Lecture Notes for Analytic Topology

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November 22, 2026

1 Disclaimer

These are **draft** lecture notes in expanded form. *If you would like to have a coherent, checked and correct account of Analytic Topology, you are better served by reading the relevant part of various textbooks.* These notes are **not** carefully checked (so will contain errors), may contain material which is not part of the course and may not contain all material I will cover in the lectures. In particular, motivation and order of concepts may differ and I may choose other (equivalent) definitions.

2 Basics

I expect this to have been covered in a first Topology course.

Definition 2.1. A topology on a set X is a collection τ of subsets of X such that τ is closed under finite intersections and arbitrary unions.

A topological space is a pair (X, τ) where X is a set and τ is a topology on X.

Note that I use the convention that $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset$ is everything (where everything depends on the context).

Definition 2.2. Suppose (X, τ) is a topological space and $A \subseteq X$.

A is *X-open* or *an open subset of X* if and only if $A \in \tau$.

A is *X-closed* if and only if $X \setminus A \in \tau$.

The *closure* of A, $\overline{A}$, is the intersection of all closed subsets of X containing A. If the space/topology with respect to which the closure is taken is unclear we will use $\overline{A}^X$.

The *interior* of A, $\operatorname{int}(A)$, is the union of all open subsets of X contained in A. (I hope I never use this.)

When a proof invokes 'duality' this means that we take complements as required to get from open sets to closed sets or vice versa.

Lemma 2.3. *Suppose X is a topological space and $A \subseteq X$.*

1. *Finite unions and arbitrary intersections of closed sets are closed.*
2. $\overline{A}$ *is closed and*

$$\overline{A} = \{x \in X : \forall \text{ open } U \ni x \; U \cap A \neq \emptyset\}.$$

Hence $\overline{A}$ *is the smallest closed set containing* A.

3. $\operatorname{int}(A)$ *is open and*

$$\operatorname{int}(A) = \{x \in X : \exists \text{ open } U \subseteq X \; x \in U \subseteq A\}.$$

Hence $\operatorname{int}(A)$ *is the largest open set contained in* A.

Definition 2.4. Suppose X, Y are topological spaces and $f : X \to Y$ a function.

f is *continuous* if and only if the preimage of every Y-open set under f is X-open.

f is a homeomorphism if and only if f is a continuous bijection with continuous inverse.

Lemma 2.5. *Suppose* X, Y *are topological spaces and* $f : X \to Y$ *a function. TFAE:*

1. f *is continuous.*
2. *the preimage of every* Y*-closed set under* f *is* X*-closed.*
3. *for every* $A \subseteq X$, $f\left(\overline{A}\right) \subseteq \overline{f(A)}$.

Definition 2.6. Suppose X is a topological space with topology τ and $A \subseteq X$.

The *subspace topology* on A is $\{U \cap A : U \in \tau\}$.

Lemma 2.7. *Suppose* X *is a topological space,* $Y \subseteq X$ *and* $A \subseteq Y$.

Then $\overline{A}^Y = Y \cap \overline{A}^X$.

Lemma 2.8. *Suppose* X, Y *are topological spaces,* $f : X \to Y$ *a function,* $A \subseteq X$ *and* $B \subseteq Y$ *(equipped with the subspace topology).*

If $f(X) \subseteq B$ *then* f *is continuous if and only if* $f' : X \to B; x \mapsto f(x)$ *is continuous.*

If f *is continuous then* $f|_A$ *is continuous.*

Lemma 2.9 (Pasting Lemma)**.** *Suppose* X, Y *are topological spaces,* A, B *closed* $\subseteq X$ *such that* $A \cup B = X$, $f : A \to Y, g : B \to Y$ *functions.*

If f, g *are continuous and agree on* $A \cap B$ *(i.e.* $f|_{A\cap B} = g|_{A\cap B}$*) then the function* $h = f \cup g : X \to Y$ *given by* $x \mapsto \begin{cases} f(x); & x \in A \\ g(x); & x \in B \end{cases}$ *is well-defined and continuous.*

Lemma 2.10 (Continuity is local)**.** *Suppose* X, Y *are topological spaces and* $f : X \to Y$ *is a function.*

If for every $x \in X$ *there is open* $U \ni x$ *such that* $f|_U : U \to Y$ *is continuous at* x *then* f *is continuous.*

3 Bases, Subbases

Definition 3.1. Suppose (X, τ) is a topological space. A collection $\mathcal{B}$ of subsets of X is a *basis* (of open sets) for τ if and only if

(B0) $\mathcal{B} \subseteq \tau$;

(B1) $\forall U \in \tau\ \exists \mathcal{B}' \subseteq \mathcal{B} :\ U = \bigcup \mathcal{B}'$.

We say that $\mathcal{B}$ generates τ or that τ is generated by (the basis) $\mathcal{B}$.

If a basis for a topological space has been fixed, we call its elements *basic open* sets.

Note that (B1) can be restated as $\forall U \in \tau\ \forall x \in U\ \exists B \in \mathcal{B}\ x \in B \subseteq U$.

Lemma 3.2. *Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X such that*

(B0') $X = \bigcup \mathcal{B}$;

(B1') $\forall A, B \in \mathcal{B}\ \exists \mathcal{B}' \subseteq \mathcal{B}\ A \cap B = \bigcup \mathcal{B}'$.

Then $\mathcal{B}$ is a basis for a unique topology τ on X, namely

$$\left\{\bigcup \mathcal{B}' \colon \mathcal{B}' \subseteq \mathcal{B}\right\}.$$

Proof. It is straightforward to check that the given τ is a topology (note that we use the convention that $\bigcup \emptyset = \emptyset$), that $\mathcal{B}$ is a basis for it and that if σ is a topology on X with basis $\mathcal{B}$ then $\tau = \sigma$. □

Note that (B1') is satisfied if $\mathcal{B}$ is closed under binary (finite) intersections. Also (B1') is of course equivalent to

$$\forall A, B \in \mathcal{B}\ \forall x \in A \cap B\ \exists C \in \mathcal{B}\ x \in C \subseteq A \cap B.$$

Definition 3.3. Suppose (X, τ) is a topological space. A collection $\mathcal{S}$ of subsets of X is a *subbasis* (of open sets) for τ if and only if

$$\left\{\bigcap \mathcal{F} \colon \mathcal{F} \text{ finite } \subseteq \mathcal{S}\right\}$$

is a basis for (X, τ).

We say that $\mathcal{S}$ generates τ or that τ is generated by (the subbasis) $\mathcal{S}$.

If a subbasis for a topological space has been fixed, we call its elements *subbasic open* sets.

Lemma 3.4. *Suppose X is a set and $\mathcal{S}$ a collection of subsets of X.*

Then

$$\mathcal{B}_{\mathcal{S}} = \left\{\bigcap \mathcal{F} \colon \mathcal{F} \textit{ finite } \subseteq \mathcal{S}\right\}$$

satisfies [B0'] and [B1'] and hence $\mathcal{S}$ is a subbasis for a topology τ on X.

Moreover, this topology is uniquely determined by $\mathcal{S}$.

Proof. Note that $\bigcap \emptyset = X$ (by convention in this course) so that $\mathcal{B}_{\mathcal{S}}$ satisfies [B0']. By construction $\mathcal{B}_{\mathcal{S}}$ is closed under binary intersections so that [B1'] holds as well.

Uniqueness follows from 3.2. □

Lemma 3.5. *Suppose X, Y are topological spaces, $\mathcal{S}$ is a subbasis for Y and $f\colon X \to Y$ is a function. Then f is continuous if and only if every subbasic open set (of Y) has open preimage under f.*

Proof. Preimages commute with unions and intersections. □

3.1 Examples

1. If (X, d) is a metric space then $\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ is a basis for X.
2. $\{(a, b)\colon a, b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$
3. $\{[a, b)\colon a, b \in \mathbb{R}\}$ is a basis for the Sorgenfrey line.
4. If X is a linearly ordered space, $\{(-\infty, a)\colon a \in X\} \cup \{(a, \infty)\colon a \in X\}$ is a subbasis for the order topology on X.
5. $\{U \times \mathbb{R}\colon U \text{ open } \subseteq \mathbb{R}\} \cup \{\mathbb{R} \times U\colon U \text{ open } \subseteq \mathbb{R}\}$ is a subbasis for $\mathbb{R}^2$.

4 The Lattice of Topologies

Definition 4.1. A partially ordered set is a pair $(P, \leq)$ where P is a set and $\leq$ is a binary relation on P that is reflexive ($\forall p \in P\ p \leq p$), transitive ($\forall p, q, r \in P\ p \leq q$ and $q \leq r \Rightarrow p \leq r$) and anti-symmetric ($\forall p, q \in P\ p \leq q$ and $q \leq p \Rightarrow p = q$).

A partially ordered set is totally ordered if any two elements are comparable, i.e. $\forall p, q \in P\ p \leq q$ or $q \leq p$.

If $A \subseteq P$ and $p \in P$ then $p = \inf A$ if and only if p is a lower bound for A ($\forall a \in A\ p \leq a$) and whenever q is a lower bound for A then $q \leq p$ ($\forall q \in P(\forall a \in A\ q \leq a) \Rightarrow q \leq p$).

Similarly $p = \sup A$ if and only if p is an upper bound for A and whenever q is an upper bound for A then $p \leq q$.

A partially ordered set is a lattice if it has finite infima and finite suprema. This includes $\inf \emptyset$ which is necessarily the greatest element of P and $\sup \emptyset$ which is necessarily the least element of P.

A lattice is complete if it has arbitrary infima and suprema.

Note that in a partially ordered set $p = \inf A$ is not the same as a maximal lower bound, i.e. a lower bound such that there isn't a strictly greater lower bound. This corresponds to the Approximation Property and is only equivalent in totally ordered sets. The same warning applies to $p = \sup A$.

Example 4.2. $\mathbb{R}$ with its usual order form a partially ordered set which is in fact totally ordered.

If X is any set then $(\mathcal{P}(X), \subseteq)$ is a complete lattice and usually (if $|X| \geq 2$) not totally ordered.

Any subset of a partially ordered set is partially ordered when restricting the order relation to it. Similarly any subset of a totally ordered set is totally ordered when restricting the order relation to it.

Lemma 4.3. *Suppose $(P, \leq)$ is a partially ordered set with greatest element, least element and arbitrary infima (respectively arbitrary suprema). Then P has arbitrary suprema (respectively infima) and thus is a complete lattice.*

Proof. Let $A \subseteq P$ and define $S = \{p \in P \colon \forall a \in A \; a \leq p\}$, the set of upper bounds of A. Note that S is non-empty since the greatest element of P is contained in S.

We claim $\alpha = \inf S$ is the supremum of A. It is enough to show α is an upper bound for A since then $\alpha \in S$ gives that it must be the least upper bound for A. So fix $a \in A$ and note that by construction a is a lower bound for S and hence $a \leq \inf S = \alpha$ (by definition of inf). □

Definition 4.4. Suppose X is a set.

Write $\mathcal{T}$ for the set of all topologies on X and define a relation $\leq$ on $\mathcal{T}$ by $\tau \leq \sigma \iff \tau \subseteq \sigma$, in which case we say that τ is *coarser/weaker/smaller* than σ and σ is *finer/stronger/greater* than τ.

Lemma 4.5. *If $\tau_i, i \in I$ are topologies on X then so is $\tau = \bigcap_{i \in I} \tau_i$ where we take $\bigcap \emptyset = \mathcal{P}(X)$. Moreover, τ is the greatest topology contained in each of the τ_i.*

Proof. It is straightforward to verify the conditions and check the last sentence. □

Lemma 4.6. *$(\mathcal{T}, \leq)$ is a complete lattice.*

Proof. That $\leq$ is a partial order (reflexive, transitive, anti-symmetric) is immediate from $\mathcal{T} \subseteq \mathcal{P}(\mathcal{P}(X))$. That infs exist follows from Lemma 4.5 (i.e. $\inf \mathcal{T}' = \bigcap \mathcal{T}'$). The indiscrete topology $\{\emptyset, X\}$ is the least element and the discrete topology $\mathcal{P}(X)$ is the greatest topology. □

Note that $\bigcup \tau_i$ is, in general, not a topology but only a subbasis for the topology $\sup \tau_i$. So even though $\mathcal{T}$ is a complete lattice and a subset of $\mathcal{P}(\mathcal{P}(X))$ with the order inherited, they do not have the same sup operation.

Lemma 4.7. *The topology generated by a subbasis is the smallest topology containing this subbasis.*

Proof. Let τ be the topology generated by a subbasis $\mathcal{S}$. Clearly $\mathcal{S} \subseteq \tau$ since $\bigcap \{A\} = A$ for each A.

If σ is a topology containing $\mathcal{S}$ then σ must contain all finite intersections of elements of $\mathcal{S}$, i.e. $\mathcal{B}_{\mathcal{S}}$. But every element of τ is the union of elements of $\mathcal{B}_{\mathcal{S}}$ and hence must be in σ. Thus $\tau \subseteq \sigma$ as required. □

5 Initial Topology and Products

Note: In MT 2026 I have only lectured on the specific case of products. Thus the general theory of initial topologies is not examinable as bookwork and only the specific case of products of topological space is examinable as bookwork. Besides the language, it is exactly the same.

Theorem 5.1. *Suppose X is a set, $(Y_i, \tau_i), i \in I$ is a family of topological spaces and $f_i \colon X \to Y_i, i \in I$ is a family of functions. There is a unique topology τ on X, called the* initial topology *on X with respect to the family $\{f_i \colon i \in I\}$, such that:*

(Test Condition) for every topological space Z and for every function $f \colon Z \to X$, f is continuous with respect to τ if and only if for every $i \in I$ $f_i \circ f$ is continuous.

This unique topology τ is generated by the subbasis $\left\{ {f_i}^{-1}(U) : i \in I, U \in \tau_i \right\}$. Moreover, τ is the coarsest topology on X such that all the f_i are continuous.

Proof. We first show uniqueness: suppose σ and τ are two topologies satisfying the Test Condition and consider the map $\mathrm{id}_{\tau,\tau} \colon (X, \tau) \to (X, \tau)$. Since $\mathrm{id}_{\tau,\tau}$ is continuous and τ satisfies the Test Condition, each $f_i = f_i \circ \mathrm{id}_{\tau,\tau}$ is continuous (from τ to τ_i). Now consider $\mathrm{id}_{\tau,\sigma} \colon (X, \tau) \to (X, \sigma)$. Since $f_i \circ \mathrm{id}_{\tau,\sigma} = f_i$ is continuous (from τ to τ_i) and σ satisfies the Test Condition, $\mathrm{id}_{\tau,\sigma}$ is continuous so that $\sigma \subseteq \tau$. By symmetry $\tau \subseteq \sigma$ and hence uniqueness follows.

Clearly each f_i is continuous from the given τ.

For existence, we only need to verify that the given τ satisfies the Test Condition: so suppose that Z is a topological space and $f \colon Z \to X$ is a function. If f is continuous (into (X, τ)), then, since each f_i is continuous, $f_i \circ f$ is a composition of continuous functions so continuous. Conversely, if each $f_i \circ f$ is continuous and S is subbasic open then $S = {f_i}^{-1}(U)$ for some open $U \subseteq Y_i$ so that $f^{-1}(S) = (f_i \circ f)^{-1}(U)$ is Z-open.

The final sentence follows from elementary set algebra. □

Definition 5.2. Suppose $X_i, i \in I$ are topological spaces. The *(Tychonoff) product topology* on $P = \prod_{i \in I} X_i = \left\{ f \colon I \to \bigcup_{i \in I} X_i \colon \forall i \in I \ f(i) \in X_i \right\}$ is the initial topology with respect to the natural projections $\pi_i \colon P \to X_i$ given by $\pi_i(f) = f(i)$.

Lemma 5.3. *Suppose $X_i, i \in I$ and Y are topological spaces.*

The product topology has basis consisting of all sets of the form $\prod_{i \in I} U_i$ where each U_i is an open subset of X_i and for all but finitely many $i \in I$ we have $U_i = X_i$. When no explicit basis for the product topology is given, we assume that this basis has been selected.

A function $f \colon Y \to \prod_i X_i$ is continuous if and only if all of the compositions $\pi_i \circ f \colon Y \to X_i$ are continuous and the product topology is the unique topology on $\prod_i X_i$ satisfying this result.

Proof. Straightforward verification: we note that

$$\prod_{i\in I} U_i = \bigcap_{i\in I} {\pi_i}^{-1}(U_i) = \bigcap_{i\in I: U_i \neq X_i} {\pi_i}^{-1}(U_i)$$

so that the given collection of sets is indeed a basis for the product topology.

The sentence about functions follows immediately from the fact that the product topology is the initial topology with respect to the π_i. □

Definition 5.4. Suppose X and $Y_i, i \in I$ are sets and $f_i \colon X \to Y_i, i \in I$ are functions. The *diagonal map* of the f_i is the map

$$\Delta_{i\in I} f_i \colon X \to \prod Y_i; x \mapsto (f_i(x))_{i\in I}.$$

Lemma 5.5. *If all the f_i are continuous then so is the diagonal map.*

Proof. By the Test Condition for initial topologies (recall that the product topology is the initial topology with respect to the π_i) it is enough to observe that for each $i \in I$

$$\pi_i \circ \Delta = f_i.$$

□

6 Low Separation Properties

Definition 6.1. Suppose X is a topological space.

X is T_0 if and only if for any distinct $x, y \in X$ there is open U containing exactly one of x and y.

X is T_1 if and only if for any ordered pair $(x, y) \in X^2 \setminus \{(t, t) \colon t \in X\}$ of distinct points there is open $U \subseteq X$ such that $x \in U \not\ni y$ (which is equivalent to $x \in U \subseteq X \setminus \{y\}$).

X is T_2 (*Hausdorff*) if and only if for any distinct $x, y \in X$ there are disjoint open $U \ni x$, $V \ni y$ if and only if for any distinct $x, y \in X$ there is open U such that

$$x \in U \subseteq \overline{U} \subseteq X \setminus \{y\}.$$

X is T_3 (*regular*) if and only if X is T_1 and for any $x \notin C$ closed $\subseteq X$ there are disjoint open $U \ni x, V \supseteq C$ if and only if X is T_1 and for any $x \notin C$ closed $\subseteq X$ there is open U such that

$$x \in U \subseteq \overline{U} \subseteq X \setminus C.$$

X is $T_{3.5}$ (*Tychonoff*) if and only if X is T_1 and for any $x \notin C$ closed $\subseteq X$ there is a continuous $f \colon X \to [0, 1]$ such that $f(x) = 0$, $f(C) \subseteq \{1\}$ if and only if X is T_1 and for any $x \in U$ open $\subseteq X$ there is a continuous $f \colon X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus U) \subseteq \{1\}$.

