**You said:**

I've written a document containing some information about the theta method for solving ODEs. Please can you look at it and check the maths is right, check the spelling and grammar are accurate and give feedback on the structure?

**ChatGPT said:**

Of course! Please **upload your document** (Word, PDF or another readable format), and I'll review it carefully.

I'll focus on three main areas:

1. **Mathematical accuracy**

- Check the derivation and formulation of the $\theta$-method for solving ordinary differential equations (ODEs).
- Verify equations, algebraic manipulations and numerical examples.
- Check statements about consistency, order of accuracy, stability and convergence, where applicable.
- Identify any mathematical errors, missing assumptions or explanations that could be misleading.

2. **Spelling, grammar and mathematical notation**

- Correct spelling, punctuation and grammatical errors using British English.
- Check that mathematical terminology and notation are consistent.
- Suggest improvements to sentence clarity while preserving your writing style.

3. **Structure and presentation**

- Assess whether the material follows a logical progression.
- Check whether definitions, derivations, examples and conclusions are introduced in an appropriate order.
- Identify sections that need more explanation or could be made more concise.
- Suggest improvements to make the document easier for its intended audience to understand.

**How I'll present the feedback**

I'll give you a structured review that distinguishes between:

- **Mathematical errors** that should be corrected.

- **Language and notation corrections** that improve accuracy and readability.
- **Optional improvements** to the explanations, structure and presentation.

For each significant issue, I'll reference the relevant section or equation, explain the problem and suggest a specific correction.

I'll also give an **overall assessment** of the document, highlighting what works well and which areas would benefit most from revision.

One useful detail: **who is the document intended for?** For example, first-year undergraduates, second-year mathematics students or a more advanced numerical analysis course. This will help me judge whether the mathematical explanations are sufficiently rigorous and appropriately detailed.

Upload it whenever you're ready.

**You said:**

PDF **bad_document.pdf**
PDF

I've uploaded the document. The summary is just meant to be a reminder of the key points of the method.

Worked for 14s >

**ChatGPT said:**

I've reviewed your six-page document, *Using the θ-method to solve ODEs*, including the mathematical derivations, numerical results, spelling, grammar and overall structure.

Given your clarification that this is intended as a **concise reminder of the key points**, rather than a comprehensive introduction to numerical methods, I think the level of detail is broadly appropriate. You don't need to add substantial background material.

My main finding is that **the central mathematics is largely correct, but there are a few errors and some places where the mathematical reasoning needs to be tightened**. There are also several straightforward spelling, grammar and LaTeX referencing issues.

## 1. Mathematical accuracy

**Overall assessment** Corrections needed

The θ-method formula, the leading truncation-error term and the overall convergence argument are correct. The main concerns are the treatment of remainder terms, the assumptions required for the error bound, and the interpretation of the numerical results.

### 1.1 Definition of the θ-method — page 2

Your formula is correct:

$$\frac{U_{n+1} - U_n}{\Delta t} = \theta f(t_{n+1}, U_{n+1}) + (1-\theta) f(t_n, U_n).$$

The three special cases are also correctly identified:

- $\theta = 0$: explicit Euler.
- $\theta = 1$: implicit Euler.
- $\theta = \frac{1}{2}$: Crank–Nicolson (also called the trapezoidal method).

One small mathematical clarification: you specify $N \geq 2$, but the method is also defined for $N = 1$. There is no need to impose $N \geq 2$.

### 1.2 Truncation-error derivation — pages 2–3

Your final expression is correct:

$$\boxed{T_n = \frac{\Delta t}{2}(1-2\theta)u''(t_{n+1/2}) + O(\Delta t^2).}$$

However, there is an issue in the substitution step on page 3, labelled equation (4).

You expand $u(t_n)$ and $u(t_{n+1})$ to second order, with remainders $O(\Delta t^3)$. When their difference is divided by $\Delta t$, these remainders become $O(\Delta t^2)$, not $O(\Delta t^3)$.

Furthermore, your expansions of $u'(t_n)$ and $u'(t_{n+1})$ already have remainders of order $O(\Delta t^2)$.

Consequently, the remainder in equation (4) should be

$$\boxed{O(\Delta t^2)}$$

rather than $O(\Delta t^4)$.

This is an important correction because the remainder determines the order of accuracy of Crank–Nicolson.

**A further improvement:** your statement that the $O(\Delta t^2)$ terms do not cancel for any $\theta$ is a little too strong without qualification. They do not cancel *identically for general smooth solutions*, but they can vanish for particular solutions.

In fact, with sufficient smoothness, the midpoint expansion gives the more informative expression

$$T_n = \frac{1-2\theta}{2}\Delta t\, u''(t_{n+1/2}) - \frac{\Delta t^2}{12}u'''(t_{n+1/2}) + O(\Delta t^3).$$

The coefficient of $u'''$ is independent of $\theta$. This confirms that $\theta = \frac{1}{2}$ is second-order accurate in general.

