Professor Joyce **C3.4 Algebraic Geometry** **MT 2026**

PRELIMINARY VERSION. MAY BE UPDATED.

Problem Sheet 4

Section A: introductory questions, not for handing in

1. (a) Show that if $\mathrm{char}(k)$ does not divide d, then the hypersurface

$$\mathbb{V}(x_0^d + \ldots + x_n^d) \subset \mathbb{P}^n$$

is nonsingular.

(b) Under the same assumption on k, find the singular locus of the hypersurface

$$\mathbb{V}(x_0^d + \ldots + x_{n-1}^d) \subset \mathbb{P}^n.$$

2. Let X be an irreducible affine variety. Show that for any open set $U \subset X$ and point $p \in X$, the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ are subrings of the function field $k(X)$.

Hint. You need to explain first how to include $\mathcal{O}_X(U) \subset k(X)$ and $\mathcal{O}_{X,p} \subset k(X)$.

Section B: core questions, to go over in class (do not hand in)

3. Let $X_1 = \mathbb{V}(xy(x-y)) \subset \mathbb{A}^2$ and $X_2 = \mathbb{V}(xy, yz, zx) \subset \mathbb{A}^3$.
   (a) Decompose these varieties into irreducible components.
   (b) By computing the dimension of tangent spaces at various points, show that X_1 and X_2 are not isomorphic.

4. (a) Let $F : X \dashrightarrow Y$ be a rational map of quasi-projective varieties, with X irreducible. If (U, f) is a representation for F, with U affine, show that

$$\{(u, f(u)) : u \in U\} \subset U \times Y$$

   is a closed subvariety. Define the *graph* Γ_F of F to be the (Zariski) closure of $\{(u, f(u)) : u \in U\} \subset X \times Y$. Show that the projection from the graph $\Gamma_F \to X$ is a birational equivalence.
   (b) Define $F : \mathbb{A}^2 \to \mathbb{A}^1$ by $F(x, y) = \frac{y}{x}$. Find the equation defining $\Gamma_F \subset \mathbb{A}^3$.

5. Recall that the projective varieties $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic. Find a birational equivalence $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

6. In these examples, attempt a resolution of singularities of the given variety by successively blowing up singular points.
   (a) Find a resolution of singularities of the affine curve $C_1 = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$. Deduce that C_1 is rational, in other words birationally equivalent to $\mathbb{P}^1$.

   Hint. Try blowing up the curve at the origin, and consider the map that projects to the exceptional divisor.
   (b) Desingularise the affine curve $C_2 = \mathbb{V}(y^2 - x^4 - x^5) \subset \mathbb{A}^2$. Draw a picture of the series of blow-ups.
   (c) Desingularise the affine surface $S_2 = \mathbb{V}(-xy + z^2) \subset \mathbb{A}^3$.

Section C: optional questions

7. (a) Find an inductive procedure to desingularise the affine surface

$$S_n = \mathbb{V}(-xy + z^n) \subset \mathbb{A}^3$$

for a positive integer $n > 2$.

(b) By successively blowing up singular points, desingularise the surface

$$T_1 = \{x^2 + y^3 + z^3 = 0\} \subset \mathbb{A}^3.$$

(c) Investigate whether it is possible to desingularise the affine surface

$$T_2 = \{x^2 - y^2 z = 0\} \subset \mathbb{A}^3$$

by a procedure with begins by blowing up the origin $(0, 0, 0) \in T_2$. How is this example different from all the earlier examples of singular affine surfaces?

8. Assume that the base field $k = \mathbb{C}$. Show that the projective surface

$$X = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \subset \mathbb{P}^3$$

contains at least 27 lines.

Hint. Move two of the summands to the other side, and factorise both sides of the resulting equality.

Comment. It is possible to show that there exists a union of six disjoint lines

$$E = L_1 \cup \ldots \cup L_6 \subset X$$

and a birational morphism $\pi\colon X \to \mathbb{P}^2$ such that π is the blowup of six points in $\mathbb{P}^2$ and $E \subset \mathbb{P}^2$ is the exceptional locus of π. For details, see the last chapter of Reid: Undergraduate Algebraic Geometry.