Lemma 6.2. *Suppose X is a topological space and $\mathcal{B}$ a basis for X.*

Then X is regular (resp. Tychonoff) if and only if it is T_1 and the condition holds for basic open sets.

Also in the Tychonoff condition, we can swap the roles of 0 *and* 1.

Proof. The forward directions are clear.

For the backwards directions: if $x \in U$ open $\subseteq X$ find basic open B with $x \in B \subseteq U$, apply the condition to the pair x, B and observe that the open set (resp. continuous function) also works for the pair x, U.

For the last sentence, note that $x \mapsto 1 - x$ is a homeomorphism of $[0, 1]$ to itself swapping 0 and 1. □

Lemma 6.3. *X is T_1 if and only if every singleton $\{x\}$ of X is closed.*

Thus $T_{3.5} \implies T_3 \implies T_2 \implies T_1 \implies T_0$.

Proof. Suppose X is T_1 and $x \in X$. For each $y \in X \setminus \{x\}$ find $U_y \ni y$ such that $x \in X \setminus U_y$. Then

$$\{x\} = \bigcap_{y \neq x} X \setminus U_y$$

is an intersection of closed sets, so closed.

For the converse, let $x, y \in X$ be distinct. By assumption $U = X \setminus \{y\}$ is open and clearly $x \in U \not\ni y$.

For the (non-trivial) implications: If X is $T_{3.5}$, $x \in X$ and C closed $\subseteq X$ with $x \notin C$, let $f \colon X \to [0, 1]$ be a continuous function with $f(x) = 0$ and $f(C) \subseteq \{1\}$. Then $U = f^{-1}([0, 1/3))$ and $V = f^{-1}((2/3, 1])$ are the required open sets.

If X is T_3 then it is T_1 so singletons are closed. If $x, y \in X$ are distinct then apply regularity to $x \notin \{y\}$ closed $\subseteq X$. □

Lemma 6.4. *For $i \leq 3.5$ all the T_i are preserved by subspaces and products.*

Proof. Suppose X and $X_k, k \in K$ are topological spaces satisfying T_i (for some $i \leq 3.5$) and $A \subseteq X$. We write $P = \prod_k X_k$ for the Tychonoff product.

T_0, subspace: If $x, y \in A$ are distinct, let U be X-open containing exactly one of x and y. Then $A \cap U$ is as required.

T_1, subspace: As for T_0.

T_2, subspace: As for T_0 (intersect both the open sets).

T_3, subspace: Suppose $x \in A$ and $C \subseteq A$ is A-closed and $x \notin C$. Then $x \notin \overline{C}^X = D$ since $C = \overline{C}^A = A \cap \overline{C}^X$ and $x \in A$. Thus we can apply regularity to $x \notin D$ closed $\subseteq X$ and intersect the open sets we obtain with A.

$T_{3.5}$**, subspace:** As for T_3, except we of course restrict the continuous map to A.

T_0**, products:** If $x, y \in P$ are distinct, then there is some $k \in K$ such that $x_k = \pi_k(x), y_k = \pi_k(y)$ are distinct. Thus in X_k we can find an open U containing exactly one of x_k, y_k. Then ${\pi_k}^{-1}(U)$ is as required.

T_1**, products:** As for T_0.

T_2**, products:** Similar to T_0 but take disjoint X_k-open $U \ni x_k, V \ni y_k$ and consider ${\pi_k}^{-1}(U), {\pi_k}^{-1}(V)$.

T_3**, products:** It is enough to check the condition for standard basic open sets. So let $x \in P$ and $U = \bigcap_{k \in F} {\pi_k}^{-1}(U_k)$ be basic open (i.e. each U_k is an open subset of X_k and F is a finite subset of K). If $x \in U$ then for $k \in F$, $x_k = \pi_k(x) \in U_k$ so that there is X_k-open V_k and X_k-closed $D_k = \overline{V_k}$ with $x_k \in V_k \subseteq D_k \subseteq U_k$. Then

$$x \in V = \bigcap_{k \in F} {\pi_k}^{-1}(V_k) \subseteq D = \bigcap_{k \in F} {\pi_k}^{-1}(D_k) \subseteq U$$

and V is open and D closed so that $\overline{V} \subseteq D$ as well.

$T_{3.5}$**, products:** It is enough to check the condition for standard basic open sets. So let $x \in P$ and $U = \bigcap_{k \in F} {\pi_k}^{-1}(U_k)$ be basic open (i.e. each U_k is an open subset of X_k and F is a finite subset of K). If $x \in U$ then for $k \in F$, $x_k = \pi_k(x) \in U_k$ so that there is a continuous function $f_k \colon X_k \to [0,1]$ such that $f_k(x_k) = 1$ and $f_k(X_k \setminus U_k) \subseteq \{0\}$ (note that 0 and 1 have interchanged roles).

Now consider the functions

$$\begin{aligned} \pi_F \colon P &\to \prod_{k \in F} X_k \\ g &\mapsto g|_F, \end{aligned}$$

$$\begin{aligned} \prod_{k \in F} f_k \colon \prod_{k \in F} X_k &\to [0,1]^F \\ (x_k)_{k \in F} &\mapsto (f_k(x_k))_{k \in F} \end{aligned}$$

and

$$\begin{aligned} m \colon [0,1]^F &\to [0,1] \\ (r_k) &\mapsto \prod_k r_k. \end{aligned}$$

Each of them is continuous (the first two by the Test Condition, the last by elementary Analysis) and hence so is their composition

$$h = m \circ \prod_k f_k \circ \pi_F \colon P \to [0,1].$$

It is straightforward to check that $h(x) = 1$ and $h(P \setminus U) \subseteq \{0\}$. □

Theorem 6.5 (The Embedding Lemma)**.** *Suppose X is a topological space, $Y_i, i \in I$ are topological spaces and $f_i \colon X \to Y_i, i \in I$ are continuous such that*

- *the f_i separate points, i.e. for distinct $x, y \in X$ there is $i \in I$ with $f_i(x) \neq f_i(y)$;*
- *$\left\{ {f_i}^{-1}(U) : i \in I, U \text{ open } \subseteq Y_i \right\}$ is a basis for X.*

Then the diagonal $\Delta = \Delta_{i \in I} f_i$ is a homeomorphic embedding of X into $\prod_{i \in I} Y_i$, i.e. a continuous map which is a homeomorphism onto its image.

Proof. The diagonal is continuous and as the f_i separate points, it is injective. It remains to check that it is open onto $\Delta(X)$: so let $U \subseteq X$ be basic open, i.e. of the form ${f_i}^{-1}(V)$ for some $i \in I$, V open $\subseteq Y_i$. Then by construction $\Delta(U) \subseteq {\pi_i}^{-1}(V) \cap \Delta(X)$. On the other hand, if $y \in {\pi_i}^{-1}(V) \cap \Delta(X)$ then we can find $x \in X$ with $\Delta(x) = y$ and $f_i(x) = \pi_i(\Delta(x)) = \pi_i(y) \in V$ so that $x \in {f_i}^{-1}(V)$ giving $\supseteq$. Thus basic open sets are mapped to $\Delta(X)$-open sets. Finally, images and unions commute so the result follows. □

Corollary 6.6. *If X is T_1 then the two conditions in the previous theorem follow from*

- *the f_i separate points from closed sets, i.e. for $x \notin C$ closed $\subseteq X$ there is $i \in I$ and disjoint open $U, V \subseteq Y_i$ with $f_i(x) \in U$, $f_i(C) \subseteq V$.*

Definition 6.7. The *Sierpinski space* $\mathbb{S}$ is $(\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$.

Theorem 6.8. *A topological space is T_0 if and only if it is homeomorphic to a subspace of some $\mathbb{S}^\kappa$.*

A topological space is $T_{3.5}$ if and only if it is homeomorphic to a subspace of some $[0,1]^\kappa$.

Proof. For the first result, apply the Embedding Lemma to the family of indicator functions χ_U, U open $\subseteq X$.

For the second result, apply the Embedding Lemma to the family of all $[0,1]$-valued continuous functions. □

Lemma 6.9. *A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \colon x \in X\}$ is closed in $X \times X$.*

Proof. Suppose X is Hausdorff. If $(x,y) \in X^2 \backslash \Delta$ then $x \neq y$ so by Hausdorffness there are disjoint X-open $U \ni x, V \ni y$. As U, V are disjoint $U \times V \cap \Delta = \emptyset$ so that $(x,y) \in U \times V \subseteq X^2 \setminus \Delta$. As (x,y) was arbitrary this shows that Δ is closed.

Suppose now that Δ is closed and $x, y \in X$ are distinct. Then $(x,y) \notin \Delta$ so there is a basic open $W = U \times V$ (in X^2) with $(x,y) \in W \subseteq X^2 \setminus \Delta$ and U, V open in X. Then $U \cap V = \emptyset$ so that $U \ni x, V \ni y$ are as required. □

Lemma 6.10. *Suppose $f, g \colon X \to Y$ are continuous functions and Y a Hausdorff space. Then $Eq(f,g) = \{x \in X \colon f(x) = g(x)\}$ is closed in X.*

Proof.

$$\{x \in X \colon f(x) = g(x)\} = (f \Delta g)^{-1} (\Delta) .$$

□

7 Normality

Definition 7.1. Suppose X is a topological space and $A, B \subseteq X$.

The sets A and B are *separated* (in X) if and only if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. They are separated by open sets if and only if there are disjoint open sets $U \supseteq A$, $V \supseteq B$.

Definition 7.2. Suppose X is a topological space.

X is T_4 (*normal*) if and only if X is T_1 and disjoint closed sets are separated by open sets, i.e. for every pair C, D of disjoint closed subsets of X there are disjoint open subsets U, V of X such that $C \subseteq U$ and $D \subseteq V$.

Equivalently (by duality) X is normal if and only if X is T_1 and for every closed $C \subseteq U$ open $\subseteq X$ there is open $V \subseteq X$ with $C \subseteq V \subseteq \overline{V} \subseteq U$.

Definition 7.3. Suppose X is a topological space.

X is *functionally normal* if and only if X is T_1 and for every pair C, D of disjoint closed subsets of X there is a continuous function $f \colon X \to [0,1]$ such that $f(C) \subseteq \{0\}$ and $f(D) \subseteq \{1\}$. We call f a Urysohn function for the pair C, D.

Equivalently (by duality) X is functionally normal if and only if X is T_1 and for every closed $C \subseteq U$ open $\subseteq X$ there is a continuous function $f \colon X \to [0,1]$ with $f(C) \subseteq \{0\}$ and $f(X \setminus U) \subseteq \{1\}$.

Lemma 7.4. *Suppose X is a functionally normal topological space. Then X is normal.*

Proof. For disjoint closed C, D with Urysohn function f consider $U = f^{-1}([0, 1/2))$, $V = f^{-1}((1/2, 1])$. □

Theorem 7.5 (Urysohn's Lemma)**.** *Suppose X is a topological space.*

If X is normal then X is functionally normal.

Proof. Suppose we have a closed $C \subseteq U$ open $\subseteq X$.

Constructing the Onion Slices Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of $(0,1)$ (e.g. $D = \mathbb{Q} \cap (0,1)$ or D as the dyadic rationals) and write $D_{-1} = \{0,1\}$ and $D_n = \{d_0, \ldots, d_n\} \cup \{0,1\} = \{d_n\} \cup D_{n-1}$ for $n \in \omega$. Let $\overline{U_0} = C$ and $U_1 = U$. (Note U_0 is not defined but is not referred to in the remainder of the proof. We will only ever make use of $\overline{U_0}$.)

By (strong) induction on n we will construct open sets $U_{d_n}, n \in \omega$ such that

Induction Hypothesis for $r, s \in D_n$ with $r < s$ we have $\overline{U_r} \subseteq U_s$

(note that we never talk about U_0 or $\overline{U_1}$ here!).

Suppose we have defined U_{d_k} for $k < n$.

Let

$$L = \{d \in D_{n-1} : d < d_n\} \quad G = \{d \in D_{n-1} : d_n < d\}$$

and note that these are both finite non-empty sets. Let $l = \max L$ and $g = \min G$ and apply normality to $\overline{U_l} \subseteq U_g$ (induction hypothesis) to obtain open U_{d_n} with $\overline{U_l} \subseteq U_{d_n} \subseteq \overline{U_{d_n}} \subseteq U_g$. By choice of l and g (and transitivity of $\subseteq$) the inductive hypothesis has been preserved.

Defining the Urysohn function We define $f, g : X \to [0,1]$ by

$$f(x) = \inf \{d \in D : x \in U_d\}$$
$$g(x) = \inf \left\{d \in D : x \in \overline{U_d}\right\}$$

(treating $\inf \emptyset = 1$).

Next we claim that $f = g$: clearly $g(x) \leq f(x)$ (as if $x \in U_d$ then $x \in \overline{U_d}$). Next, if $g(x) < r \in [0,1]$ then there are $e, d \in D$ with $g(x) < e < d < r$ such that $x \in \overline{U_e} \subseteq U_d$, giving $f(x) \leq d < r$. Hence $f(x) \leq g(x)$, as required.

Continuity For $r \in (0,1)$ we see: if $f(x) \in [0,r)$ then there is $d \in D$ with $f(x) < d < r$ so that $x \in U_d$ and $f(U_d) \subseteq [0,r)$; if on the other hand $f(x) = g(x) \in (r,1]$ then there is $d \in D$ with $r < d < g(x)$ so that $x \notin \overline{U_d}$ and $g\left(X \setminus \overline{U_d}\right) \subseteq (r,1]$. Hence f is continuous.

Finally observe that if $x \in C = \overline{U_0}$ then $x \in U_d$ for all $d \in D$ so that $f(x) = 0$ and if $x \notin U = U_1$ then $x \notin \overline{U_d}$ for all $d \in D$ so that $g(x) = 1$. □

The following example shows that subspaces of normal spaces need not be normal. Note that you can replace $\aleph_1$ with any uncountable set.

Example 7.6 (Simplified Tychonoff Plank)**.** Let $T = \aleph_1 \cup \{r\}$ with topology $\mathcal{P}(\aleph_1) \cup \{T \setminus F : F \text{ finite } \subseteq T\}$ and let $R = \aleph_0 \cup \{t\}$ with topology $\mathcal{P}(\aleph_0) \cup \{R \setminus F : F \text{ finite } \subseteq R\}$.

Note that T and R are compact Hausdorff so that $T \times R$ is compact Hausdorff and hence normal.

Let $X = T \times R \setminus \{(r,t)\}$.

X is not normal, specifically the disjoint X-closed subsets $C = \aleph_1 \times \{t\}$ and $D = \{r\} \times \aleph_0$ cannot be separated by open sets. For suppose $U \supseteq D$

is open. For each $n \in \aleph_0$ we can then choose some finite $F_n \subseteq \aleph_1$ such that $(r, n) \in (T \setminus F_n) \times \{n\} \subseteq U$. Letting $\alpha \in \aleph_1 \setminus \bigcup_{n \in \aleph_0} F_n$ (the latter is non-empty as $\aleph_1$ is uncountable but $\bigcup_n F_n$ is a countable union of finite sets so countable), we see that $\{\alpha\} \times \aleph_0 \subseteq U$ and hence $(\alpha, t) \in \overline{U} \cap C$.

8 Strong Separation Properties

Note: in MT 2026/2027 these are just for note and not examinable as bookwork. I assume you have seen the techniques involved, but you do not have to learn the details of the definitions, their relationships or the proofs for bookwork purposes.

Definition 8.1. Suppose X is a topological space.

X is T_5 (*hereditarily normal*) if and only if X is T_1 and every subspace of X is normal.

X is T_6 (*perfectly normal*) if and only if X is T_1 and every closed set is functionally closed, i.e. for every closed $C \subseteq X$ there is a continuous $f \colon X \to [0, 1]$ such that $C = f^{-1}(0)$.

Theorem 8.2. *X is hereditarily normal if and only if X is T_1 and if separated sets can be separated by open sets, i.e. if $A, B \subseteq X$ such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ then there are disjoint open sets $U \supseteq A$, $V \supseteq B$.*

Proof. Problem Sheet 2. □

Theorem 8.3. *X is perfectly normal if and only if X is normal and every closed set is a G_δ, i.e. for every closed set C there are open sets $U_n, n \in \mathbb{N}$ with $C = \bigcap_n U_n$.*

Proof. Assume X is perfectly normal.

It is T_1 and if C, D are closed find $f_C, f_D \colon X \to [0, 1]$ such that $C = {f_C}^{-1}(0)$, $D = {f_D}^{-1}(0)$. Since C and D are disjoint the function $f = \frac{f_C}{f_C + f_D} \colon X \to [0, 1]$ is well-defined, continuous and $C \subseteq f^{-1}(0)$, $D \subseteq f^{-1}(1)$ so X is functionally normal, so normal. Also

$$C = {f_C}^{-1}(0) = {f_C}^{-1}\left(\bigcap_n [0, 2^{-n})\right) = \bigcap_n {f_C}^{-1}\left([0, 2^{-n})\right)$$

and by continuity f_C, the RHS is a countable intersection of open sets

Conversely, assume X is normal and every closed set is a G_δ. Clearly X is T_1. If C is closed, then find $U_n, n \in \mathbb{N}$ open such that $C = \bigcap_n U_n$ and Urysohn functions $f_n \colon X \to [0, 1]$ such that $f_n(C) \subseteq \{0\}$ and $f_n(X \setminus U_n) \subseteq \{1\}$. Then by the M-test $f = \sum_n \frac{1}{2^{n+1}} f_n$ is a continuous $[0, 1]$-valued function with $C = f^{-1}(0)$. □

Lemma 8.4. *Subspaces of perfectly normal spaces are perfectly normal.*

Proof. IF $X \subseteq Y$ and Y is perfectly normal, then X is T_1 as the subspace of a T_1 space. Also, for X-closed C find Y-closed $\hat{C}$ with $C = \hat{C} \cap X$ and continuous $f\colon Y \to [0,1]$ such that $\hat{C} = f^{-1}(0)$ and observe that $f|_X$ is continuous and $C = {f|_X}^{-1}(0)$ as required. □

Corollary 8.5. *Perfectly normal spaces are hereditarily normal.*
Hereditarily normal spaces are normal.

