Infinite groups: Sheet 1, Solutions

2026

Section A

Exercise 1. (i) If $N \trianglelefteq H$ and $H \trianglelefteq G$, with N of finite index in H, and H finitely generated, then N contains a finite index subgroup K which is normal in G.

(ii) Show that if H is characteristic in G then we can take K to be characteristic in G as well.

Solution.

(i) Let $m = [H : N]$. Since H is finitely generated, it has only finitely many subgroups of index m. Because $H \trianglelefteq G$, every conjugate gNg^{-1} is a subgroup of H of index m. Hence the set of conjugates of N by elements of G is finite. If these conjugates are $N_1 = N, N_2, \ldots, N_r$, put

$$K = \bigcap_{j=1}^{r} N_j.$$

Then $K \leq N$ and K has finite index in H. Conjugation by an element of G permutes the subgroups N_j, so it preserves their intersection. Thus $K \trianglelefteq G$.

(ii) If H is characteristic in G, let K be the intersection of *all* subgroups of H of index $m = [H : N]$. There are only finitely many of them, so K has finite index in H, and $K \leq N$. Every automorphism of G preserves H and permutes its subgroups of index m. Hence it preserves K, so K is characteristic in G.

Exercise 2. Find the commutator subgroup and the abelianization for the finite dihedral group D_{2n} and for the infinite dihedral group D_∞.

Solution. Write

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ srs^{-1} = r^{-1} \rangle.$$

Since $[s,r]=r^{-2}$, the subgroup $\langle r^2\rangle$ is contained in D'_{2n}. Conversely, $D_{2n}/\langle r^2\rangle$ is abelian, because in this quotient $r=r^{-1}$ and hence r commutes with s. Therefore

$$D'_{2n}=\langle r^2\rangle.$$

If n is odd then $\langle r^2\rangle=\langle r\rangle$, so $D_{2n}^{\mathrm{ab}}\cong\mathbb{Z}_2$. If n is even, then $D_{2n}^{\mathrm{ab}}\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$.

For

$$D_\infty=\langle r,s\mid s^2=1,\ srs^{-1}=r^{-1}\rangle$$

the same argument gives

$$D'_\infty=\langle r^2\rangle,\qquad D_\infty^{\mathrm{ab}}\cong\mathbb{Z}_2\oplus\mathbb{Z}_2.$$

Exercise 3. Show that

(i) The dihedral group D_{2n} is isomorphic to a semi-direct product $\mathbb{Z}_n\rtimes_\varphi\mathbb{Z}_2$ and the infinite dihedral group D_∞ is isomorphic to a semi-direct product $\mathbb{Z}\rtimes_\varphi\mathbb{Z}_2$.

(ii) The permutation group S_n is the semidirect product of A_n and $\mathbb{Z}_2=\{\mathrm{Id},(12)\}$.

Solution.

(i) In D_{2n} the rotation subgroup $\langle r\rangle\cong\mathbb{Z}_n$ is normal and the quotient is generated by the image of a reflection, hence is $\mathbb{Z}_2$. The exact sequence

$$1\longrightarrow\langle r\rangle\longrightarrow D_{2n}\longrightarrow\mathbb{Z}_2\longrightarrow 1$$

splits by sending the non-trivial element of $\mathbb{Z}_2$ to s. Conjugation by s acts on $\langle r\rangle$ by inversion. Thus $D_{2n}\cong\mathbb{Z}_n\rtimes\mathbb{Z}_2$. The identical argument, with $\langle r\rangle\cong\mathbb{Z}$, gives $D_\infty\cong\mathbb{Z}\rtimes\mathbb{Z}_2$.

(ii) The sign homomorphism $\mathrm{sgn}:S_n\to\mathbb{Z}_2$ has kernel A_n. It splits by sending the non-trivial element of $\mathbb{Z}_2$ to the transposition (12). Hence

$$S_n\cong A_n\rtimes\langle(12)\rangle.$$

Exercise 4. (i) Show that the torsion-free abelian group $\mathbb{Q}$ is not a free abelian group.

