Infinite groups: Sheet 2, Solutions

2026

Section A

Exercise 1. 1. Prove that if S and $\bar{S}$ are two finite generating sets of G, then the word metrics dist_S and $\mathrm{dist}_{\bar{S}}$ on G are bi-Lipschitz equivalent, i.e. there exists $L > 0$ such that

$$\frac{1}{L}\mathrm{dist}_S(g,g') \leqslant \mathrm{dist}_{\bar{S}}(g,g') \leqslant L\mathrm{dist}_S(g,g')\,, \forall g,g' \in G\,. \tag{1}$$

2. Prove that an isomorphism between two finitely generated groups is a bi-Lipschitz map when the two groups are endowed with word metrics.

Solution.

1. By left invariance it is enough to compare $|g|_S$ and $|g|_{\bar{S}}$. Put

$$L = \max\Big(\max_{s\in S}|s|_{\bar{S}}, \max_{\bar{s}\in\bar{S}}|\bar{s}|_S\Big).$$

If $g = s_1\cdots s_m$ is a shortest word in $S\cup S^{-1}$, replacing each s_i by a word of length at most L in $\bar{S}\cup\bar{S}^{-1}$ gives $|g|_{\bar{S}} \le L|g|_S$. Interchanging S and $\bar{S}$ gives $|g|_S \le L|g|_{\bar{S}}$. Applying this to $g^{-1}g'$ yields the desired bi-Lipschitz inequalities.

2. If $\varphi : G \to H$ is an isomorphism and S is a finite generating set of G, then $\varphi(S)$ is a finite generating set of H and φ is an isometry for the corresponding word metrics. Any other finite word metric on H is bi-Lipschitz equivalent to this one by Part (1), and similarly on G. Thus φ is bi-Lipschitz for arbitrary word metrics coming from finite generating sets.

Exercise 2. Consider the *integer Heisenberg group*

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \; ; \; x_1,\dots,x_n,y_1,\dots,y_n,z\in\mathbb{Z} \right\}.$$

Prove that $H_{2n+1}(\mathbb{Z})$ is nilpotent of class 2.

Solution. Write an element as $h(x,y,z)$, where $x,y\in\mathbb{Z}^n$ and $z\in\mathbb{Z}$ (with the order of the y–coordinates as in the displayed matrix). Direct multiplication gives

$$h(x,y,z)h(x',y',z') = h\bigl(x+x', y+y', z+z'+\langle x,y'\rangle\bigr),$$

for the corresponding integral bilinear pairing. Hence

$$[h(x,y,z),h(x',y',z')] = h\bigl(0,0,\langle x,y'\rangle - \langle x',y\rangle\bigr).$$

Thus the commutator subgroup is contained in

$$Z = \{h(0,0,z) : z\in\mathbb{Z}\},$$

and Z is central. Therefore $C^3H_{2n+1}(\mathbb{Z}) = 1$. The group is not abelian: choosing $x = e_1$ and a suitable $y' = e_1$ gives a non-trivial commutator. Hence its nilpotency class is exactly 2.