9 Urysohn's Metrization Theorem

Definition 9.1. A topological space X is metrizable if and only if there is a metric on X that induces the topology of the space.

Theorem 9.2. *A compact Hausdorff space is metrizable if and only if it is second countable.*

Proof. For the forward direction note that compact implies Lindelöf and Lindelöf metric spaces are second countable (Sheet 1).

We now prove the reverse implication. By Sheet 0, compact Hausdorff spaces are normal, hence functionally normal and Tychonoff.

Thus

$$\mathcal{B} = \left\{ f^{-1}(U) : f\colon X \to [0,1] \text{ continuous }, U \text{ open } \subseteq [0,1] \right\}$$

is a basis for X.

By Sheet 2, Q1 and the fact that X has a countable basis, there is a countable $\mathcal{B}' \subseteq \mathcal{B}$ which is still a basis for X. Hence there are countably many continuous functions $f_i\colon X \to [0,1], i \in \mathbb{N}$ such that

$$\mathcal{B}' \subseteq \left\{ {f_i}^{-1}(U) : i \in \mathbb{N}, U \text{ open } \subseteq [0,1] \right\} =: \mathcal{B}''.$$

Observe that $\mathcal{B}''$ is thus still a basis for X and X is T_1 so that $f_i, i \in I$ satisfies the conditions for the Embedding Lemma.

Thus $\Delta_{i \in \mathbb{N}} f_i\colon X \to [0,1]^{\mathbb{N}}$ is a homeomorphic embedding. But $[0,1]^{\mathbb{N}}$ is metrizable (Sheet 1) so that X is homeomorphic to a subspace of a metrizable space and hence metrizable. □

Note that we can replace 'compact Hausdorff' with 'Lindelöf regular' or 'separable normal' in the above theorem.

Alternative proof. Instead of appealing to Sheet 2, Q1, fix a countable basis $\mathcal{B}$ of X. Note that there are only countably many pairs $(U,V) \in \mathcal{B}^2$. For each such pair which satisfies $\overline{U} \subseteq V$, use Urysohn's Lemma (i.e. functional normality) to find a continuous $f = f_{U,V}\colon X \to [0,1]$ such that $f\left(\overline{U}\right) \subseteq \{0\}$ and $f(X \setminus V) \subseteq \{1\}$. Note that then $\overline{U} \subseteq f^{-1}([0,1/2)) \subseteq V$.

We then claim that

$$\left\{ {f_{U,V}}^{-1}([0,1/2)) : (U,V) \in \mathcal{B}^2, \overline{U} \subseteq V \right\}$$

is a basis for X and hence the family $f_{U,V}, (U,V) \in \mathcal{B}^2$ with $\overline{U} \subseteq V$ satisfies the conditions of the Embedding Lemma.

To prove the claim, let $x \in W$ open $\subseteq X$. Find $V \in \mathcal{B}$ with $x \in V \subseteq W$. Use regularity to find open W' such that $x \in W' \subseteq \overline{W'} \subseteq V$ and then find $U \in \mathcal{B}$ such that $x \in U \subseteq W'$. We thus have $(U,V) \in \mathcal{B}^2$ such that $x \in U \subseteq \overline{U} \subseteq {f_{U,V}}^{-1}([0,1/2)) \subseteq V \subseteq W$ as required.

□

10 Paracompactness

10.1 General Theory of Paracompactness

Definition 10.1. Suppose X is a set and $\mathcal{A}$ is a collection of subsets of X.

A collection $\mathcal{A}'$ is a *refinement* of $\mathcal{A}$ if and only if $\forall A' \in \mathcal{A}' \exists A \in \mathcal{A}\ A' \subseteq A$.

Definition 10.2. Suppose X is a topological space.

A collection $\mathcal{A}$ of subsets of X is *locally finite* if and only if

$$\forall x \in X \exists \text{ open } U \ni x\ |\{A \in \mathcal{A} \colon A \cap U \neq \emptyset\}| < \aleph_0.$$

A collection $\mathcal{A}$ of subsets of X is *discrete* if and only if

$$\forall x \in X \exists \text{ open } U \ni x\ |\{A \in \mathcal{A} \colon A \cap U \neq \emptyset\}| \leq 1.$$

A collection $\mathcal{A}$ of subsets of X is *closure preserving* if and only if for every $\mathcal{A}' \subseteq \mathcal{A}$, $\overline{\bigcup_{A \in \mathcal{A}'} A} = \bigcup_{A \in \mathcal{A}'} \overline{A}$.

Lemma 10.3. *Suppose X is a topological space and $\mathcal{A}$ a collection of subsets of X.*

If $\mathcal{A}$ is discrete then it is locally finite.

If $\mathcal{A}$ is locally finite then it is closure preserving.

Proof. The first statement is trivial ($1 < \aleph_0$).

For the second, suppose $\mathcal{A}' \subseteq \mathcal{A}$. As $\overline{\bigcup \mathcal{A}'}$ is a closed set containing each $A \in \mathcal{A}'$ it contains each $\overline{A}$ for $A \in \mathcal{A}'$ yielding $\overline{\bigcup \mathcal{A}'} \supseteq \bigcup_{A \in \mathcal{A}'} \overline{A}$.

Thus all we need to show that $\bigcup_{A \in \mathcal{A}'} \overline{A}$ is closed in X. So let $x \in X \backslash \bigcup_{A \in \mathcal{A}'} \overline{A}$ and find open $U \ni x$ such that $\mathcal{A}_U = \{A \in \mathcal{A}' \colon U \cap A \neq \emptyset\}$ is finite. Note that since U is open, $U \cap A \neq \emptyset \iff U \cap \overline{A} \neq \emptyset$ so that $V = U \setminus \bigcup_{A \in \mathcal{A}_U} \overline{A}$ is an open set containing x which is contained in $X \setminus \bigcup_{A \in \mathcal{A}'} \overline{A}$. □

Definition 10.4. Suppose X is a topological space.

X is paracompact if and only if every open cover of X has a locally finite open refinement covering X.

Lemma 10.5. *A compact space is paracompact.*

Proof. Subcovers are refinements and finite families are locally finite. □

Lemma 10.6. *Suppose X is a paracompact topological space and C a closed subset of X.*

Then every X-open cover of C has an X-locally finite X-open refinement covering C.

Proof. Suppose $\mathcal{U}$ is an X-open cover of C. Then $\mathcal{U} \cup \{X \setminus C\}$ is an X-open cover of X, so has a locally finite open refinement $\mathcal{V}$ covering X. We claim that $\mathcal{W} = \{V \in \mathcal{V} \colon V \cap C \neq \emptyset\}$ is the required X-locally finite X-open refinement of $\mathcal{U}$ covering C.

The only non-trivial claim is that $\mathcal{W}$ is a refinement of $\mathcal{U}$. For this, note that if $V \cap C \neq \emptyset$ then $V \not\subseteq X \setminus C$ so there is $U \in \mathcal{U}$ with $V \subseteq U$. □

Lemma 10.7. *A closed subspace of a paracompact space is paracompact.*

Proof. Suppose C is a closed subspace of a paracompact space X. Let $\mathcal{U}$ be a C-open cover of C and for each $U \in \mathcal{U}$ choose X-open $V_U \subseteq X$ such that $V_U \cap C = U$. Then $\{V_U \colon U \in \mathcal{U}\}$ is an X-open cover of C and by the previous lemma has a X-locally finite open refinement $\mathcal{V}$ covering C. Let $\mathcal{W} = \{V \cap C \colon V \in \mathcal{V}\}$ and check that this is the required C-locally finite C-open refinement of $\mathcal{U}$ covering C. □

Theorem 10.8. *A paracompact regular space is normal.*

Proof. Suppose X is paracompact regular and that $C \subseteq W$ where C is a closed and W an open subset of X. For $c \in C$ use regularity to find open U_c such that $c \in U_c \subseteq \overline{U_c} \subseteq W$. Then $\{U_c \colon c \in C\}$ is an X-open cover of C and thus has a locally finite open refinement $\mathcal{V}$ covering C. We then note that $\mathcal{V}$ is closure preserving and since $\mathcal{V}$ refines $\{U_c \colon c \in C\}$ we have

$$C \subseteq \bigcup \mathcal{V} \subseteq \overline{\bigcup \mathcal{V}} = \bigcup_{V \in \mathcal{V}} \overline{V} \subseteq \bigcup_{c \in C} \overline{U_c} \subseteq W$$

as required. □

This last result together with example 7.6 shows that an open subspace of a paracompact space need not be paracompact.

Definition 10.9. Suppose X is a topological space and $\mathcal{P}$ is a property of families of subsets of X (e.g. 'locally finite').

A family $\mathcal{A}$ of subsets of X is σ-$\mathcal{P}$ if and only if there are families $\mathcal{A}_n, n \in \mathbb{N}$ of subsets of X such that each $\mathcal{A}_n$ is $\mathcal{P}$ and $\mathcal{A} = \bigcup_n \mathcal{A}_n$.

For example a family $\mathcal{A}$ is σ-locally finite if and only if it can be written as a countable union of locally finite families. It is σ-discrete if and only if it can be written as a countable union of discrete families.

Lemma 10.10. *Suppose X is a regular space.*

TFAE:

1. *X is paracompact.*

2. *every open cover of X has a σ-locally finite open refinement covering X.*

3. *every open cover of X has a locally finite refinement covering X.*

4. *every open cover of X has a locally finite closed refinement covering X.*

Proof. Problem Sheet 3. □

10.2 Paracompactness and Metrizability

Theorem 10.11 (Stone's Theorem)**.** *Suppose X is a metric space.*

Then X is paracompact and in fact every open cover of X has a σ-discrete open refinement covering X.

Proof. Suppose $\mathcal{U}$ is an open cover of X. Well order $\mathcal{U}$ by $\leq$ (by the well-ordering principle).

For each $U \in \mathcal{U}$ and $n \in \omega$ we define

$$S_U = \{x \in U : \forall V < U \; x \notin V\},$$

$$S_{U,n} = \left\{x \in S_U : B_{3/2^n}(x) \subseteq U\right\}$$

and

$$W_{U,n} = \bigcup_{x \in S_{U,n}} B_{1/2^n}(x).$$

We claim that for each $n \in \omega$, $\mathcal{W}_n = \{W_{U,n} : U \in \mathcal{U}\}$ is a discrete family of open subsets of X refining $\mathcal{U}$. The only non-trivial claim is discreteness. So assume $y_1 \in W_{U_1,n}, y_2 \in W_{U_2,n}$ with $U_1 \neq U_2$ and without loss of generality $U_1 \leq U_2$. For each $i = 1, 2$ pick $x_i \in S_{U_i,n}$ such that $y_i \in B_{1/2^n}(x_i)$. Since $x_2 \in S_{U_2,n} \subseteq S_{U_2}$ we have $x_2 \notin U_1$ and hence $d(x_1, x_2) \geq 3/2^n$. But then by the triangle law

$$d(y_1, y_2) \geq d(x_1, x_2) - d(y_1, x_1) - d(y_2, x_2) \geq 1/2^n.$$

Hence the $B_{1/2^{n+1}}(x), x \in X$ witness discreteness of $\mathcal{W}_n$.

Next observe that $\bigcup_n \mathcal{W}_n$ is a cover of X: if $x \in U \in \mathcal{U}$ for U $\leq$-minimal so that $x \in S_U$ then for sufficiently large n we have $B_{3/2^n}(x) \subseteq U$ and so $x \in W_{U,n}$.

Hence $\bigcup_n \mathcal{W}_n$ is a σ-discrete open refinement of X covering X as required.

As σ-discrete certainly implies σ-locally finite, the previous lemma shows that X must be paracompact.

□

It is possible to define $W'_{U,n}$ such that they form a locally finite family: we define by recursion on n

$$W'_{U,n} = \bigcup \left\{B_{1/2^n}(x) : x \in S_{U,n} \setminus \bigcup \left\{W'_{U_1,k} : k < n, U_1 \in \mathcal{U}\right\}\right\}.$$

We can then manually check that this is locally finite - see Engelking Theorem 4.4.1 for details.

Lemma 10.12. *Every metric space has a σ-discrete basis.*

Proof. For every $n \in \mathbb{N}$, consider the open cover $\mathcal{U}_n = \{B_{2^{-n}}(x) \colon x \in X\}$ and use the previous theorem to obtain a σ-discrete open refinement $\mathcal{W}_n$ of $\mathcal{U}_n$.

Then observe that $\bigcup_n \mathcal{W}_n$ is still a basis for X (easy) and is σ-discrete. □

Lemma 10.13. *Suppose X is a normal space with a σ-discrete basis $\mathcal{B} = \bigcup_n \mathcal{B}_n$ (where each $\mathcal{B}_n$ is discrete).*

For every open $U \subseteq X$ there is a continuous $f \colon X \to [0,1]$ such that $U = f^{-1}((0,1])$.

Proof. Let U be an open subset of X and for each $n \in \mathbb{N}$ let

$$U_n = \bigcup \left\{ B \in \mathcal{B}_n \colon \overline{B} \subseteq U \right\}.$$

Since $\mathcal{B}$ is a basis and X is regular, $U = \bigcup_n U_n$. But as each $\mathcal{B}_n$ is discrete and hence locally finite, $\overline{U_n} = \bigcup \left\{ \overline{B} \colon \overline{B} \subseteq U \right\} \subseteq U$ so that $\bigcup_n \overline{U_n} = U$. Thus every open set is a countable union of closed sets (and, by duality, every closed set is a countable intersection of open sets).

Now apply Urysohn's Lemma for each $n \in \mathbb{N}$ to choose continuous $f_n \colon X \to [0, 2^{-n}]$ such that $f_n\left(\overline{U_n}\right) \subseteq \{2^{-n}\}$ and $f_n(X \setminus U) \subseteq \{0\}$. By the M-test $f = \sum_n f_n$ is continuous and $f(X \setminus U) \subseteq \{0\}$ and $f(U) \subseteq (0,1]$ as claimed. □

Definition 10.14. Define $H_\emptyset = \{0\}$ with its unique topology.

For I a non-empty set, the hedgehog of spininess I, H_I is the set quotient of $[0,1] \times I$ obtained by identifying all the points $(0,i), i \in I$ with the metric given by

$$d((x,i),(y,j)) = \begin{cases} y & x = 0 \\ x & y = 0 \\ |x-y| & i = j \\ x+y & i \neq j \end{cases}.$$

where we write 0 for the equivalence class $\{(0,i) \colon i \in I\}$. Hence a countable product of hedgehogs is metrizable.

Proof. Straightforward verification that this is a metric. □

Theorem 10.15. *Every normal space with a σ-discrete basis is a subspace of a product of countably many hedgehogs and thus metrizable.*

Proof. Suppose X is a normal space with a σ-discrete basis $\mathcal{B} = \bigcup_n \mathcal{B}_n$ where each $\mathcal{B}_n$ is a non-empty discrete family of open sets and write $\kappa_n = \mathcal{B}_n$.

By the lemma above, for each $B \in \mathcal{B}_n$ we can find a continuous $f_B \colon X \to [0,1]$ such that $B = {f_B}^{-1}((0,1])$.

Define the functions $f_n \colon X \to H_{\kappa_n}$ by

$$f_n(x) = \begin{cases} (f_B(x), B) & x \in B \in \mathcal{B}_n \\ 0 & \text{otherwise} \end{cases}.$$

First note that by discreteness of $\mathcal{B}_n$, f_n is well-defined (each x belongs to at most one $B \in \mathcal{B}_n$).

Next, f_n is continuous: fix $x \in X$ and open $U \ni x$ such that U meets at most one $B \in \mathcal{B}_n$. If U meets no $B \in \mathcal{B}_n$ then $f_n = 0$ on U and hence is continuous at x. If U meets a unique $B \in \mathcal{B}_n$ then $f_n = f_B \Delta(x \mapsto B)$ on U and again f_n is continuous at x.

By definition of f_n, we have $\mathcal{B}_n \subseteq \left\{ {f_n}^{-1}((0,1] \times \{\alpha\}) : \alpha \in \kappa_n \right\}$ and hence the family $f_n, n \in \mathbb{N}$ satisfies the conditions of the Embedding Lemma (noting that X is T_1).

Thus $\Delta_n f_n \colon X \to \prod_n H_{\kappa_n}$ is an embedding of X into the product of countably many hedgehogs. □

We can summarize the result of this section by:

Theorem 10.16. *TFAE:*

1. *X is metrizable*
2. *X is normal and has a σ-discrete base.*
3. *X is a subspace of a product of countably many hedgehogs.*

Note: In 2026/2027 I have not lectured the full proof given here. Rather I only showed

Theorem 10.17 (Bing's Theorem)**.** *TFAE:*

1. *X is metrizable*
2. *X is perfectly normal and has a σ-discrete base.*
3. *X is a subspace of a product of countably many hedgehogs.*

This is the relevant statement for examination purposes.

Examining the proofs carefully, we can do better: we can replace σ-discrete by σ-locally finite and we can insist that each of the hedgehogs has spininess $w(X)$.

Finally, with a bit of work we can show that regular spaces that have a σ-locally finite base are normal (essentially having a σ-locally finite base implies that the space is paracompact) and hence we can replace normality in the second condition by regularity to obtain:

Theorem 10.18 (Bing-Nagata-Smirnov Metrization Theorem)**.** *A topological space is metrizable if and only if it is regular and has a σ-locally finite base.*

11 Filters

11.1 Abstract Filters

Definition 11.1. Suppose X is a set.

A filter on X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

- $\emptyset \notin \mathcal{F} \neq \emptyset$;
- $\mathcal{F}$ is closed under finite intersection;
- $\mathcal{F}$ is closed under supersets.

A filter $\mathcal{U}$ which is maximal with respect to $\subseteq$ is called an ultrafilter.

Two collections $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}(X)$ mesh, written $\mathcal{F}\#\mathcal{G}$, if and only if $\forall F \in \mathcal{F} \forall G \in \mathcal{G}\ F \cap G \neq \emptyset$. In this case we also use $\mathcal{F}\#\mathcal{G}$ for $\{F \cap G \colon F \in \mathcal{F}, G \in \mathcal{G}\}$. If $\mathcal{F}$ and/or $\mathcal{G}$ are singletons, we may omit the curly braces, i.e. if $\mathcal{F} = \{F\}$, we write $F\#\mathcal{G}$ instead of $\mathcal{F}\#\mathcal{G}$.