(ii) Let $0\to A\to B\xrightarrow{r}C\to 0$ be a short exact sequence of abelian groups, where C is free abelian. Then the sequence splits and $B\cong A\oplus C$.

(iii) Let F be a free abelian group of rank n and let $B=\{x_1,\dots,x_n\}$ be a generating set of F. Show that B is a basis of F. Conclude that n equals the minimal cardinality of all generating sets of F.

Solution.

(i) Suppose that $\mathbb{Q}$ were free abelian and let b be an element of a basis. Since $\mathbb{Q}$ is divisible, there is $q \in \mathbb{Q}$ with $2q = b$. Writing q as a finite integral linear combination of basis elements and comparing the coefficient of b in the equality $2q = b$ gives an even integer equal to 1, a contradiction. Thus $\mathbb{Q}$ is not free abelian.

(ii) Let $(c_i)_{i \in I}$ be a basis of the free abelian group C. For each i choose $b_i \in B$ with $r(b_i) = c_i$. By the universal property of free abelian groups, $c_i \mapsto b_i$ extends to a homomorphism $s : C \to B$ and $r \circ s = \mathrm{id}_C$. Thus the sequence splits. Every $b \in B$ can be written uniquely as

$$b = (b - s(r(b))) + s(r(b)),$$

with the first term in $A = \ker r$ and the second in $s(C)$; moreover $A \cap s(C) = 0$. Hence $B \cong A \oplus C$.

(iii) Choose a basis $Y = \{y_1, \dots, y_n\}$ of F. The map $y_i \mapsto x_i$ extends to a surjective endomorphism $f : F \to F$. After tensoring with $\mathbb{Q}$ we obtain a surjective linear map $\mathbb{Q}^n \to \mathbb{Q}^n$, hence an isomorphism. Therefore $\ker f$ has rank 0. But $\ker f$ is a subgroup of the free abelian group F, hence torsion-free; consequently $\ker f = 0$ and f is an automorphism. Thus B is a basis. Finally, a generating set with m elements would give a surjection $\mathbb{Q}^m \to \mathbb{Q}^n$ after tensoring with $\mathbb{Q}$, so $m \geq n$. Therefore n is the minimum number of generators.

Section B

Exercise 5. Prove that if $K \leq H \leq G$, and if H is a normal subgroup of G and K is a characteristic subgroup of H, then K is a normal subgroup of G. Is the same true if K is only normal in H? Provide a proof in the affirmative case, a counterexample in the negative case.

Solution. If $g \in G$, normality of H implies that conjugation by g restricts to an automorphism of H. Since K is characteristic in H, this automorphism preserves K. Thus $gKg^{-1} = K$ for every $g \in G$, and $K \trianglelefteq G$.

The statement is false if "characteristic" is replaced by "normal". Take $G = A_4$, let

$$H = V_4 = \{1, (12)(34), (13)(24), (14)(23)\},$$

and let $K = \langle (12)(34) \rangle$. Then $H \trianglelefteq A_4$ and, since H is abelian, $K \trianglelefteq H$. But conjugation by a 3–cycle permutes the three subgroups of order 2 in V_4, so K is not normal in A_4.

Exercise 6. Let G be a finitely generated group and let S be an infinite set of generators of G. Show that there exists a finite subset F of S so that G is generated by F.

Solution. Let X be a finite generating set of G. For each $x \in X$, choose a word w_x in the alphabet $S \cup S^{-1}$ representing x, and let $F_x \subset S$ be the finite set of letters of S occurring in w_x. Then

$$F = \bigcup_{x \in X} F_x$$

is finite. Every element of X belongs to $\langle F \rangle$, hence $G = \langle X \rangle \leq \langle F \rangle$. Since $F \subset S \subset G$, the reverse inclusion is automatic, and $G = \langle F \rangle$.

