PRELIMINARY VERSION. MAY BE UPDATED.

Problem Sheet 3

Section A: introductory questions, not for handing in

1. **Degree of a conic, explicitly**

   Let $C = \{x_0^2 + x_1^2 + x_2^2 = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2}$. Consider lines

   $$L = \{a_0x_0 + a_1x_1 + a_2x_2 = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2}.$$

   (a) Find the maximum number of intersection points between C and L.

   (b) Find also the set of coefficients (a_0, a_1, a_2) so that the intersection $L \cap C$ *does not* consist of the maximal number of intersection points.

2. **Basics on dimension**

   (a) Show that dimension is an invariant of the isomorphism class of a projective variety.

   (b) Show that, if $X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.

Section B: core questions, to be handed in for marking

3. **Another set of equations for the twisted cubic curve**

   (a) Let $Q = \{x_0x_2 = x_1^2\} \subset \mathbb{P}^3$ and $F = \{x_0x_3^2 - 2x_1x_2x_3 + x_2^3 = 0\} \subset \mathbb{P}^3$. Prove that $C = Q \cap F \subset \mathbb{P}^3$ is the twisted cubic curve, the image of the third Veronese embedding $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$.

   Hint: multiply the second equation with x_2 and use the first equation to put it in the form of a perfect square.

   (b) Does this mean that the ideal $\mathbb{I}(\nu_3(\mathbb{P}^1))$ can in fact be generated by two elements? (Compare with Sheet 2, Question 3.)

4. **Projecting a space curve**

   Let $Q_0 = \{x_0x_3 = x_1^2\} \subset \mathbb{P}^3$ and $Q_1 = \{x_1x_3 = x_2^2\} \subset \mathbb{P}^3$. Consider the projective variety $C = Q_0 \cap Q_1 \subset \mathbb{P}^3$. Show that the formula $\pi\colon [x_0\colon x_1\colon x_2\colon x_3] \mapsto [x_0\colon x_1\colon x_2]$ defined on a Zariski open subset of C can be extended to a projective morphism

   $$\pi\colon C \to \mathbb{P}^2.$$

   Show that π maps C isomorphically to the projective variety (plane curve)

   $$D = \{x_0x_2^2 = x_1^3\} \subset \mathbb{P}^2.$$

   Hint: to extend π, try to express the ratios $x_0 : x_1 : x_2$ in a different way using the equations of C.

5. **Dimension, degree and Hilbert polynomial**

   (a) Show carefully that if X is a reducible projective variety with equidimensional irreducible components X_i, then $\deg X = \sum_i \deg X_i$.

   (b) Compute the degree $\deg \nu_d(\mathbb{P}^1)$ directly from the definition.

   (c) Let F be a homogeneous irreducible polynomial of degree d in $k[x_0, \dots, x_n]$, and let $X = \mathbb{V}(F) \subset \mathbb{P}^n$. Find the Hilbert polynomial of X. Deduce that, as expected, $\dim X = n - 1$ and $\deg X = d$.

6. **Affine and quasi-projective varieties**

   (a) Find an open affine cover of $\mathbb{A}^2 \setminus \{(0,0)\}$.

   (b) Show that $\mathrm{GL}(n, k)$ is an affine variety, i.e. that it is isomorphic as a quasi-projective variety to a Zariski closed subset of an affine space.

   (c) Let X be an affine variety and $f \in k[X]$. Show that f vanishes nowhere on X if and only if f is invertible in $k[X]$.

Section C: optional questions

7. **The Veronese image**

   Find the Hilbert polynomial of the Veronese image $\nu_d(\mathbb{P}^n)$. Find the dimension and degree of this projective variety.

8. **Complete intersection of two hypersurfaces**

   (a) Let $f, g \in k[x_0, \ldots, x_n]$ be two essentially distinct irreducible homogeneous polynomials (i.e. not scalar multiples of each other) of degrees e, f such that the homogeneous ideal $I = \langle f, g \rangle$ is radical. Compute the Hilbert function of $X = \mathbb{V}(I) \subset \mathbb{P}^n$. Deduce the dimension and degree of X.

   (b) Conclude that the twisted cubic $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$ is not a *complete intersection*: its (radical) vanishing ideal cannot be generated by two homogeneous polynomials. Compare with Question 3.