A basis for a filter $\mathcal{F}$ is a collection $\mathcal{B} \subseteq \mathcal{F}$ such that $\mathcal{F} = \{F \subseteq X \colon \exists B \in \mathcal{B}\ B \subseteq F\}$.

A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ has the finite intersection property if and only if for every finite $\mathcal{C}' \subseteq \mathcal{C}$ we have $\bigcap \mathcal{C}' \neq \emptyset$.

The following lemmas are elementary:

Lemma 11.2. *Suppose X is a set. A non-empty collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a filter basis for a unique filter $\mathcal{F}$ on X if and only if $\emptyset \notin \mathcal{B}$ and*

$$\forall A, B \in \mathcal{B}\ \exists C \in \mathcal{B}\ C \subseteq A \cap B,$$

in which case $\mathcal{F} = \{A \subseteq X \colon \exists B \in \mathcal{B}\ B \subseteq A\}$. We call $\mathcal{F}$ the filter generated by $\mathcal{B}$.

Lemma 11.3. *Suppose X is a set. A non-empty collection $\mathcal{S} \subseteq \mathcal{P}(X)$ is contained in a smallest (wrt $\subseteq$) filter $\mathcal{F}$ if and only if $\emptyset \notin \mathcal{S}$ and $\mathcal{S}$ has the finite intersection property, in which case $\mathcal{F}$ is the filter generated by the filter basis $\{\bigcap \mathcal{S}' \colon \mathcal{S}' \text{ finite } \subseteq \mathcal{S}\}$. We call $\mathcal{F}$ the filter generated by $\mathcal{S}$.*

Lemma 11.4. *Suppose X is a set.*

Two non-empty filter bases $\mathcal{F}, \mathcal{G}$ of subsets of X mesh if and only if there is a filter containing both of them.

In the following we make no explicit distinction between a filter and a basis generating it.

Lemma 11.5. *Suppose X is a set, $\mathcal{F}$ a filter on X and $\emptyset \neq \mathcal{G} \subseteq \mathcal{P}(X)$ closed under finite intersections. If $\mathcal{G}\#\mathcal{F}$ then $\mathcal{G}\#\mathcal{F}$ is a filter basis on X (and we also use $\mathcal{G}\#\mathcal{F}$ for the filter generated by this basis).*

Lemma 11.6. *Suppose X is a set. A filter $\mathcal{U}$ on X is an ultrafilter on X if and only if for every $A \subseteq X$ we have $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof. Suppose $\mathcal{U}$ is maximal wrt $\subseteq$ and $A \subseteq X$ such that $A \notin \mathcal{U}$. If $U \in \mathcal{U}$ then $U \not\subseteq A$ so that $U\#(X \setminus A)$. Hence $(X \setminus A)\#\mathcal{U}$ is a filter containing $\mathcal{U}$ and by maximality the two must be equal, giving $X \setminus A \in \mathcal{U}$.

Conversely, let $\mathcal{U}$ be a filter on X satisfying the condition and suppose $\mathcal{G}$ is a strictly bigger filter on X. Then $A \in \mathcal{G} \setminus \mathcal{U}$ for some $A \subseteq X$. By the condition $X \setminus A \in \mathcal{U} \subseteq \mathcal{G}$ so that $(X \setminus A) \cap A = \emptyset \in \mathcal{G}$, a contradiction. □

Lemma 11.7. *Suppose X is a set and that $\mathcal{U}$ is a filter on X. TFAE*

1. *$\mathcal{U}$ is an ultrafilter on X;*
2. *for every $A \subseteq X$ one of $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$;*
3. *for every $A, B \subseteq X$, if $A \cup B \in \mathcal{U}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$;*
4. *$A, B \subseteq X$, if $A, B \notin \mathcal{U}$ then $A \cup B \notin \mathcal{U}$.*

Proof. The equivalence of the first two statements is the previous lemma.

For the third, suppose $\mathcal{U}$ is an ultrafilter on X and $A \cup B \in \mathcal{U}$. If $A \notin \mathcal{U}$ then $X \setminus A \in \mathcal{U}$ and hence $\mathcal{U} \ni (A \cup B) \cap X \setminus A \subseteq B$ so that $B \in \mathcal{U}$.

Conversely, suppose $\mathcal{U}$ is a filter and the condition holds. For $A \subseteq X$, note that since $\mathcal{U} \ni X = A \cup (X \setminus A)$, one of A or $X \setminus A$ belongs to $\mathcal{U}$.

The last statement is simply the contrapositive of the previous one. □

Example 11.8. Suppose X is a set, $x \in X$. The following are filters on X:

- $\mathcal{P}_x = \{A \subseteq X : x \in A\}$, the principal filter at x (it is in fact an ultrafilter);
- for infinite X, $\mathrm{cof} = \{X \setminus F : F \text{ finite } \subseteq X\}$ the cofinite (or Frechet) filter on X;
- if X is a topological space with topology τ then $\mathcal{N}_x = \{N \subseteq X : \exists U \in \tau\ x \in U \subseteq N\}$ is the neighbourhood filter at x.

Theorem 11.9 (Ultrafilter Extension Lemma)**.** *Suppose X is a set. Then every filter can be extended to an ultrafilter.*

The proof is non-examinable.

Proof. Observe that the union of an increasing chain of filters is a filter (which is an upper bound in the partial order of filters on X with order $\subseteq$) and apply Zorn's Lemma. □

The following fact is non-examinable: We write ZF for the axioms of set theory without the Axiom of Choice.

Theorem 11.10. *It is consistent with* ZF *that the only ultrafilters (on any set) are the principal ultrafilters. In particular, it is consistent with* ZF *that the cofinite filter on ω cannot be extended to an ultrafilter.*

Lemma 11.11. *Suppose X, Y are sets, $f : X \to Y$ is a function.*

If $\mathcal{F}$ is a filter on X, then $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is a filter-basis for a filter $\mathcal{G}$ on Y (which we also, in general, denote by $f(\mathcal{F})$) and $\mathcal{G} = \left\{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\right\}$. Moreover, if $\mathcal{F}$ is an ultrafilter on X then $\mathcal{G}$ is an ultrafilter on Y.

Proof. Computing $f(F_1) \cap f(F_2) \supseteq f(F_1 \cap F_2)$ easily implies that $f(\mathcal{F})$ is a filter basis.

Next, if $B \subseteq Y$ such that $f^{-1}(B) \in \mathcal{F}$ then $f\left(f^{-1}(B)\right) \subseteq B$ so that $B \in \mathcal{G}$. Conversely, if $B \in \mathcal{G}$ then there is $F \in \mathcal{F}$ with $f(F) \subseteq B$ so that $F \subseteq f^{-1}(B)$ and hence $f^{-1}(B) \in \mathcal{F}$.

Finally, if $\mathcal{F}$ is an ultrafilter on X and $B \subseteq Y$ then since $f^{-1}(B) \cup f^{-1}(Y \setminus B) = X$ one of B or $Y \setminus B$ belongs to $\mathcal{G}$ as required. □

11.2 Topological filters

Example 11.12. Suppose X is a set and $a : \mathbb{N} \to X; k \mapsto a_k$ is a sequence in X.

Writing $T_n = \{a_k : k \geq n\}$ for the n-tail of (a_n), we have that $\{T_n : n \in \mathbb{N}\}$ is a filter basis for a filter $\mathcal{F}_{(a_n)}$ on X.

We note that if X is a metric space then $a_n \to x \in X$ if and only if $\mathcal{N}_x \subseteq \mathcal{F}_{(a_n)}$ since $a_n \to x$ means that every open $U \ni x$ contains some tail T_n.

Definition 11.13. Suppose X is a topological space, $\mathcal{F}$ a filter on X and $x \in X$.

We say that $\mathcal{F}$ *converges to* x, written $\mathcal{F} \to x$, if and only if $\mathcal{N}_x \subseteq \mathcal{F}$.

We write $\lim \mathcal{F} = \{x \in X : \mathcal{F} \to x\}$ and abuse notation to write $\lim \mathcal{F} = x$ instead of $\lim \mathcal{F} = \{x\}$.

Lemma 11.14. *Suppose X is a topological space, $\mathcal{F}, \mathcal{G}$ are filters on X and $x \in X$.*

If $\mathcal{F} \to x$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} \to x$.

Lemma 11.15. *Suppose X is a topological space, $A \subseteq X$ and $x \in X$.*

TFAE:

1. *$x \in \overline{A}$.*
2. *$\mathcal{N}_x \# A$.*
3. *There is a filter $\mathcal{F}$ on X containing A that converges to x.*
4. *There is an ultrafilter $\mathcal{U}$ on X containing A that converges to x.*

Proof. Since

$$\overline{A} = \{x \in X : \forall \text{ open } U \ni x \; U \cap A \neq \emptyset\}$$

the first two conditions are equivalent.

Equivalence of the other three conditions is then obvious (using the Ultrafilter Extension Lemma). □

Lemma 11.16. *Suppose X, Y are topological spaces and $f : X \to Y$ is a function.*

TFAE:

1. *f is continuous.*
2. *For every filter $\mathcal{F}$ on X and $x \in \lim \mathcal{F}$, $f(x) \in \lim f(\mathcal{F})$.*
3. *For every ultrafilter $\mathcal{U}$ on X and $x \in \lim \mathcal{U}$, $f(x) \in \lim f(\mathcal{U})$.*

Proof. Fix $x \in X$ and a Y-open $V \ni f(x)$. Then $x \in f^{-1}(V)$ open in X by continuity so $f^{-1}(V) \in \mathcal{F}$ and thus $V \in f(\mathcal{F})$. Thus $\mathcal{N}_{f(x)} \subseteq f(\mathcal{F})$ as required.

The second implication is obvious.

Finally if $x \in \overline{A}$ then we can find an ultrafilter $\mathcal{U} \to x$ that contains A. Hence $f(\mathcal{U}) \to f(x)$ and $f(A) \in f(\mathcal{U})$ so that $f(x) \in \overline{f(A)}$. Hence $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ and as $A \subseteq X$ is arbitrary f is continuous. □

Lemma 11.17. *A topological space is Hausdorff if and only if every filter converges to at most one point if and only if every ultrafilter converges to at most one point.*

Proof. Suppose X is a topological space.

If $\mathcal{F}$ is a filter on X converging to two distinct points $x, y \in X$ then $\mathcal{N}_x, \mathcal{N}_y \subseteq \mathcal{F}$ which implies $\mathcal{N}_x \# \mathcal{N}_y$ which in turn implies that any two open sets containing x and y respectively must meet.

Conversely if X is not Hausdorff then let $x, y \in X$ be distinct such that any two open sets containing x and y meet. Then $\mathcal{N}_x \# \mathcal{N}_y$ and hence the generated filter converges to x and y. By the Ultrafilter Extension Lemma we can extend it to an ultrafilter converging to both x and y. □

12 Compactness

12.1 Prerequisites

I assume that this material has been seen before.

Definition 12.1. Suppose X is a set.

A cover of X is a collection $\mathcal{U}$ of subsets of X such that $\bigcup \mathcal{U} \supseteq X$.

A subcover of a cover $\mathcal{U}$ of X is a family $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is a cover of X.

We use adjectives to specify the kinds of sets which appear in a cover. For example, we will use 'open cover' to refer to a cover consisting of open sets.

Definition 12.2. X is compact if and only if every open cover $\mathcal{U}$ has a finite subcover, i.e. there is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $X \subseteq \bigcup \mathcal{V}$.

A subset A of X is compact as a subset if and only if every X-open cover of A has a finite subcover.

Lemma 12.3. *Suppose X is a topological space.*

X is compact if and only if every family of closed subsets with the f.i.p. has non-empty intersection.

Proof. Duality together with the fact that $\mathcal{U}$ is a cover of X if and only if $\bigcap_{U \in \mathcal{U}} X \setminus U = \emptyset$. □

In view of the following lemma, we drop the distinction between 'compact as a subset' and 'compact (with respect to the subspace topology)'.

Lemma 12.4. *Suppose X is a topological space and A is a subset of X.*

A is compact as a subset if and only if A is compact with respect to the subspace topology.

Proof. In this proof 'cover' refers to a cover of A.

Suppose A is compact as a subset. Let $\mathcal{U}$ be an A-open cover. For each $U \in \mathcal{U}$ choose X-open V_U such that $U = V_U \cap A$. Then $\mathcal{V}_\mathcal{U} = \{V_U : U \in \mathcal{U}\}$ is an X-open

cover of A so has a finite subcover which we can write as $\{V_{U_i} : i = 0, \dots, n\}$ for some n and $U_i \in \mathcal{U}$. Then $\{U_i : i = 0, \dots, n\}$ is the required subcover of $\mathcal{U}$.

For the converse suppose A is compact with respect to the subspace topology. Let $\mathcal{U}$ be an X-open cover of A. Then $\{U \cap A : U \in \mathcal{U}\}$ is an open (wrt the subspace topology) cover of A and thus has a finite subcover which we can write as $\{U_i \cap A : i = 0, \dots, n\}$ for some $U_i \in \mathcal{U}$. Then $\{U_i : i = 0, \dots, n\}$ is the required subcover of $\mathcal{U}$. □

Lemma 12.5. *A closed subset of a compact space is compact.*

Proof. Suppose X is a compact space and $C \subseteq X$ is closed.

Suppose $\mathcal{U}$ is an X-open cover of C. Then $\mathcal{U} \cup \{X \setminus C\}$ is an open cover of X and hence has a finite subcover $\mathcal{V}$. We then note that $\mathcal{V} \cap \mathcal{U}$ is the required finite subcover of $\mathcal{U}$. □

12.2 Compactness via ultrafilters

Lemma 12.6. *A topological space is compact if and only if every ultrafilter on X converges.*

Proof. Suppose X is a topological space.

⇒: Suppose $\mathcal{U}$ is an ultrafilter on X that does not converge. For each $x \in X$ we can thus choose X-open $U_x \ni x$ such that $U_x \notin \mathcal{U}$. Then $\{U_x : x \in X\}$ is an open cover of X so has a finite subcover $\mathcal{V}$. By finiteness and Lemma 11.7 $X = \bigcup \mathcal{V} \notin \mathcal{U}$, a contradiction.

⇐: Suppose that every ultrafilter on X converges. Let $\mathcal{C}$ be a family of closed sets with the f.i.p.. Then $\mathcal{C}$ generates a filter which can be extended to an ultrafilter $\mathcal{U}$ converging to some $x \in X$. As each $C \in \mathcal{C}$ belongs to $\mathcal{U}$ we have that $x \in \overline{C} = C$ for each $C \in \mathcal{C}$ and thus $x \in \bigcap \mathcal{C} \neq \emptyset$ as required. □

Theorem 12.7 (Tychonoff's Theorem)**.** *A product of compact spaces is compact.*

Proof. Suppose $X_i, i \in I$ are compact spaces and let $\mathcal{U}$ be an ultrafilter on $\prod_i X_i$. Then $\mathcal{U}_i = \pi_i(\mathcal{U})$ are ultrafilters on X_i and hence converge to some $x_i \in X_i$.

Let $x = (x_i)_{i \in I} \in \prod_i X_i$. For every $i \in I$ and X_i-open $U_i \ni x_i$ we have $U_i \in \mathcal{U}_i$ so that ${\pi_i}^{-1}(U_i) \in \mathcal{U}$. Hence all subbasic open sets containing x belong to $\mathcal{U}$ and thus $\mathcal{N}_x \subseteq \mathcal{U}$ so that $\mathcal{U} \to x$ as required. □

12.3 Tychonoff's Theorem implies the Axiom of Choice

This subsection is non-examinable and has not been lectured in MT 2026.

Note that we employed the Axiom of Choice twice in the proof of Tychonoff's Theorem. First to show that compactness is equivalent to ultrafilters converging

(Ultrafilter Extension Lemma) and secondly to choose some $x_i \in \lim \pi_i(\mathcal{U})$. This second 'choice' is not needed in Hausdorff spaces (since then $\lim \pi_i(\mathcal{U})$ is at most a singleton anyway).

To understand the proof, you need to accept (prove) that 'finitely many choices do not require the Axiom of Choice'.

Theorem 12.8. *Tychonoff's Theorem implies the Axiom of Choice.*

Proof. Suppose $Y_i, i \in I$ are non-empty sets. For each i, let $X_i = Y_i \cup \{\star_i\}$ where $\star_i \notin Y_i$ (e.g. $\star_i = Y_i$ to avoid choice). The topology on X_i is $\{\emptyset, Y_i, \{\star_i\}, X_i\}$.

Clearly each X_i is compact and Y_i is a closed subset of X_i. By Tychonoff's Theorem, $\prod_i X_i$ is compact and $\mathcal{C} = \left\{{\pi_i}^{-1}(Y_i) : i \in I\right\}$ is thus a collection of closed sets. We need to verify the f.i.p., i.e.

$$\forall F \text{ finite } \subseteq I,\ C_F = \bigcap_{i \in F} {\pi_i}^{-1}(Y_i) \neq \emptyset.$$

So fix a finite $F \subseteq I$ and note that by induction $\prod_{i \in F} Y_i \neq \emptyset$ (as $A, B \neq \emptyset \to A \times B \neq \emptyset$). Let $y \in \prod_{i \in F} Y_i$ and define $x \in \prod_i X_i$ by $x = y \cup \{i \mapsto \star_i : i \in I \setminus F\} \in C_F$.

Note that the 'choice' of y does not need the Axiom of Choice. We have shown that $\exists y\ y \in \prod_{i \in F} Y_i$ and then simply appeal to existential quantifier elimination (or the equivalent method in your favourite first order logic proof calculus).

By compactness $\bigcap \mathcal{C} \neq \emptyset$ and it is easy to verify that $\bigcap \mathcal{C} \subseteq \prod_i Y_i$. □

13 Compactifications

13.1 An Example

In Lemma 13.29 we show how to construct the metric completion of a metric space by taking the set of Cauchy sequences, declaring two Cauchy sequences as equivalent if their 'difference' tends to 0 and equipping this space with the natural metric. In some sense for each Cauchy sequence of our original space we ensure that there is a point it will converge to. We then of course have to work reasonably hard to ensure that the 'new' Cauchy sequences also converge via a diagonal argument.

Below we carry out a similar construction to obtain a compact Hausdorff space which contains $\mathbb{N}$ (densely): $\mathbb{N}$ is not compact because the non-principal ultrafilters do not converge. For each such ultrafilter we add a point and identify an element $n \in \mathbb{N}$ with the principal ultrafilter at n. We equip this set of ultrafilters with a suitable topology to obtain our compactification of $\mathbb{N}$. Because we only want a topology, it is not necessary to take any quotients/equivalence classes, so the construction is a little simpler. Also, the proof of compactness is simpler as the diagonal argument from Lemma 13.29 disappears by using our previous results.

