

## Geometric Group Theory

### Problem Sheet 2

#### Section C

1. Give an example of a residually finite group which is not Hopf.

*Solution* An infinite direct sum of  $Z$ 's or a free group of infinite rank will do.

2. Show that every cyclic subgroup of  $F_n$  (the free group of rank  $n$ ) is separable.

*Solution* Enough to do for  $n = 2$ . Let  $v \in F_2$  and let  $w \notin \langle v \rangle$ . Certainly we can find a homomorphism to  $Sym(X)$  as in the notes so that  $f(w) \neq f(v)$ . The issue is to make sure that  $f(v^n) \neq f(w)$  for any  $n$ . We may assume  $v$  is cyclically reduced (otherwise just replace  $v$  by a cyclically reduced conjugate of it  $gvg^{-1}$ -and replace  $w$  by  $gwg^{-1}$  as well). Now consider  $k$  such that  $N = |v|^k > |w|$  and let  $X$  be the set of reduced words of length  $N$ . We define as in the notes maps  $\alpha, \beta \in Sym(X)$  acting as the generators  $a, b$  on reduced words of length  $\leq N - 1$ . In fact slightly more generally we define  $\alpha(g) = ag$  for all  $g$  in  $X$  such that  $ag \in X$ - and similarly for  $\beta$ .

Now we identify the elements  $v^k$  and  $v^{-k}$  in  $X$ . We need to check that this is possible as if some permutation say  $\alpha$  is already defined on these two elements and it is defined in different ways then this identification is not possible.

Note that if  $v^k = a_1 \dots a_r$  ( $a_i \in \{a^{\pm 1}, b^{\pm 1}\}$ ) then the permutation corresponding to the letter  $a_1^{-1}$  is defined already on  $v^k$ . Similarly the permutation corresponding to  $a_r$  is already defined on  $v^{-k} = a_r^{-1}(a_{r-1}^{-1} \dots a_1^{-1})$ . Since  $v$  is cyclically reduced  $a_1^{-1} \neq a_r$  so it is possible to identify  $v^k$  and  $v^{-k}$ . Note that after this identification the permutations corresponding to  $a_1^{-1}, a_r$  are both defined on the new point.

It follows that the maps  $\alpha, \beta$  are still well defined after this identification. Finally we extend  $\alpha, \beta$  to the rest of  $X$  in any way. Then  $v^n \cdot e = v^r$  where  $r \equiv n \pmod{2k}$ ,  $r \in [-k, k]$ , so  $v^r \cdot e \neq w \cdot e$ , therefore  $f(v^n) \neq f(w)$  for any  $n$ .

3. Show that if  $A, B$  are residually finite then  $A * B$  is also residually finite.

*Solution* If  $w = c_1 \dots c_n$  is a reduced word in  $A * B$  define homomorphisms  $f : A \rightarrow A_1$ ,  $g : B \rightarrow B_1$  ( $A_1, B_1$  finite) such that if  $c_i \in A$ ,  $f(c_i) \neq 1$  and if  $c_i \in B$   $g(c_i) \neq 1$ . By the universal property of the amalgam there

is a homomorphism  $F : A * B \rightarrow A_1 * B_1$  such that  $F$  restricted to  $A$  is  $f$  and restricted to  $B$  is  $g$ . So  $F(w) \neq 1$ . We remark now that  $A_1 * B_1$  has a finite index free subgroup so  $A_1 * B_1$  is residually finite. So there is a homomorphism  $G : A_1 * B_1 \rightarrow C$ ,  $C$  finite, such that  $G(F(w)) \neq 1$ .