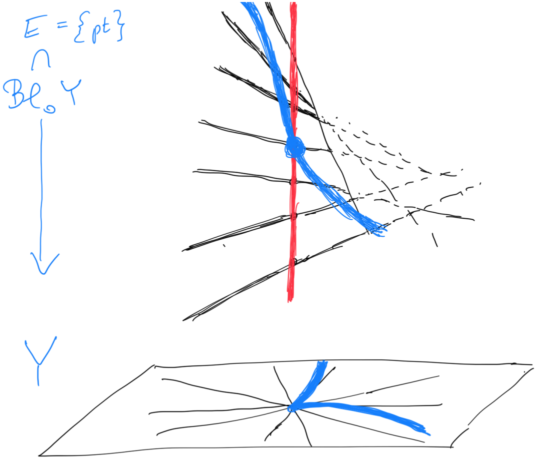
Both are nonsingular, so $B_0 Y$ is a nonsingular curve.
Finally the blowup map

$$\pi \colon B_0 \mathbb{A}^2 \to \mathbb{A}^2$$

induces a surjective, birational morphism

$$\pi|_{B_0 Y} \colon B_0 Y \to Y.$$

Another example: the cuspidal curve



Strategy for computing a general blowup

Consider in general

$$0 \in X \subset \mathbb{A}^n.$$

The blowup

$$B_0 X \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

can be computed into the following steps.

1. Cover $\mathbb{P}^{n-1}$ by the n standard affine open pieces. Introduce sensible names for the local coordinates.
2. Use the simplified quadric equations of $B_0\mathbb{A}^n$ to make substitutions into the equations of X.
3. Remove extra factors of local equations of the exceptional set to find the Zariski closure $(B_0X)_i$ in this local chart.
4. Check for singularities, etc of the resulting open pieces $(B_0X)_i$.

Strategy for computing a general blowup

The whole blowup B_0X will be covered by the n affine pieces

$$B_0X = \bigcup_{i=0}^{n-1} (B_0X)_i.$$

We get a surjective, birational morphism

$$\pi|_{B_0X} \colon B_0X \to X.$$

In some cases, the singularities of B_0X will be better than the singularities of X.

In particularly favourable cases, an iterated blowup of points of X may lead to a chain of surjective, birational morphisms

$$X_m \to X_{m-1} \to \ldots \to X_1 = B_pX \to X$$

with $p \in X$ a singular point, and X_m a *nonsingular* quasi-projective variety.

Resolution of singularities

Here is the general result, a cornerstone of 20th century algebraic geometry.

Theorem 16.1 (Hironaka's Resolution of Singularities Theorem)

Assume $\mathrm{char}\, k = 0$. *Given a quasi-projective variety* $X \supset Z = \mathrm{Sing}(X)$, *there exists a surjective, birational morphism*

$$f : Y \to X,$$

from a non-singular *quasiprojective variety* Y, *such that with* $E = f^{-1}(Z)$, f *induces an isomorphism*

$$Y \setminus E \cong X \setminus Z.$$

In other words, X is unchanged outside its singular locus.
If X is projective, Y can also be chosen to be projective.
The map f is a composite of certain kinds of blowups (not just that of closed points).