This method can be generalized to any Tychonoff space, although the details are a little more complicated. Look up 'Wallman compactification' if you are interested.

13.1.1 The Space

Let

$$\beta\omega = \{\mathcal{U} \colon \mathcal{U} \text{ is an ultrafilter on } \mathbb{N}\}.$$

Two ultrafilters could be considered to be similar if they contain lots of the same subsets of $\mathbb{N}$, so if for $A \subseteq \mathbb{N}$, we define

$$A^\star = \{\mathcal{U} \in \beta\omega \colon A \in \mathcal{U}\}$$

the collection $\{A^\star \colon A \subseteq \mathbb{N}\}$ is a reasonable subbasis for a topology τ on $\beta\omega$.

Note that since the $\mathcal{U}$ are filters, for $A, B \subseteq \mathbb{N}$,

$$A^\star \cap B^\star = \{\mathcal{U} \in \beta\omega \colon A, B \in \mathcal{U}\} = \{\mathcal{U} \in \beta\omega \colon A \cap B \in \mathcal{U}\} = (A \cap B)^\star.$$

Thus

$$\mathcal{B} = \{A^\star \colon A \subseteq \mathbb{N}\}$$

is in fact a basis for τ.

Also note that

$$\beta\omega \setminus A^\star = (\mathbb{N} \setminus A)^\star$$

since the $\mathcal{U}$ are ultrafilters on $\mathbb{N}$ (i.e. $A \notin \mathcal{U} \iff \mathbb{N} \setminus A \in \mathcal{U}$), so the basic open sets are in fact clopen.

13.1.2 $\beta\omega$ is Hausdorff

Next, if $\mathcal{U}, \mathcal{V}$ are distinct ultrafilters then there is $A \subseteq \mathbb{N}$ such that wlog $A \in \mathcal{U} \setminus \mathcal{V}$. As they are ultrafilters this gives $\mathbb{N} \setminus A \in \mathcal{V} \setminus \mathcal{U}$ so that $A^\star$ and $(\mathbb{N} \setminus A)^\star$ are disjoint open sets containing $\mathcal{U}$ and $\mathcal{V}$ respectively.

13.1.3 $\beta\omega$ is compact

Note that the basic closed sets (complements of basic open sets) are

$$A_\star = \{\mathcal{U} \in \beta\omega \colon A \notin \mathcal{U}\} = (\mathbb{N} \setminus A)^\star$$

so can be written as $C^\star$ for suitable $C \subseteq \mathbb{N}$. Let $\mathcal{C}^\star = \{C_i^\star \colon i \in I\}$ be a collection of basic closed sets with the finite intersection property. As $C_i^\star \cap C_j^\star = (C_i \cap C_j)^\star$ we see that $\mathcal{C} = \{C_i \colon i \in I\}$ is a family of subsets of $\mathbb{N}$ with the finite intersection property. Thus $\mathcal{C}$ is a subbasis for a filter on $\mathbb{N}$ (i.e. the finite intersection of elements of $\mathcal{C}$ forms a filter basis) which can be extended to an ultrafilter $\mathcal{U} \supseteq \mathcal{C}$. Then $\mathcal{U} \in \bigcap \mathcal{C}^\star$ showing that $\bigcap \mathcal{C}^\star \neq \emptyset$.

As it is enough to consider basic open covers (and hence families of basic closed sets with the f.i.p.) to prove that a space is compact, this gives the claim.

13.1.4 $n \mapsto \mathcal{P}_n$ is a dense homeomorphic embedding

Let $\beta\colon \mathbb{N} \to \beta\omega$ be given by $n \mapsto \mathcal{P}_n$. This is a continuous ($\mathbb{N}$ is discrete so every map from $\mathbb{N}$ is continuous) injection. By noting that

$$\beta(A) = A^\star \cap \beta(\mathbb{N})$$

we see that β is open onto its image and hence a homeomorphic embedding.

Now let $A^\star$ be a non-empty basic open subset of $\beta\omega$. Then $A \neq \emptyset$ and picking $n \in A$ we see that $\mathcal{P}_n \in A^\star \cap \beta(\mathbb{N})$ showing that $\beta(\mathbb{N})$ is dense as claimed.

13.1.5 Summary

We have constructed a compact Hausdorff space $\beta\omega$ which contains (a copy of) $\mathbb{N}$ as a dense subspace. The extra points we added (the elements of the remainder $\beta\omega \setminus \beta(\mathbb{N})$) correspond to the non-converging ultrafilters on $\mathbb{N}$.

Although we have used the factually correct notation $\beta\omega$ for this compactification which will denote the Stone-Čech compactification of X, we have not in fact proved that this is the Stone-Čech compactification of X. **Do not use the fact that it is the Stone-Čech compactification on Problem Sheets or in the exam unless explicitly instructed otherwise.**

13.2 General Theory of Compactifications

Definition 13.1. Suppose X is a topological space.

A *compactification* of X is a pair (h, Y) such that Y is a compact Hausdorff topological space and $h\colon X \to Y$ is a homeomorphic embedding such that $h(X)$ is dense in Y.

The *remainder* of a compactification (h, Y) of X is the subspace $Y \setminus h(X)$ of Y.

We sometimes emphasize the fact that Y is Hausdorff by writing 'Hausdorff compactification' instead of 'compactification'.

Definition 13.2. Suppose X is a topological space and (h, Y), (g, Z) are compactifications of X.

We say that (g, Z) is larger than (h, Y), written $(h, Y) \leq (g, Z)$, if and only if there is a continuous function $\pi\colon Z \to Y$ such that $\pi \circ g = h$.

We say that (h, Y) and (g, Z) are equivalent compactifications of X, written $(h, Y) \sim (g, Z)$ if and only if there is a homeomorphism $\pi\colon Z \to Y$ such that $\pi \circ g = h$.

Lemma 13.3. *Suppose X is a topological space and (h, Y), (g, Z) are compactifications of X.*

If $(h, Y) \leq (g, Z)$ and $(g, Z) \leq (h, Y)$ then $(g, Z) \sim (h, Y)$. Moreover the witnessing maps $\pi\colon Z \to Y$ and $\rho\colon Y \to Z$ are homeomorphisms.

Thus $\leq$ induces a partial order on the collection of equivalence classes (wrt $\sim$) of compactifications of X.

Proof. Let $\pi\colon Z \to Y$ witness $(h, Y) \leq (g, Z)$ and $\rho\colon Y \to Z$ witness $(g, Z) \leq (h, Y)$. We claim that π and ρ are inverses of each other so that both are homeomorphisms as required.

To show this, it is sufficient to show that $\pi \circ \rho = \mathrm{id}_Y$ and $\rho \circ \pi = \mathrm{id}_Z$ (which will follow by symmetry).

Since $h(X)$ is dense in Y and Y is Hausdorff, it is sufficient to show that $\pi \circ \rho$ is the identity on $h(X)$. But $\pi(\rho(h(x))) = \pi(g(x)) = h(x)$ as required. □

Lemma 13.4. *Suppose X is a topological space and (h, Y) and (g, Z) are compactifications of X. If $\pi\colon Z \to Y$ witnesses that $(h, Y) \leq (g, Z)$ then π maps the remainder onto the remainder, i.e.*

$$\pi(Z \setminus g(X)) = Y \setminus h(X).$$

Proof. We first note that the image of the compact space Z under π contains the dense subset $h(X)$ of Y and hence must be all of Y (as compact subsets are closed in Hausdorff spaces).

In the following we will use that if $A \subseteq Z$ then $g(X) \cap A = g\left(g^{-1}(A)\right)$ which follows from injectivity of g

Now assume that there is $z \in Z \setminus g(X)$ such that $\pi(z) \in h(X)$ and fix $x \in X$ such that $\pi(z) = h(x)$. Since $z \neq g(x)$ we can find an Z-open $U \ni z$ such that $g(x) \notin \overline{U}$. Since $g(X)$ is dense in Z, $z \in \overline{U \cap g(X)} \not\ni g(x)$ and $x \notin \overline{g^{-1}(U)}$ since g is a homeomorphic embedding (i.e. if $x \in \overline{g^{-1}(U)}$ then $g(x) \in \overline{g(g^{-1}(U))} = \overline{U \cap g(X)}$). Since h is a homeomorphic embedding, $h(x) \notin \overline{h(g^{-1}(U))}^{h(X)} = h(X) \cap \overline{h(g^{-1}(U))}^{Y}$. But

$$h(x) = \pi(z) \in \pi\left(\overline{U \cap g(X)}\right) \subseteq \overline{\pi(U \cap g(X))} = \overline{\pi(g(g^{-1}(U)))} = \overline{h(g^{-1}(U))}$$

since $\pi \circ g = h$ and we have the required contradiction. □

Theorem 13.5. *A topological space has a Hausdorff compactification if and only if it is Tychonoff.*

Proof. If a topological space X has a Hausdorff compactification, it is homeomorphic to a subset of a compact Hausdorff space. But compact Hausdorff spaces are normal, hence functionally normal (Urysohn's Lemma) hence Tychonoff and subspaces of Tychonoff spaces are Tychonoff.

Conversely if X is Tychonoff then there is a homeomorphic embedding $h\colon X \to [0,1]^\kappa$ for some κ. Thus $(h, \overline{h(X)})$ is a Hausdorff compactification of X. □

Theorem 13.6. *The partially ordered set of (equivalence classes of) compactifications of a Tychonoff space has suprema of non-empty sets.*

Proof. Suppose $(g_i, Y_i), i \in I$ are compactifications of X. Let $\Delta = \Delta_i g_i\colon X \to \prod_i Y_i$ be the diagonal. Since one (in fact every) g_i is a homeomorphic embedding, Δ is a homeomorphic embedding. Writing $S = \overline{\Delta(X)}$ and Δ_S for the map

Δ with co-domain restricted to S, we see that (Δ_S, S) is a compactification of X. Clearly each $\pi_i \colon S \to Y_i$ witnesses that $(g_i, Y_i) \leq (\Delta_S, S)$.

Now suppose that (h, Z) is a compactification of X such that for all $i \in I, (g_i, Y_i) \leq (h, Z)$ as witnessed by maps $\sigma_i \colon Z \to Y_i$. Then $H = \Delta_i \sigma_i \colon Z \to \prod_i Y_i$ is continuous and if $x \in X$ then for each $i \in I$, $\Delta_S(x)_i = g_i(x) = \sigma_i(h(x)) = H(h(x))_i$ so that $\Delta_S = H \circ h$ as required. It remains to show that H maps Z into S: for this we note that

$$H(Z) = H\left(\overline{h(X)}\right) \subseteq \overline{H(h(X))} = \overline{\Delta_S(X)} = S.$$

Hence $(\Delta_S, S) \leq (h, Z)$ and thus $(\Delta_S, S) = \sup_i (g_i, Y_i)$. □

13.3 The Stone-Čech Compactification

Lemma 13.7. *Suppose Y is a Hausdorff topological space.*

If X is a dense subspace of Y then $|Y| \leq 2^{2^{|X|}}$.

Proof. Let $N \colon Y \to \mathcal{P}(\mathcal{P}(X))$ be given by $N(y) = \{U \cap X \colon y \in U \text{ open } \subseteq Y\}$. Since X is dense in Y, each non-empty Y-open set meets X. Hence each element of $N(y)$ is non-empty and each $N(y)$ is a filter basis. Since distinct points of Y have disjoint open sets containing them, N is an injection as required. □

Theorem 13.8. *Suppose X is a Tychonoff topological space.*

Then X has a greatest compactification, denoted by $(\beta, \beta X)$ and called the Stone-Čech-compactification of X.

Proof. By the previous lemma every equivalence class of compactifications has a representative where the space has cardinality at most $2^{2^{|X|}}$. Choose one representative for each equivalence class and taking the supremum over all of these representatives gives the result (note that there are only set-many such representatives). □

Definition 13.9. Suppose X is a topological space, (g, Y) a compactification of X and $\mathcal{C}$ a class of topological spaces.

(g, Y) satisfies the *Stone-Čech property* for $\mathcal{C}$ if and only if for every $Z \in \mathcal{C}$ and every continuous $f \colon X \to Z$ there is a continuous $F \colon Y \to Z$ such that $f = F \circ g$. We say that F *extends* f.

Usually we drop the 'for $\mathcal{C}$' as this is clear from the context.

Theorem 13.10. *Suppose X is a Tychonoff topological space.*

The Stone-Čech-compactification of X is the unique compactification satisfying the Stone-Čech property for compact Hausdorff spaces.

Proof. We first show uniqueness: suppose (g, Y) and (h, Z) are compactifications for X that satisfy the Stone-Čech property for compact Hausdorff spaces. Since (g, Y) satisfies the Stone-Čech property for compact Hausdorff spaces there is a continuous $H \colon Y \to Z$ such that $h = H \circ g$. Hence $(h, Z) \leq (g, Y)$.

By symmetry $(g, Y) \leq (h, Z)$ and by a previous lemma (g, Y) and (h, Z) are thus equivalent compactifications of X.

We next show that the Stone-Čech-compactification satisfies the Stone-Čech-property for compact Hausdorff spaces: let Z be compact Hausdorff and $f\colon X \to Z$ be continuous. Since $\beta\colon X \to \beta X$ is an embedding, so is $\beta\Delta f\colon X \to \beta X \times Z$ and hence $(\beta\Delta f, \overline{\beta\Delta f\,(X)})$ is a compactification of X. Thus there is $\pi\colon \beta X \to \overline{\beta\Delta f\,(X)}$ with $\beta\Delta f = \pi \circ \beta$. Then $\pi_Z \circ \pi\colon \beta X \to Z$ is continuous and extends f. □

Theorem 13.11. *Suppose X is a Tychonoff topological space.*

The Stone-Čech-compactification of X is the unique compactification satisfying the Stone-Čech property for $\{[0, 1]\}$.

Proof. It is sufficient to prove that the Stone-Čech property for $\{[0, 1]\}$ implies the Stone-Čech property for compact Hausdorff spaces.

So let (g, Y) be a compactification of X satisfying the Stone-Čech property for $\{[0, 1]\}$, Z a compact Hausdorff space and $f\colon X \to Z$ a continuous map.

Since Z is normal so Tychonoff, there is a homeomorphic embedding of Z in $[0, 1]^\kappa$ for some κ. We identify Z with its image under this homeomorphism. For $i \in \kappa$ note that $\pi_i \circ f\colon X \to [0, 1]$ is continuous so that there is a continuous $F_i\colon Y \to [0, 1]$ with $\pi_i \circ f = F_i \circ g$. Thus $F = \Delta_i F_i\colon Y \to [0, 1]^\kappa$ is continuous and $\pi_i \circ F \circ g = \pi_i \circ f$ showing that F extends f.

We need to check that $F(Y) \subseteq Z$: for this we compute

$$F(Y) = F\left(\overline{g(X)}^Y\right) \subseteq \overline{F(g(X))}^{[0,1]^\kappa} = \overline{f(X)}^{[0,1]^\kappa} \subseteq \overline{Z}^{[0,1]^\kappa} = Z$$

since Z is a compact subset of the Hausdorff space $[0, 1]^\kappa$ and hence closed in $[0, 1]^\kappa$. □

13.4 The One-point Compactification

Theorem 13.12. *Suppose X is a Tychonoff space.*

TFAE:

1. *$X^\star = \beta X \setminus \beta(X)$ is closed in βX.*
2. *There is a compactification (g, Y) of X such that $Y \setminus g(X)$ is closed in Y.*
3. *For every compactification (g, Y) of X, $Y \setminus g(X)$ is closed in Y.*

Proof. We prove (3) ⇒ (2) ⇒ (1) ⇒ (3).

(3) ⇒ (2): Since X is Tychonoff, it has a compactification which by (3) has the required property.

(2) ⇒ (1): Suppose (g, Y) is as in (2). Let $\pi\colon \beta X \to Y$ witness $(g, Y) \leq (\beta, \beta X)$ and note that since remainders are mapped to remainders (and of course the image of X to the image of X) we have $\pi^{-1}(Y \setminus g(X)) = X^\star$. As π is continuous, the result follows.

(1) ⇒ (3): Suppose (g, Y) is a compactification of X. Let $\pi\colon \beta X \to Y$ witness $(g, Y) \leq (\beta, \beta X)$ and note that since remainders are mapped to remainders (and of course the image of X to the image of X) and π is onto, we have $Y \setminus g(X) = \pi(X^\star)$. As π is continuous, it preserves compactness and since Y is Hausdorff and compact subsets of Hausdorff spaces are closed the result follows. □

Definition 13.13. Suppose X is a non-compact Hausdorff topological space with topology τ and $\infty \notin X$.

The *one-point* (not necessarily Hausdorff) *compactification* of X is the set $\omega X = X \cup \{\infty\}$ with topology

$$\sigma = \tau \cup \{\omega X \setminus C\colon C \text{ compact } \subseteq X\}$$

and dense embedding $\omega\colon X \to \omega X; x \mapsto x$.

Proof. We need to check that σ is a topology on ωX. This is straightforward using that compact subsets of X are closed in X and the intersection of compact sets in a Hausdorff space is compact.

We also need to check that ω is a dense embedding. Again, using that compact subsets of X are closed in X this is straightforward. Density follows from non-compactness of X.

Finally, we need to check that ωX is compact. So let $\mathcal{U}$ be an open cover. Find $U \in \mathcal{U}$ such that $\infty \in U$ so that $U = \omega X \setminus C$ for some compact $C \subseteq X$. As C is compact (both wrt τ and wrt σ since $C \subseteq X$) there is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup \mathcal{V} \supseteq C$. Then $\mathcal{V} \cup \{\omega X \setminus C\}$ is the required finite subcover of $\mathcal{U}$. □

Definition 13.14. Suppose X is a topological space.

X is locally compact if and only if every neighbourhood filter has a filter basis of compact sets, i.e. for every $x \in X$ and open $U \ni x$ there is open V and compact K such that $x \in V \subseteq K \subseteq U$.

Lemma 13.15. *A compact Hausdorff space is locally compact.*

Proof. Suppose X is compact Hausdorff and $x \in U$ open $\subseteq X$.