Exercise 7. Suppose that we have a short exact sequence of groups

$$\{1\} \longrightarrow G_1 \xrightarrow{i} G_2 \xrightarrow{\pi} G_3 \longrightarrow \{1\} . \tag{1}$$

such that the groups G_1, G_3 are finitely generated. Prove that G_2 is also finitely generated.

Solution. Let S_1 and S_3 be finite generating sets of G_1 and G_3. For each $\bar{s} \in S_3$ choose a lift $s \in G_2$ with $\pi(s) = \bar{s}$, and let T be the finite set of these lifts. We claim that

$$i(S_1) \cup T$$

generates G_2. Indeed, if $g \in G_2$, write $\pi(g)$ as a word $\bar{s}_1^{\varepsilon_1} \cdots \bar{s}_m^{\varepsilon_m}$ in S_3. Then

$$g s_m^{-\varepsilon_m} \cdots s_1^{-\varepsilon_1} \in \ker \pi = i(G_1).$$

The latter element is a word in $i(S_1)$, so g is a word in $i(S_1) \cup T$.

Exercise 8. Let H be the group of permutations of $\mathbb{Z}$ generated by the transposition $t = (01)$ and the translation map $s(i) = i + 1$. Let H_i be the group of permutations of $\mathbb{Z}$ supported on $[-i, i] = \{-i, -i+1, \dots, 0, 1, \dots i-1, i\}$, and let H_ω be the group of finitely supported permutations of $\mathbb{Z}$ (i.e. the group of bijections $f : \mathbb{Z} \to \mathbb{Z}$ such that f is the identity outside a finite subset of $\mathbb{Z}$),

$$H_\omega = \bigcup_{i=0}^{\infty} H_i.$$

Prove that H_ω is a normal subgroup in H and that $H/H_\omega \simeq \mathbb{Z}$. Prove that H_ω is not finitely generated.

Solution. For every $k \in \mathbb{Z}$,

$$s^k t s^{-k} = (k\ k+1).$$

The adjacent transpositions generate every finite symmetric group, so they generate all finitely supported permutations of $\mathbb{Z}$. Hence $H_\omega \leq H$. Conjugation by s shifts supports and therefore preserves H_ω; conjugation by t also preserves it because $t \in H_\omega$. Since $H = \langle s, t \rangle$, we get $H_\omega \trianglelefteq H$.

The quotient H/H_ω is generated by the image of s. No non-zero power s^m is finitely supported, so this image has infinite order. Therefore $H/H_\omega \cong \mathbb{Z}$.

Finally, if H_ω were generated by finitely many permutations $g_1, \ldots, g_r$, all their supports would lie in $[-N, N]$ for some N. Then $\langle g_1, \ldots, g_r \rangle \leq H_N$, whereas H_ω contains, for example, the transposition $(N+1\ N+2)$. This is a contradiction.

Exercise 9. 1. Let $a_i, i \in I$ and $c_j, j \in J$ be sets of generators of A and C, respectively. Prove that the set of elements $(1, c_j), j \in J$ and $(\delta_{a_i}, 1), i \in I$, generate $G_A := A \wr C$. In particular, if A and C are finitely generated, so is $A \wr C$.

2. Let G be the wreath product $\mathbb{Z} \wr \mathbb{Z} \cong N \ltimes \mathbb{Z}$, where N is the (countably) infinite direct sum of copies of $\mathbb{Z}$. Prove that G is 2-generated and that the normal subgroup N is not finitely generated.

Solution.

1. Let $B = \bigoplus_C A$. Conjugating $(\delta_a, 1)$ by $(1, c)$ moves its support from $1 \in C$ to $c \in C$ (with the convention for the action used in the statement). Hence the proposed generators give, at every coordinate $c \in C$, a copy of every generator a_i of A. They therefore generate all finitely supported functions $C \to A$, i.e. the base group B. The elements $(1, c_j)$ generate the factor C, so together the stated elements generate $B \rtimes C = A \wr C$.