As X is compact Hausdorff, it is regular so there is open V and compact K such that $x \in V \subseteq K \subseteq U \subseteq X$ as required. □

Lemma 13.16. *An open subspace of a locally compact Hausdorff space is locally compact.*

Proof. Suppose X is locally compact Hausdorff and Y is an open subspace of X. Let $x \in U$ where U is a Y-open subset of Y. As Y is open in X, U is in fact X-open and hence there are X-open V and compact K with $x \in V \subseteq K \subseteq U \subseteq Y$. But then $V = V \cap Y$ is Y-open and $K = K \cap Y$ is compact as required. □

Theorem 13.17. *Suppose X is a non-compact Hausdorff topological space.*

The one-point compactification ωX of X is Hausdorff if and only if X is locally compact.

Proof. We note that X is open in its one-point compactification.

⇐: Let $x, y \in \omega X$ be distinct. If $x, y \in X$ then Hausdorffness of X gives disjoint X-open sets $U \ni x, V \ni y$ which are ωX-open by definition of ωX. So assume that $y = \infty$. By local compactness (with $U = X$) find X-open V and compact $K \subseteq X$ such that

$$x \in V \subseteq K \subset X.$$

Then $V \ni x$ and $\omega X \setminus K \ni \infty$ are the required disjoint ωX-open sets.

⇒: Since X is open in ωX and ωX is compact Hausdorff, X is locally compact by the two previous lemmas. □

Corollary 13.18. *A locally compact Hausdorff space is Tychonoff.*

Corollary 13.19. *Suppose X is a Hausdorff space.*

X is locally compact if and only if one (and hence all) of the three conditions from 13.12 holds.

Theorem 13.20. *Suppose X is a Tychonoff non-compact space and $\infty \notin X$.*

X is locally compact if and only if X has a smallest compactification.

In this case, the smallest compactification is the one-point compactification.

Proof. We prove both directions separately

⇒: Suppose X is locally compact. We will show that the one-point compactification $(\omega, \omega X)$ is the smallest compactification of X. So assume (g, Y) is any compactification of X and define

$$\pi \colon Y \to \omega X; y \mapsto \begin{cases} \omega\left(g^{-1}(y)\right), & y \in g(X) \\ \infty, & \text{otherwise} \end{cases}.$$

Clearly $\omega = \pi \circ g$ so that we only need to verify that π is continuous. So let U be open in ωX. If $\infty \notin U$ then U is an open subset of X and hence $\pi^{-1}(U) = g(U)$ is open in $g(X)$ which is open in Y (since X is locally compact Hausdorff). If $\infty \in U$ then $U = \omega X \setminus C$ for some compact subset of X. Hence $\pi^{-1}(U) = Y \setminus g(C)$ and C is compact so $g(C)$ is compact so closed in Y.

⇐: Suppose (g, Y) is the smallest compactification of X. We will show that $Y \setminus g(X)$ is a singleton so closed in the Hausdorff space Y, giving the result.

So assume for a contradiction that there are distinct $y_1, y_2 \in Y \setminus g(X)$. Then $Y' = Y \setminus \{y_1, y_2\}$ is an open subset of Y so is locally compact Hausdorff and non-compact (by density of $X \subseteq Y'$) and hence has a one-point compactification (h, Z) which is smaller than the two-point compactification (id, Y) of Y' (in the lattice of compactifications of Y') as witnessed by some $\pi \colon Y \to Z$.

Since

$$Z = \overline{h(Y')} \subseteq \overline{h\left(\overline{g(X)}^{Y'}\right)} \subseteq \overline{\overline{h(g(X))}} = \overline{h \circ g(X)}$$

we can see that $(h \circ g, Z)$ is a compactification of X. But then π witnesses that $(h \circ g, Z) \leq (g, Y)$. As (g, Y) is the smallest compactification of X it must be

equivalent to $(h \circ g, Z)$ and by Lemma 13.3 π has to be a homeomorphism. But π maps both of y_1 and y_2 to ∞, a contradiction. □

2026: The remainder of the chapter i.e. Čech-completeness and the Baire Category Theorem is not examinable.

13.5 Čech-completeness

Definition 13.21. Suppose Z is a topological space and $X \subseteq Z$.

X is a G_δ-subset of Z if and only if X is a countable intersection of Z-open sets, i.e. there are Z-open U_n, $n \in \mathbb{N}$, such that $X = \bigcap_{n\in\mathbb{N}} U_n$.

X is an F_σ-subset of Z if and only if X is a countable union of Z-closed sets, i.e. there are Z-closed C_n, $n \in \mathbb{N}$, such that $X = \bigcup_{n\in\mathbb{N}} C_n$.

Lemma 13.22. *Suppose Z is a topological space and $X \subseteq Z$.*

X is a G_δ-subset of Z if and only if $Z \setminus X$ is an F_σ-subset of Z.

Proof. Duality. □

Definition 13.23. A topological space X is Čech-complete if and only if X is $T_{3.5}$ and $\beta X \setminus \beta(X)$ is an F_σ-subset of βX.

Lemma 13.24. *Suppose X is a $T_{3.5}$ topological space.*

TFAE:

1. *X is Čech-complete.*
2. *For every compactification (h, Y) of X, $Y \setminus h(X)$ is an F_σ-subset of Y.*
3. *For some compactification (h, Y) of X, $Y \setminus h(X)$ is an F_σ-subset of Y.*

Proof. As for locally compact: (2) implies (3) follows since X has a compactification.

For (3) implies (1) note that if $g\colon \beta X \to Y$ witnesses $(h, Y) \leq (\beta, \beta X)$ and $Y \setminus h(X) = \bigcup_n C_n$ for closed C_n, then $\beta X \setminus \beta(X) = g^{-1}(Y \setminus h(X)) = \bigcup_n g^{-1}(C_n)$. Hence by continuity of g the result follows.

For (1) implies (2) note that if $g\colon \beta X \to Y$ witnesses $(h, Y) \leq (\beta, \beta X)$ and $\beta X \setminus \beta(X) = \bigcup_n C_n$ for closed and hence compact C_n, then $Y \setminus h(X) = g(\bigcup_n C_n) = \bigcup_n g(C_n)$. Hence by continuity of g, the fact that images of compact sets under continuous maps are compact and the fact that compact subsets of Hausdorff spaces are closed, the result follows. □

Definition 13.25. A metric space (X, d) is complete if and only if every Cauchy sequence in X converges.

A topological space X is completely metrizable if and only if there is a metric d on X such that (X, d) is complete and d induces the topology on X.

Lemma 13.26. *A countable product of completely metrizable spaces is completely metrizable.*

A closed subset of a completely metrizable space is completely metrizable.

Proof. Sheet 0 and standard Part A result. □

Lemma 13.27. *Suppose Z is a complete metric space and X is a G_δ-subset of Z.*

Then X is completely metrizable.

Proof. Let d be a complete metric on Z inducing its topology. If $X = Z$ then X is of course completely metrizable, so assume $X \subsetneq Z$.

Let $Z \setminus X = \bigcup_n C_n$ with C_n closed non-empty in Z. Then each $d_{C_n} \colon Z \to [0,\infty); x \mapsto \inf\{d(x,c) \colon c \in C_n\}$ is continuous and hence by the Embedding Lemma and its Corollary $D = \mathrm{id}\,\Delta\Delta_n d_{C_n} \colon Z \to Z \times [0,\infty)^{\mathbb{N}}$ is a homeomorphic embedding of Z and hence of X.

We now claim that $D(X) = D(Z) \cap Z \times (0,\infty)^{\mathbb{N}}$: if $x \in X$ then $d_{C_n}(x) > 0$ for all $n \in \mathbb{N}$ giving $\subseteq$. On the other hand if $D(z) \in Z \times (0,\infty)^{\mathbb{N}}$ then $d_{C_n}(z) > 0$ for all n giving $z \notin \bigcup_n C_n$ so $z \in X$ proving $\supseteq$.

Next, $D(Z)$ is closed in $Z \times [0,\infty)^{\mathbb{N}}$: Write $\pi_n \colon Z \times [0,\infty)^{\mathbb{N}} \to [0,\infty)$ for the projection onto the $(n+1)^{st}$ coordinate (i.e. the map $(z, r_1, r_2, \dots) \mapsto r_n$) and π_Z for the projection onto Z. Then $d_{C_n} \circ \pi_Z$ and π_n are continuous for each n so that

$$Eq(d_{C_n} \circ \pi_Z, \pi_n) = \left\{p \in Z \times [0,\infty)^{\mathbb{N}} \colon d_{C_n}(\pi_Z(p)) = \pi_n(p)\right\}$$

is closed (as $[0,\infty)$ is Hausdorff). Now note that

$$D(Z) = \bigcap_n Eq(d_{C_n} \circ \pi_Z, \pi_n).$$

Hence $D(X) = D(Z) \cap Z \times (0,\infty)^{\mathbb{N}}$ is closed in $Z \times (0,\infty)^{\mathbb{N}}$ and the latter is completely metrizable since $(0,\infty)$ is homeomorphic to $\mathbb{R}$.

Hence X is homeomorphic to a closed subspace of a completely metrizable space and hence completely metrizable. □

Lemma 13.28. *For every metric space (X,d) there is a complete metric space (Y, d_∞) and a dense homeomorphic embedding $h \colon X \to Y$.*

Proof. By replacing d with $\min\{1,d\}$ we may assume that d is bounded by 1. Define $P = [0,1]^X$ with the sup-metric $d_\infty((y_x)_{x\in X}, (z_x)_{x\in X}) = \sup_{x\in X} |y_x - z_x|$ which is complete and $h \colon X \to P$ by $h(x) = (d(x,x'))_{x'\in X}$.

For $x, z, x' \in X$ the reverse triangle law gives $|d(x,x') - d(z,x')| \le d(x,z)$ and hence $d_\infty(h(x),h(z)) \le d(x,z) = |d(x,z) - d(z,z)| \le d_\infty(h(x),h(z))$ so that h is an isometric embedding of X into P. Considering $Y = \overline{h(X)}^P$ and restricting the codomain of h to Y gives the result. □

Note that the embedding above is only isometric (from the original metric) for small distances as we need the boundedness. For our purposes this is good enough because we only care about metric spaces up to homeomorphism. It is of course possible to do a more subtle argument to get a better result:

Lemma 13.29. *For every metric space (X, d) there is complete metric space $(Y, \hat{d})$ and a dense isometric embedding $h\colon X \to Y$.*

Proof. Let $Y = \{(x_n) \in X^{\mathbb{N}}\colon (x_n) \text{ is Cauchy}\} / \sim$ where $(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0$. Note that $\sim$ is indeed an equivalence relation on the set of Cauchy sequences of X. Define $\hat{d}([(x_n)], [(y_n)]) = \lim d(x_n, y_n)$, noting that this is well defined (this requires a small calculation). Checking that $\hat{d}$ is indeed a metric on Y is straightforward.

Completeness is a little fiddly. We will write sequences in X as functions $f\colon \mathbb{N} \to X$.

Let $([f_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in Y with each f_n a Cauchy sequence in X.

If f is a Cauchy sequence in X then recursively for $n \in \mathbb{N}$ we can find strictly increasing $N_n \in \mathbb{N}$ such that $m, k \geq N_n \implies d(f(m), f(k)) < 2^{-n}$. Letting $\hat{f}(n) = f(N_n)$ we obtain a Cauchy sequence $\hat{f}$ that is equivalent to f (it is a subsequence of f) and $m, k \geq N \implies d\left(\hat{f}(m), \hat{f}(k)\right) < 2^{-N}$ which we call the *exponential property*. Thus we may assume that our Cauchy sequences f_n all have this exponential property.

We now set $f(n) = f_n(n)$ and claim that this is a Cauchy sequence in X:

let $\epsilon > 0$ and using Cauchyness of $([f_n])_n$ find $N \in \mathbb{N}$ such that for $n, m \geq N$ we have $\lim_{k\to\infty} d(f_n(k), f_m(k)) = \hat{d}([f_n], [f_m]) < \epsilon/3$ and $1/2^N < \epsilon/3$.

Now fix $n, m \geq N$ and choose $k \geq \max\{n, m, N\}$ (possibly depending on n, m of course) such that $d(f_n(k), f_m(k)) < \epsilon/3$. Then

$$\begin{aligned} d(f(n), f(m)) &= d(f_n(n), f_m(m)) \\ &\leq d(f_n(n), f_n(k)) + d(f_n(k), f_m(k)) + d(f_m(k), f_m(m)) \\ &< 2^{-n} + \epsilon/3 + 2^{-m} \quad \text{by the exponential property and choice of } k \\ &\leq \epsilon \quad \text{since } n, m \geq N \text{ and choice of } N \end{aligned}$$

showing that f is indeed Cauchy.

Also

$$\begin{aligned} d(f(m), f_n(m)) &= d(f_m(m), f_n(m)) \\ &\leq d(f_m(m), f_m(k)) + d(f_m(k), f_n(k)) + d(f_n(k), f_n(m)) \\ &\leq 2^{-m} + \epsilon/3 + 2^{-m} \leq \epsilon. \end{aligned}$$

For any $n \geq N$ this gives $\lim_{k\to\infty} d(f(k), f_n(k)) \leq \epsilon$ and thus $\hat{d}([f], [f_n]) \leq \epsilon$.

Thus $\hat{d}([f], [f_n]) \to 0$ as $n \to \infty$ and thus $([f_n])_n$ converges (in Y) to $[f]$.

Finally observe that $h\colon X \to Y; h(x) = (x)$ where (x) is the sequence with constant value x is an isometric embedding. Either note that $h(X)$ is dense or take $\overline{h(X)}$ to complete the proof. □

We call (h, Y) a completion of X (and could show that Y is unique up to the natural definition of equivalence) and identify X with its image under h.

Lemma 13.30. *If a metrizable space X is Čech-complete then it is completely metrizable.*

Proof. Let Y be a completion of X and $(\beta, \beta Y)$ its Stone-Čech compactification. As X is dense in Y, $\beta(X)$ will be dense in βY, so $(\beta|_X, \beta Y)$ is also a compactification of X. Thus $\beta(X)$ is a G_δ-subset of βY and hence a G_δ-subset of $\beta(Y)$ (simply intersect the witnessing open sets with $\beta(Y)$). Hence X is a G_δ-subset of Y and thus completely metrizable. □

Lemma 13.31. *If a metric space (X, d) is complete and $\mathcal{F}$ is a filter on X that contains sets of arbitrarily small diameter, then $\mathcal{F}$ converges (to a point of X).*

Proof. For each $n \in \mathbb{N}$ choose $C_n \in \mathcal{F}$ with $\operatorname{diam}(C_n) \leq 2^{-n}$. By taking closures we may assume that each C_n is closed and by taking finite intersections of $C_1, \ldots, C_n$ that the C_n form a decreasing sequence of closed sets. For each $n \in \mathbb{N}$, we can thus choose $x_n \in C_n$ and observe that if $n \leq m$ then $x_n, x_m \in C_n$ so that $d(x_n, x_m) \leq 2^{-n+1}$. Hence (x_n) is Cauchy and thus converges to some $\hat{x} \in X$. But then $\hat{x} \in \bigcap C_n$ (since each C_n is closed and a tail of (x_m) belongs to C_n). Therefore $B_{2^{-n+2}}(\hat{x}) \supseteq C_n$ and thus $\mathcal{F} \to \hat{x}$ as required. □

Lemma 13.32. *If a space X is completely metrizable then it is Čech-complete.*

Proof. Let d be a complete metric on X inducing the original topology on X. Without loss of generality we may assume that d is bounded by 1 (otherwise take $\min\{d, 1\}$ which will still be complete). For each $x \in X$, let $d_x \colon X \to [0, 1]$ be given by $d_x(y) = d(x, y)$, which is clearly continuous so extends to a continuous function $\beta d_x \colon \beta X \to [0, 1]$.

For each $x \in X$, let $V_{n,x} = {\beta d_x}^{-1}([0, 2^{-n}))$ and observe that $X \cap V_{n,x} = B_{2^{-n}}(x)$ which has diameter $\leq 2^{-n+1}$.

For each $n \in \mathbb{N}$ we let

$$U_n = \bigcup_{x \in X} V_{n,x}$$

which is open in βX and claim $X = \bigcap_n U_n$.

That $X \subseteq \bigcap_n U_n$ is clear.

Now let $z \in \bigcap_n U_n$. Then $X \# \mathcal{N}_z$ by density of X in βX.

Thus $\{X \cap N \colon N \in \mathcal{N}_z\}$ generates a filter $\mathcal{F}$ on X and a filter $\mathcal{G}$ on βX where $\mathcal{F}$ is the trace $\{X \cap A \colon A \in \mathcal{G}\}$ of $\mathcal{G}$ on X.

We clearly have $\mathcal{G} \to z$.

On the other hand for each $n \in \mathbb{N}$, some $X \cap V_{n,x} \in X \# \mathcal{N}_z$ so that $\mathcal{F}$ contains sets of arbitrarily small diameter and thus converges to some $\hat{x} \in X$. But then $\mathcal{G}$ converges to $\hat{x}$ as well.

As βX is Hausdorff the limit of $\mathcal{G}$ is unique and so $z = \hat{x} \in X$ as required. □

We summarize:

Theorem 13.33. *A metrizable space X is completely metrizable if and only if X is Čech-complete.*

13.6 The Baire Category Theorem

Theorem 13.34. *Suppose X is a compact Hausdorff space.*

If $U_n, n \in \mathbb{N}$ is a countable family of dense open sets, then $\bigcap_n U_n$ is dense.

Proof. First note that the intersection of two dense open sets is dense open: if U_1, U_2 are dense open and V is non-empty open then $V \cap U_1$ is non-empty and open and hence $V \cap U_1 \cap U_2$ is non-empty. Thus we may assume wlog (replacing them with $\bigcap_{k \leq n} U_k$) that the U_n are decreasing.

Now, fix a non-empty open V and recursively non-empty open W_n such that $\overline{W_{n+1}} \subseteq U_{n+1} \cap W_n$ with $W_0 \subseteq V \cap U_0$. This is possible: $V \cap U_0$ is non-empty by density of U_0 so pick $x \in V \cap U_0$ and then use regularity of X to obtain W_0 as required. For the inductive step, having defined W_n, we note again that $W_n \cap U_{n+1}$ is non-empty by density of U_{n+1} so pick $x \in W_n \cap U_{n+1}$ and W_{n+1} by regularity of X.

Now observe that

$$\bigcap_{n \geq 0} W_n \subseteq \bigcap_{n \geq 0} \overline{W_n} \subseteq \bigcap_{n \geq 1} \overline{W_n} \subseteq \bigcap_{n \geq 1} W_n \subseteq \bigcap_{n \geq 0} W_n$$

where the last $\subseteq$ follows from $W_1 \subseteq W_0$. Thus $\bigcap_n W_n = \bigcap_n \overline{W_n}$ and the latter is an intersection of a decreasing family of non-empty closed sets in a compact space and hence non-empty. Finally observe that since $W_0 \subseteq V$ and $\bigcap_n W_n \subseteq \bigcap_n U_n$ we have $\bigcap_n U_n \cap V \neq \emptyset$ as required. □

Definition 13.35. Suppose X is a topological space.

X is Baire if and only if a countable intersection of dense open sets is dense.

Dually, a countable union of closed, co-dense subsets of X is co-dense.

Theorem 13.36. *A dense G_δ subset of a Baire space is Baire.*

Proof. Suppose X is a dense G_δ subset of the Baire space Y. Let $V_n, n \in \mathbb{N}$ be Y-open sets such that $X = \bigcap_n V_n$ and $W_n, n \in \mathbb{N}$ be X-dense, X-open subsets of X. Then each W_n is Y-dense (as X is dense in Y) and we can find Y-open U_n such that $W_n = X \cap U_n$. Then $\{U_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$ is a countable family of dense open sets.

Now assume that W is X-open and non-empty. Find Y-open U such that $W = X \cap U$. As Y is Baire,

$$\emptyset \neq U \cap \bigcap_n V_n \cap \bigcap_n U_n = U \cap X \cap \bigcap_n U_n = W \cap \bigcap_n X \cap U_n = W \cap \bigcap_n W_n$$

as required. □

Corollary 13.37. *Čech-complete Tychonoff spaces and hence completely metrizable spaces are Baire.*

Proof. By definition a Čech-complete Tychonoff space is homeomorphic to a dense G_δ-subset of a compact Hausdorff space. □

Corollary 13.38. *$\mathbb{Q}$ is not a G_δ-subset of $\mathbb{R}$.*

Proof. Note that $\mathbb{Q}$ is dense in $\mathbb{R}$ which is completely metrizable, so Baire. If $\mathbb{Q}$ were a G_δ-subset of $\mathbb{R}$ it would be Baire. Enumerating $\mathbb{Q}$ as $\{q_n : n \in \mathbb{N}\}$ and considering the closed co-dense sets $\{q_n\}, n \in \mathbb{N}$ gives a contradiction. □

Lemma 13.39. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function.*

The set of continuity of f, $\{x \in \mathbb{R} : f$ is continuous at $x\}$ is a G_δ-subset of $\mathbb{R}$.

Proof. For $x \in \mathbb{R}, \delta > 0$ we let

$$\omega(x, \delta) = \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

(where this sup may be ∞ if $f(B_\delta(x))$ is unbounded) and

$$\omega(x) = \lim_{\delta \to 0} \omega(x, \delta).$$

Since for fixed $x \in \mathbb{R}$ the function $\omega(x, \delta)$ decreases as δ decreases the limit (taken in $[0, \infty]$) exists.

We now note that for $\epsilon > 0$ the set

$$\Omega_\epsilon = \{x \in \mathbb{R} : \omega(x) < \epsilon\}$$

is open: if $x \in \Omega_\epsilon$ find $\delta > 0$ such that $\omega(x, \delta) < \epsilon$. If $y \in B_\delta(x)$ we can then find $\delta' > 0$ with $B_{\delta'}(y) \subseteq B_\delta(x)$ so that $\omega(y) \leq \omega(y, \delta') \leq \omega(x, \delta) < \epsilon$ and hence $B_\delta(x) \subseteq \Omega_\epsilon$.

But then the points of continuity of f are exactly $\bigcap_{\epsilon > 0} \Omega_\epsilon = \bigcap_{n \in \mathbb{N}} \Omega_{2^{-n}}$, a G_δ as required. □

The proof above works for any function between metric spaces. The introduced function $\omega(x)$ is known as the oscillation at x although the details of its definition may vary between authors.

Corollary 13.40. *There is no function $f : \mathbb{R} \to \mathbb{R}$ whose set of continuity is exactly $\mathbb{Q}$.*

14 Connectedness

14.1 Recall

I assume you have seen this subsection in a previous course.

Definition 14.1. Suppose X is a topological space.

A disconnection of X is a partition of X into two non-empty open sets (these are then necessarily disjoint non-empty proper subsets of X which cover X and are also closed).

A space is disconnected if and only if it admits a disconnection.

A space is connected if and only if it does not admit a disconnection.

Note that a subset of X is disconnected if and only if it is disconnected in the subspace topology.

We need a few results from Part A:

Lemma 14.2. *A space X is connected if and only if the only clopen subsets of X are $\emptyset$ and X.*

Proof. If $\emptyset \neq D \neq X$ is clopen then $D, X \setminus D$ is a disconnection of X. □

Lemma 14.3. *A space is connected if and only if every continuous $\{0,1\}$-valued map is constant.*

Proof. If U, V is a disconnection, then the indicator function 1_U on X is a continuous non-constant map.

Conversely, if $f\colon X \to \{0,1\}$ is a continuous surjection, then $f^{-1}(0)$ and $f^{-1}(1)$ form a disconnection. □

We will write 2 for $\{0,1\}$ (with the discrete topology).

Corollary 14.4. *The one point space is connected.*

Lemma 14.5. *Suppose X is a topological space and $A, A_i, i \in I$ are connected subsets of X.*

If $\forall i \in I\ A_i \cap A \neq \emptyset$ then $A \cup \bigcup_i A_i$ is connected.

Proof. Wlog $X = A \cup \bigcup_i A_i$. Consider a continuous $f\colon X \to 2$. Note that $f|_A$ and $f|_{A_i}$ $(i \in I)$ are continuous and thus constant (by assumption), with values c and $c_i, i \in I$. Fix $i \in I$, $a \in A \cap A_i$ and observe that $c = f(a) = c_i$. Hence f is constant. □

Lemma 14.6. *Suppose X is a topological space and A a connected subset of X and B a subset of X.*

If $A \subseteq B \subseteq \overline{A}$ then B is connected.

Proof. Assume $f\colon B \to 2$ is continuous. Then $f|_A$ is a continuous 2-valued map on the connected space A so constant. But since $B \subseteq \overline{A}^X$ we have $B = \overline{A}^B$ and so by continuity

$$f(B) = f(\overline{A}^B) \subseteq \overline{f(A)}^2 = f(A)$$

is a singleton. □

14.2 Components and Quasicomponents

Definition 14.7. Suppose X is a topological space and $x \in X$.

The component of x (in X) is

$$C_X(x) = \bigcup \{A \subseteq X\colon x \in A \text{ and } A \text{ is connected}\}.$$

The quasicomponent of x (in X) is

$$Q_X(x) = \bigcap \{C \subseteq X\colon x \in C \text{ and } C \text{ is clopen}\}.$$

Lemma 14.8. *Suppose X is a topological space and $x \in X$.*

The component of x is connected and contained in the quasicomponent of x and both the component and the quasicomponent of x are closed in X.

Proof. Write

$$C_X(x) = \{x\} \cup \bigcup \{A \subseteq X : x \in A \text{ and } A \text{ is connected}\}$$

and apply the earlier lemma to see that $C_X(x)$ is connected. But then $\overline{C_X(x)}$ is also connected so contained in $C_X(x)$ and hence they are equal thus closed.

If D is clopen in X and $x \in D$ then $C_X(x) \cap D$ is clopen in $C_X(x)$ and non-empty so must be all of $C_X(x)$. Hence $C_X(x) \subseteq D$, giving $C_X(x) \subseteq Q_X(x)$.

Finally $Q_X(x)$ is an intersection of closed sets, so closed. □

Lemma 14.9. *If X is compact, $\mathcal{C}$ is a family of closed subsets of X and U is open such that $\bigcap \mathcal{C} \subseteq U$ then there is finite $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigcap \mathcal{C}' \subseteq U$.*

Proof. Suppose $\mathcal{C}$ and U are as in the lemma.

Note that $\mathcal{C} \cup \{X \setminus U\}$ is a family of closed subsets of X with empty intersection so that there is a finite subfamily $\mathcal{C}' \cup \{X \setminus U\}$ with empty intersection. But this gives $\bigcap \mathcal{C}' \subseteq U$ as required. □

Theorem 14.10 (Sura-Bura Lemma)**.** *In a compact Hausdorff space, components and quasicomponents coincide.*

Proof. Suppose X is compact Hausdorff and $x \in X$. Write $Q = Q_X(x)$.

It is sufficient to show that Q is connected. So assume that A, B form a Q-disconnection, i.e. A, B are disjoint, non-empty, Q-open and $Q = A \cup B$. Since Q is X-closed, A, B are X-closed and as X is compact Hausdorff so normal, there are disjoint X-open U, V such that $A \subseteq U, B \subseteq V$. Note that in particular $\overline{U} \cap V = \emptyset = U \cap \overline{V}$. Then

$$Q = \bigcap \{C \subseteq X : x \in C \text{ clopen } \} = A \cup B \subseteq U \cup V$$

so by the preceding lemma there is a finite collection $\mathcal{C}$ of clopen sets containing x such that $A \cup B \subseteq C = \bigcap \mathcal{C} \subseteq U \cup V$. Note that C is X-clopen (as a finite intersection of X-clopen sets).

Hence

$$\overline{U \cap C} \subseteq \overline{U} \cap C = \overline{U} \cap ((U \cup V) \cap C) = U \cap C$$

so that $U \cap C$ and (similarly) $V \cap C$ are X-clopen. Since these are disjoint one of them contains x, wlog $x \in U \cap C$ and thus $Q \subseteq U \cap C$ by definition of $Q = Q_X(x)$. But then $B \subseteq V$ so that $B \cap U \cap C = \emptyset$. Hence $B = B \cap Q \subseteq B \cap U \cap C = \emptyset$ contradicting non-emptiness of B. □

15 Disconnectedness

Definition 15.1. Suppose X is a topological space.

X is totally disconnected if and only if every component of X is a singleton.

X is zero-dimensional if and only if X has a basis of clopen sets.

Lemma 15.2. *In a compact Hausdorff space, total disconnectedness and zero-dimensionality are equivalent.*

Proof. Suppose X is compact Hausdorff.

If X is zero-dimensional, let $\mathcal{B}$ be a clopen basis and for each $x \in X$ let $\mathcal{B}_x = \{B \in \mathcal{B} \colon x \in B\}$. As X is Hausdorff we have

$$Q_X(x) \subseteq \bigcap \mathcal{B}_x = \{x\}$$

and since $C_X(x) \subseteq Q_X(x)$ the space X is totally disconnected.

Conversely, assume that X is totally disconnected. We claim that

$$\{C \subseteq X \colon C \text{ clopen } \}$$

is a basis for X.

Let $x \in X$ and $U \ni x$ be open. By the Sura-Bura Lemma

$$Q_X(x) = C_X(x) = \{x\} \subseteq U$$

and, recalling that $Q_X(x)$ is an intersection of clopen sets in the compact Hausdorff space, by Lemma 14.9 there are finitely many clopen sets $\mathcal{C}_x$ (each containing x) with $x \in C = \bigcap \mathcal{C}_x \subseteq U$ and C is clopen. □

16 Stone Duality

Note: this is not a course on Boolean Algebras. I do not expect you to remember the precise axioms and the technical details which are purely Boolean Algebra theoretic for the exam. It is also not a course on Category Theory, so the Category Theoretic statements are just for your information. In the same sense I will ignore the subtle distinction between a space and a homeomorphic copy of it and similarly a Boolean Algebra and an isomorphic copy of it and will just refer to 'relabelling'.

A lot of the material in the remainder of the section is 'routine' algebraic manipulation. I recommend focusing on the operations of the operation (functor) $. \mapsto \mathcal{B}_.$ which turns topological spaces and continuous maps into Boolean Algebras and homomorphisms in the opposite direction and the operation (function) $. \mapsto \mathcal{S}(.)$ which turns Boolean Algebras and homomorphisms into topological spaces and continuous maps again reversing the direction of the maps. These two operations interact in the nicest possible way when restricting the topological side to compact Hausdorff zero-dimensional spaces.

16.1 Boolean Algebras

Definition 16.1. A Boolean Algebra is a set B together with the following functions, constants and relations:

- two binary associative and commutative functions $\wedge, \vee \colon B^2 \to B$ usually written in infix notation $a \wedge b$ instead of $\wedge(a, b)$ and called *conjunction* and *disjunction*;
- a unary *complement* or *negation* $\neg \colon B \to B$ usually written without brackets $\neg a$;
- a constant (element of B) $\mathbf{1}$ and a constant (element of B) $\mathbf{0}$;
- a partial order $\leq$;

Below a, b, c should be read as universally quantified over B.

We require conjunction and disjunction to distribute over each other, i.e. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and satisfy $a \wedge a = a = a \vee a$.

The negation should be self-inverse, i.e. $\neg\neg a = a$, turn conjunctions into disjunctions i.e. $\neg(a \wedge b) = (\neg a) \vee (\neg b)$ and thus also disjunctions into conjunctions.

For our constants we require $a \wedge \neg a = \mathbf{0}$, $a \vee \neg a = \mathbf{1}$, $a \wedge \mathbf{1} = a$, $a \vee \mathbf{1} = \mathbf{1}$, $a \vee \mathbf{0} = a$, $a \wedge \mathbf{0} = \mathbf{0}$ as well as $\neg\mathbf{0} = \mathbf{1}$ and hence $\neg\mathbf{1} = \mathbf{0}$.

The partial order $\leq$ should have greatest element $\mathbf{1}$, least element $\mathbf{0}$ and satisfy $\inf\{a, b\} = a \wedge b$ and $\sup\{a, b\} = a \vee b$ as well as $a \leq b \Leftrightarrow \neg b \leq \neg a$.

We can in fact recover the order from the operations:

Lemma 16.2. *Suppose we have a set B with conjunction, disjunction and complement satisfying the above.*

In this case

$$a = a \wedge b \Leftrightarrow a \wedge \neg b = \mathbf{0} \Leftrightarrow b = a \vee b.$$

Proof. We prove the first if and only if first:

$\Rightarrow$: $a \wedge \neg b = (a \wedge b) \wedge \neg b = a \wedge (b \wedge \neg b) = a \wedge \mathbf{0} = \mathbf{0}$

$\Leftarrow$: $a = a \wedge \mathbf{1} = a \wedge (b \vee \neg b) = (a \wedge b) \vee (a \wedge \neg b) = (a \wedge b) \vee \mathbf{0} = a \wedge b$.

For the second if and only if:

$\Rightarrow$: $b = b \vee \mathbf{0} = b \vee (a \wedge \neg a) = (b \vee a) \wedge (b \vee \neg a) = (b \vee a) \wedge \neg(\neg b \wedge a) = (b \vee a) \wedge \neg\mathbf{0} = (b \vee a) \wedge \mathbf{1} = b \vee a$.

$\Leftarrow$: $\quad a \wedge \neg b = a \wedge \neg(a \vee b) = a \wedge (\neg a \wedge \neg b) = (a \wedge \neg a) \wedge \neg b = \mathbf{0} \wedge \neg b = \mathbf{0}$. $\square$

Lemma 16.3. *If we have a set B with conjunction, disjunction and complement satisfying the above then defining $a \leq b :\Leftrightarrow a = a \wedge b$ gives a partial order turning B into a Boolean algebra.*

Moreover if B was a Boolean Algebra with partial order $\leq'$ then $\leq = \leq'$.

Proof. That $\leq$ is a partial order with greatest element $\mathbf{1}$ and least element $\mathbf{0}$ is straightforward.

That $a \wedge b$ is a lower bound for $\{a, b\}$ is immediate from the definitions. Now suppose $c \leq a$ and $c \leq b$ then $c = a \wedge c = b \wedge c$ so that $c = a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and thus $c \leq a \wedge b$. Hence $a \wedge b$ is the greatest lower bound for $\{a, b\}$ as claimed.

For $a \vee b = \sup\{a, b\}$ use the previous lemma and a similar proof.

The equality $\leq = \leq'$ for Boolean Algebras follows from $a \leq' b \Leftrightarrow a = \inf_{\leq'}\{a, b\}$. $\square$

We can also start with an ordered set and recover the algebra operations.

Definition 16.4. A distributive lattice $(B, \leq)$ is a set B with partial order $\leq$ that has finite infima and suprema (including infima and suprema of $\emptyset$) and thus a least element $\mathbf{0} = \sup\emptyset$ and greatest element $\mathbf{1} = \inf\emptyset$ such that finite infima and suprema distribute over one another.

A distributive lattice B is complemented if and only if $\forall b \in B\ \exists! a \in B\ (a \wedge b = \mathbf{0}$ and $a \vee b = \mathbf{1})$.

Lemma 16.5. *If $(B, \leq)$ is a complemented distributive lattice with greatest element $\mathbf{1}$ and least element $\mathbf{0}$ then setting $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ and defining $\neg a$ as the unique element such that $a \wedge \neg a = \mathbf{0}$ and $a \vee \neg a = \mathbf{1}$ gives a Boolean algebra.*

Proof. Straightforward. $\square$

Definition 16.6. A Boolean Algebra homomorphism is a function $f\colon B \to A$ from a Boolean Algebra B into a Boolean Algebra A that preserves all functions, constants and relations, i.e.

$$\begin{aligned} f(b \wedge_B b') &= f(b) \wedge_A f(b') \\ f(b \vee_B b') &= f(b) \vee_A f(b') \\ f(\neg_B b) &= \neg_A f(b) \\ f(\mathbf{0}_B) &= \mathbf{0}_A \\ f(\mathbf{1}_B) &= \mathbf{1}_A \\ b \leq_B b' &\Rightarrow f(b) \leq_A f(b'). \end{aligned}$$

A Boolean Algebra isomorphism is a Boolean Algebra homomorphism which has an inverse that is also a Boolean Algebra homomorphism.

We will usually drop the subscripts if it is clear from the context which operation we are referring to.

Lemma 16.7. *If f preserves $\wedge$,$\vee$ and $\neg$ then it is a Boolean Algebra homomorphism.*

If f is a bijective Boolean Algebra homomorphism then it is an isomorphism.

Proof. The first statement comes from the construction of $\leq$ from the operations noting that $f(\mathbf{0}_B) = f(b \wedge \neg b) = f(b) \wedge \neg f(b) = \mathbf{0}_A$ and $f(\mathbf{1}_B) = f(b \vee \neg b) = f(b) \vee \neg f(b) = \mathbf{1}_A$ (for any $b \in B$ for example $b = \mathbf{0}_B$).