2. For $\mathbb{Z} \wr \mathbb{Z} = N \rtimes \langle t \rangle$, let a denote the element of N which is $1 \in \mathbb{Z}$ at the coordinate 0 and 0 elsewhere. The conjugates $t^k a t^{-k}$, $k \in \mathbb{Z}$, generate all coordinate copies of $\mathbb{Z}$, hence generate N. Thus a and t generate the wreath product. On the other hand, $N = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ is not finitely generated: a finite set of elements has support in a fixed finite subset of $\mathbb{Z}$, and the subgroup it generates is supported in the same finite set.

Exercise 10. Prove that every short exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} F(X) \longrightarrow 1 \tag{2}$$

splits.

Solution. For each $x \in X$ choose $t_x \in G$ with $\pi(t_x) = x$. By the universal property of the free group, the map $x \mapsto t_x$ extends uniquely to a homomorphism

$$s : F(X) \longrightarrow G.$$

The composition $\pi \circ s : F(X) \to F(X)$ is the identity on the free basis X, hence is the identity homomorphism. Thus s is a section of π and the short exact sequence splits. In particular, s is injective.

Exercise 11. Suppose that $g \in Bij(X)$ is a bijection such that for some $A \subset X$,

$$g(A) \subsetneq A.$$

Prove that g has infinite order.

Solution. From $g(A) \subsetneq A$ and bijectivity of g we obtain, by repeatedly applying g,

$$g^m(A) \subsetneq g^{m-1}(A) \subsetneq \cdots \subsetneq g(A) \subsetneq A$$

for every $m \geq 1$. If g had finite order m, then $g^m(A) = A$, contradicting the strict inclusion $g^m(A) \subsetneq A$. Hence g has infinite order.

Exercise 12. Let F_2 be the free group of rank two.

- State and prove a generalization of the Ping-pong lemma to n elements $g_1, g_2, \ldots, g_n$.
- Prove that for every n, the group F_2 has a subgroup isomorphic to F_n.
- Prove that every free group of countable rank can be embedded as a subgroup of F_2.

Solution. We use the following standard n–generator version of ping–pong. Let $g_1, \ldots, g_n$ be bijections of a set X, and suppose that there are pairwise disjoint non-empty sets B_i^+, B_i^- such that

$$g_i\big(X \setminus B_i^-\big) \subset B_i^+, \qquad g_i^{-1}\big(X \setminus B_i^+\big) \subset B_i^-.$$

Then $g_1, \ldots, g_n$ freely generate a free group of rank n. Indeed, let $w = h_1 \cdots h_m$ be a non-empty reduced word in the letters $g_i^{\pm 1}$. Choose a ping–pong set different from the repelling set of the last letter. Applying the letters from right to left, reducedness ensures at each step that the current point lies outside the repelling set of the next letter, so its image lies in that letter's attracting set. The final point lies in the attracting set of the first letter, which is disjoint from the set where the point started. Hence w is not the identity.

Now let $F_2 = \langle a, b \rangle$ and, for $i \geq 1$, put

$$u_i = ab^i a.$$

The words u_i are cyclically reduced. Moreover, the $2n$ words $u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}$ are pairwise prefix-incomparable. If W_u denotes the set of reduced words beginning with the reduced word u, the corresponding sets W_{u_i} and $W_{u_i^{-1}}$ are therefore pairwise disjoint. Since u_i is cyclically reduced, reduction of $u_i^2 w$ can erase the whole second copy of u_i only when w begins with u_i^{-1}. Consequently

$$L_{u_i^2}(F_2 \setminus W_{u_i^{-1}}) \subset W_{u_i}, \qquad L_{u_i^{-2}}(F_2 \setminus W_{u_i}) \subset W_{u_i^{-1}},$$

where L_v is left multiplication by v. Ping–pong shows that $u_1^2, \ldots, u_n^2$ freely generate a subgroup isomorphic to F_n.

The same argument works simultaneously for the infinite family $(u_i^2)_{i \geq 1}$, since all the sets W_{u_i} and $W_{u_i^{-1}}$ are pairwise disjoint. Thus these elements freely generate a free group of countable rank inside F_2.