We check preservation of $\wedge$. If $a, a' \in A$ then writing $b = f^{-1}(a)$, $b' = f^{-1}(a')$ we get $f^{-1}(a \wedge a') = f^{-1}(f(b) \wedge f(b')) = f^{-1}(f(b \wedge b')) = b \wedge b'$. Preservation of $\vee$ and $\neg$ follow similarly. □

Example 16.8. If X is a set then $\mathcal{P}(X)$ with $\wedge = \cap$, $\vee = \cup$, $\neg A = X \setminus A$, $\mathbf{0} = \emptyset$, $\mathbf{1} = X$, $\leq = \subseteq$ is a Boolean Algebra.

If X, Y are sets and $f\colon X \to Y$ is a function then $\mathcal{B}_f\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ given by $\mathcal{B}_f(B) = f^{-1}(B)$ is a Boolean Algebra homomorphism.

The function $\mathcal{P}(X) \to \mathcal{P}(Y)$ given by $A \mapsto f(A)$ is not necessarily a Boolean Algebra homomorphism.

16.2 Turning a Topological Space into a Boolean Algebra

Definition 16.9. Given a topological space X, we write $\mathcal{B}_X$ for the set of clopen subsets of X and equip this with the operations as in the example at the end of the last section to obtain the *Boolean Algebra of clopen subsets of X*.

Given two topological spaces X, Y and a continuous function $f\colon X \to Y$ we let $\mathcal{B}_f\colon \mathcal{B}_Y \to \mathcal{B}_X$ be the Boolean Algebra homomorphism given by $\mathcal{B}_f(B) = f^{-1}(B)$.

Proof. The proofs of the implicit claims that $\mathcal{B}_X$ is a Boolean Algebra, that $\mathcal{B}_f$ maps into $\mathcal{B}_X$ and that it is a homomorphism are straightforward. □

Lemma 16.10. *If X, Y, Z are topological spaces and $f\colon X \to Y, g\colon Y \to Z$ are continuous maps then $\mathcal{B}_{g \circ f} = \mathcal{B}_f \circ \mathcal{B}_g$.*

Also $\mathcal{B}_{\mathrm{id}_X} = \mathrm{id}_{\mathcal{B}_X}$.

Proof. Straightforward. □

The above lemma says that $. \mapsto \mathcal{B}_.$ is a contravariant functor from the category of topological spaces with continuous maps to the category of Boolean Algebras with homomorphisms.

Example 16.11. If $X = \emptyset$ then $\mathcal{B}_X = \{\emptyset\}$ and $\mathbf{0} = \mathbf{1} (= \emptyset = X)$ on the Boolean Algebra side.

If X is a non-empty connected space then $\mathcal{B}_X = \{\emptyset, X\}$ and $\mathbf{0} = \emptyset, \mathbf{1} = X$. This Boolean Algebra is often denoted by $\mathbf{2}$.

If $X = \mathbb{N}$ (with the discrete topology) then $\mathcal{B}_X = \mathcal{P}(\mathbb{N})$.

Already with $X = 2^{\mathbb{N}}$ it is not easy to explicitly describe the Boolean Algebra of clopen subsets of X.

16.3 Turning a Boolean Algebra into a topological space

16.3.1 Filters and Ultrafilters on Boolean Algebras

Definition 16.12. A filter $\mathcal{F}$ on a Boolean Algebra B is a subset of B such that

- $\mathbf{0} \notin \mathcal{F}$ and $\mathcal{F} \neq \emptyset$;
- $a, b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F}$;
- $a \leq b$ and $a \in \mathcal{F} \Rightarrow b \in \mathcal{F}$.

An ultrafilter on B is a filter that is maximal with respect to $\subseteq$.

Proofs for the following results work in the same way as for filters on sets because of Lemma 16.2.

Lemma 16.13. *Suppose B is a Boolean Algebra and that $\mathcal{F}$ is a filter on B. TFAE*

- *$\mathcal{F}$ is an ultrafilter on B;*
- *$\forall b \in B$ $b \in \mathcal{F}$ or $\neg b \in \mathcal{F}$;*
- *$\forall a, b \in B$ $a \vee b \in \mathcal{F} \Rightarrow a \in \mathcal{F}$ or $b \in \mathcal{F}$.*

Lemma 16.14. *Suppose B is a Boolean Algebra.*
If $\mathcal{B} \subseteq B$ is a non-empty set not containing $\mathbf{0}$ and satisfying

$$a, b \in \mathcal{B} \Rightarrow \exists c \in \mathcal{B}\ c \leq a \text{ and } c \leq b$$

then

$$\{b \in B \colon \exists a \in \mathcal{B}\ a \leq b\}$$

is a filter on B containing $\mathcal{B}$.
If $\mathcal{S} \subseteq B$ is a non-empty set such that finite conjunctions of elements of $\mathcal{S}$ are non-zero then

$$\left\{\bigwedge F \colon F \text{ finite } \subseteq \mathcal{S}\right\}$$

satisfies the condition above and hence there is a filter on B containing $\mathcal{S}$.

Two key results for our purposes are

Corollary 16.15. *Suppose B is a Boolean Algebra.*
If $a, b \in B$ such that $a \not\leq b$ then there is a filter on B containing a and $\neg b$.

Lemma 16.16 (Ultrafilter Extension Lemma)**.** *Every filter on a Boolean Algebra B can be extended to an ultrafilter on B.*

16.3.2 The Stone Dual of a Boolean Algebra

With this language and these results in place we can embark on a program similar to the construction of the space of ultrafilters on $\mathbb{N}$ in section 13.1. In fact the construction of the space of ultrafilters on $\mathbb{N}$ is the special case of the below when $B = \mathcal{P}(\mathbb{N})$.

Definition 16.17. Suppose B is a Boolean Algebra.

We write $\mathcal{S}(B)$ for the set of ultrafilters on B.

For $b \in B$ we write $b^\star = \{\mathcal{U} \in \mathcal{S}(B) : b \in \mathcal{U}\}$.

Lemma 16.18. *Suppose B is a Boolean Algebra.*

For $a, b \in B$ we have:

- $(a \wedge b)^\star = a^\star \cap b^\star$;
- $(a \vee b)^\star = a^\star \cup b^\star$;
- $(\neg a)^\star = \mathcal{S}(B) \setminus a^\star$;
- $\mathbf{0}^\star = \emptyset$;
- $\mathbf{1}^\star = \mathcal{S}(B)$.

Thus $\{b^\star : b \in B\}$ is a basis of clopen sets for a compact Hausdorff zero-dimensional topology on $\mathcal{S}(B)$. The resulting topological space is called the Stone space or Stone dual of B and also denoted by $\mathcal{S}(B)$.

Moreover every clopen subset of $\mathcal{S}(B)$ is of the form $b^\star$ for some $b \in B$.

Proof. The equalities are simple calculations using our results on ultrafilters.

It follows that $\{b^\star : b \in B\}$ is a basis of clopen sets for a zero-dimensional topology on $\mathcal{S}(B)$.

This topology is Hausdorff: suppose $\mathcal{U} \neq \mathcal{V}$ are ultrafilters on B. Then swapping $\mathcal{U}$ and $\mathcal{V}$ if necessary, we can find $b \in \mathcal{U} \setminus \mathcal{V}$. As $\mathcal{V}$ is an ultrafilter we have $\neg b \in \mathcal{V} \setminus \mathcal{U}$. Thus $\mathcal{U} \in b^\star$, $\mathcal{V} \in (\neg b)^\star$ and $b^\star \cap (\neg b)^\star = (b \wedge \neg b)^\star = \mathbf{0}^\star = \emptyset$.

For compactness it is enough to check that every non-empty family of basic closed sets (complements of basic open sets) with the finite intersection property has non-empty intersection. Note that every complement of a basic open set is of the form $b^\star$ for some $b \in B$. Thus assume that $\mathcal{C}^\star = \{b_i^\star : i \in I\}$ is a family of basic closed sets with the finite intersection property. Then $\mathcal{C} = \{b_i : i \in I\}$ is a subset of B in which finite conjunctions are non-zero and is thus contained in a filter on B which can be extended to an ultrafilter $\mathcal{U}$ on B. As for every $i \in I$ we have $b_i \in \mathcal{U}$ we get that $\mathcal{U} \in \bigcap \mathcal{C}^\star$ showing that the $\bigcap \mathcal{C}^\star \neq \emptyset$.

For the final assertion, note that every clopen subset D of $\mathcal{S}(B)$ is a union of basic open sets. It is also closed in the compact space $\mathcal{S}(B)$ hence compact and thus D is a union of finitely many basic open sets giving

$$D = b_1^\star \cup \ldots \cup b_n^\star = (b_1 \vee \cdots \vee b_n)^\star$$

(for suitable $b_1, \ldots, b_n \in B$) as required. (Note that if $D = \emptyset$ then $D = \mathbf{0}^\star$.) $\square$

We now turn our attention to Boolean Algebra homomorphisms.

Lemma 16.19. *Suppose A, B are Boolean Algebras and $g\colon B \to A$ a Boolean Algebra homomorphism.*

Then the Stone dual of g,

$$\begin{aligned}&\mathcal{S}(g) : \mathcal{S}(A) \to \mathcal{S}(B)\\ &\mathcal{S}(g)(\mathcal{U}) = \{b \in B\colon g(b) \in \mathcal{U}\}\end{aligned}$$

is a well-defined continuous map between the Stone duals of A and B.

Proof. We start by showing that if $\mathcal{U}$ is an ultrafilter on A then $\mathcal{S}(g)(\mathcal{U})$ is an ultrafilter using that g is a homomorphism.

Since $g(\mathbf{0}_B) = \mathbf{0}_A \notin \mathcal{U}$ we have $\mathbf{0}_B \notin \mathcal{S}(g)(\mathcal{U})$.

Since $g(\mathbf{1}_B) = \mathbf{1}_A \in \mathcal{U}$ we have $\mathbf{1}_B \in \mathcal{S}(g)(\mathcal{U})$.

If $b_1, b_2 \in \mathcal{S}(g)(\mathcal{U})$ then $g(b_1), g(b_2) \in \mathcal{U}$ so that $g(b_1 \wedge_B b_2) = g(b_1) \wedge_A g(b_2) \in \mathcal{U}$ so that $b_1 \wedge_B b_2 \in \mathcal{S}(g)(\mathcal{U})$.

If $b_1 \in \mathcal{S}(g)(\mathcal{U})$ and $b_1 \leq b_2 \in B$ then $g(b_1) \in \mathcal{U}$ and $g(b_1) \leq g(b_2)$ so that $g(b_2) \in \mathcal{U}$ and thus $b_2 \in \mathcal{S}(g)(\mathcal{U})$.

Hence $\mathcal{S}(g)(\mathcal{U})$ is a filter on B.

Now let $b \in B$. Then $g(\neg b) = \neg g(b)$ so one of $g(b)$ and $g(\neg b)$ is in $\mathcal{U}$ and hence one of b and $\neg b$ is in $\mathcal{S}(g)(\mathcal{U})$. Thus $\mathcal{S}(g)(\mathcal{U})$ is an ultrafilter on B and $\mathcal{S}(g)$ is well-defined.

For continuity of $\mathcal{S}(g)$ we compute

$$\begin{aligned}\mathcal{S}(g)^{-1}(b^\star) &= \{\mathcal{U} \in \mathcal{S}(A) : \mathcal{S}(g)(\mathcal{U}) \in b^\star\}\\ &= \{\mathcal{U} \in \mathcal{S}(A) : b \in \mathcal{S}(g)(\mathcal{U})\}\\ &= \{\mathcal{U} \in \mathcal{S}(A) : g(b) \in \mathcal{U}\}\\ &= g(b)^\star\end{aligned}$$

and conclude that preimages of basic open sets are (basic) open so that $\mathcal{S}(g)$ is continuous. □

Lemma 16.20. *Suppose A, B, C are Boolean Algebras and $g\colon C \to B, g'\colon B \to A$ are Boolean Algebra homomorphisms. Then $\mathcal{S}(g' \circ g) = \mathcal{S}(g) \circ \mathcal{S}(g')$ and $\mathcal{S}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{S}(A)}$.*

The above results can be summarized by saying that $. \mapsto \mathcal{S}(.)$ is a contravariant functor from the category of Boolean Algebras with homomorphisms to the category of compact Hausdorff zero-dimensional spaces with continuous maps.

16.4 Going back and forth, aka equivalence of categories

Lemma 16.21. *Suppose X, Y are compact Hausdorff zero-dimensional spaces.*

The function

$$\begin{aligned} h_X : X &\to \mathcal{S}(\mathcal{B}_X) \\ h_X(x) &= \{C \in \mathcal{B}_X : x \in C\} \end{aligned}$$

is a homeomorphism. When X is clear from the context we drop the subscript.

Moreover if $f : X \to Y$ is a continuous map then

$$\mathcal{S}(\mathcal{B}_f) \circ h_X = h_Y \circ f.$$

Proof. As mentioned in the statement we drop the subscript of h in the first part of the proof.

Verifying that $h(x)$ is a filter on $\mathcal{B}_X$ is straightforward.

Since for every $C \in \mathcal{B}_X$ exactly one of $x \in C$ and $x \in X \setminus C$ is true, $h(x)$ is in fact an ultrafilter and thus h is well-defined.

It is injective since if $x \neq x'$ in X then by zero-dimensionality and Hausdorffness there is a clopen $C \subseteq X$ with $x \in C \not\ni x'$ and thus $C \in h(x) \setminus h(x')$.

It is surjective since for an ultrafilter $\mathcal{U}$ of clopen subsets of X we note that $\mathcal{U}$ consists of closed subsets of X and has the finite intersection property so that by compactness of X we can find $x \in \bigcap \mathcal{U}$ and thus $\mathcal{U} \subseteq h(x)$. Equality follows from maximality of $\mathcal{U}$.

It is continuous since for a basic clopen $C^\star = \{\mathcal{U} \in \mathcal{S}(\mathcal{B}_X) : C \in \mathcal{U}\}$ we have $h(x) \in C^\star \Leftrightarrow C \in h(x) \Leftrightarrow x \in C$ so that $h^{-1}(C^\star)$ is open in X. Thus h is continuous.

But now h is a continuous bijection from a compact space to a Hausdorff space and hence a homeomorphism as claimed.

For the moreover, fix $x \in X$ and compute

$$\begin{aligned} \mathcal{S}(\mathcal{B}_f)(h_X(x)) &= \mathcal{S}(\mathcal{B}_f)(\{C \in \mathcal{B}_X : x \in C\}) \\ &= \{D \in \mathcal{B}_Y : x \in \mathcal{B}_f(D)\} \\ &= \left\{D \in \mathcal{B}_Y : x \in f^{-1}(D)\right\} \\ &= \{D \in \mathcal{B}_Y : f(x) \in D\} \\ &= h_Y(f(x)). \end{aligned}$$

□

Note it is not difficult and a good exercise to show directly that h is open. You should also think about which parts of the proof survives if we drop one or multiple of compactness, Hausdorffness or zero-dimensionality from the assumptions.

Lemma 16.22. *Suppose A, B are Boolean Algebras.*

The function

$$\begin{aligned} g_B : B &\to \mathcal{B}_{\mathcal{S}(B)} \\ g_B(b) &= b^\star \end{aligned}$$

is an isomorphism. When B is clear from the context we drop the subscript.

Moreover, if $f\colon B \to A$ is a Boolean Algebra homomorphism then

$$\mathcal{B}_{\mathcal{S}(f)} \circ g_B = g_A \circ f.$$

Proof. Again, we drop the subscript in the first part of the proof.

For $b \in B$ we have that $b^\star$ is a clopen subset of $\mathcal{S}(B)$ and hence g is well-defined.

That g is a homomorphism follows directly from our computations for $.^\star$.

That g is surjective follows since every clopen subset of $\mathcal{S}(B)$ is of the form $b^\star$ for some $b \in B$ as shown in the Lemma defining $\mathcal{S}(B)$.

For injectivity, assume $b \neq b'$ are elements of B. Then one of $b \not\leq b'$ or $b' \not\leq b$. Switching b and b' if necessary, assume the former. In that case by Corollary 16.15 and the Ultrafilter Extension Lemma immediately afterwards we can find an ultrafilter $\mathcal{U} \ni b, \neg b'$ and then $\mathcal{U} \in b^\star \setminus (b')^\star$.

Thus g is a bijective homomorphism thus an isomorphism.

For the moreover we compute

$$\begin{aligned}
\mathcal{B}_{\mathcal{S}(f)}(g_B(b)) &= \mathcal{B}_{\mathcal{S}(f)}(b^\star) \\
&= \mathcal{S}(f)^{-1}(b^\star) \\
&= \{\mathcal{U} \in \mathcal{S}(A) : \mathcal{S}(f)(\mathcal{U}) \in b^\star\} \\
&= \{\mathcal{U} \in \mathcal{S}(A) : b \in \mathcal{S}(f)(\mathcal{U})\} \\
&= \{\mathcal{U} \in \mathcal{S}(A) : f(b) \in \mathcal{U}\} \\
&= (f(b))^\star \\
&= g_A(f(b)).
\end{aligned}$$

□

Note that by the previous two results every zero-dimensional compact Hausdorff space X is (homeomorphic to) the Stone dual of a Boolean Algebra (namely of $\mathcal{B}_X$) and every Boolean Algebra is (isomorphic to) the collection of clopen subsets of a zero-dimensional compact Hausdorff space X.

Definition 16.23. A Stone space is a zero-dimensional compact Hausdorff topological space (or equivalently homeomorphic to the Stone space of some Boolean Algebra).

Further every continuous map between Stone spaces arises as the Stone dual of a unique Boolean Algebra homomorphism and every Boolean Algebra homomorphism arises as $\mathcal{B}_f$ for a unique continuous map between Stone spaces (up to homeomorphic respectively isomorphic relabelling via h and g).

Finally the relabellings h and g commute with compositions in the appropriate ways.

Or expressed in the language of Category Theory:

Theorem 16.24 (Stone Duality)**.** *There is a contravariant equivalence between the categories of Stone spaces with continuous maps and Boolean Algebras with homomorphisms.*

Another corollary of our work is

Theorem 16.25 (Stone's Representation Theorem)**.** *Every Boolean Algebra is isomorphic to a subalgebra of* $\mathcal{P}(X)$ *for a suitable set* X*.*