You don't need to include this longer expansion in a short summary, but it is a useful way to verify your conclusion.

### 1.3 Terminology: truncation error and local error

In your introduction you write that you derive the truncation error and then use it to derive an expression for the local error.

However, Section 2.1 actually derives a bound for the **pointwise (global) error**

$$e_n = u(t_n) - U_n.$$

This is not the same as the local truncation error.

I recommend changing the introduction to say that you derive the truncation error and then use it to obtain a bound on the global error.

Your existing heading, *Pointwise Errors*, is appropriate.

### 1.4 Global error estimate — pages 3–5

This is one of the stronger parts of the document. Your approach is standard and the algebra leading to equation (10) is correct, subject to the assumptions below.

You correctly obtain

$$(1 - L\theta\Delta t)|e_{n+1}| \leq (1 + L(1-\theta)\Delta t)|e_n| + \Delta t\, T_{\max}.$$

There are three points worth correcting.

**First: the step-size restriction.**

You write that the rearrangement holds for $\Delta t \ll 1$. The precise requirement is

$$\boxed{1 - L\theta\Delta t > 0.}$$

This is necessary when dividing the inequality by $1 - L\theta\Delta t$.

You could simply write:

"Provided that $L\theta\Delta t < 1$, we may rearrange this inequality to obtain..."

**Second: the definition of $T_{\max}$.**

You define

$$T_{\max} = \max_{0 \leq n \leq N} |T_n|.$$

However, $T_n$ is only defined for $n = 0, \ldots, N-1$. Therefore, this should be

$$\boxed{T_{\max} = \max_{0 \leq n \leq N-1} |T_n|.}$$

**Third: the assumptions.**

Your introduction assumes that $f(t,u)$ is bounded, but this is not necessary for the particular error estimate you derive. More importantly, you need sufficient smoothness of the exact solution for your Taylor expansions and uniform truncation-error estimates.

For a concise summary, I would suggest stating that $f$ is Lipschitz continuous in $u$, and that the exact solution is sufficiently smooth on $[0, T]$.

Your final estimate,

$$\boxed{|e_n| \leq \frac{T_{\max}}{L}\left[\exp\left(\frac{LT}{1-L\theta\Delta t}\right)-1\right],}$$

is correct for $L > 0$, $e_0 = 0$, and the step-size restriction above.

For completeness, if $L = 0$, the corresponding bound is $|e_n| \leq TT_{\max}$.

Your conclusion that the global error has the same asymptotic order as the truncation error is justified, provided the solution is sufficiently smooth and $T$ is fixed as $\Delta t \to 0$.

**1.5 Numerical example — pages 5–6**

Your numerical example is a sensible choice for illustrating convergence.

However, there are two issues.

**Incorrect method identified on page 6.**

You write:

"whereas the implicit Euler scheme is $O(\Delta t^2)$."

This should say **Crank–Nicolson**, not implicit Euler.

Your Figure 3 caption correctly identifies Crank–Nicolson as second order.

**Reference solution and convergence plot.**

You use Crank–Nicolson with $N = 10000$ as a reference solution. This is a reasonable numerical approach, but the reference is not exact.

In Figure 3, the Crank–Nicolson error decreases to approximately $10^{-11}$ at the right-hand end of the plot. Because your reference solution also uses $N = 10000$, the smallest plotted errors may be affected by the reference solution's accuracy.

I would recommend using a more accurate independent reference, for example an adaptive high-order ODE solver with tight tolerances, particularly if you want to demonstrate second-order convergence over the full plotted range.

Alternatively, you could restrict the convergence plot to values of $N$ well below 10000.

One further point: the equation is printed as

$$\frac{du}{dt} = \log\log(4 + u^2).$$

If this is intended to mean the nested natural logarithm, it would be clearer to write

$$\frac{du}{dt} = \log\big(\log(4 + u^2)\big).$$

## 2. Spelling, grammar and notation

The writing is generally understandable, but I found several corrections worth making.

| Location | Original | Suggested correction |
|---|---|---|
| Page 1 | ODE's | ODEs |
| Page 1 | deriving it's truncation error | deriving its truncation error |
| Page 1 | rates of convergence are achieve | rates of convergence are achieved |
| Page 1 | spacial dependence | spatial dependence |
| Page 1 | $(t, u)$, then $f(u, t)$ | Use $f(t, u)$ consistently |
| Page 2 | Where $\theta$ is between 0 and 1 | where $0 \leq \theta \leq 1$ |
| Page 2 | 3 values of $\theta$ | Three values of $\theta$ |
| Page 2 | Crank Nicolson | Crank–Nicolson |
| Page 3 | re-write | rewrite |
| Page 3 | the the $O(\Delta t^2)$ terms | the $O(\Delta t^2)$ terms |
| Page 6 | first oder accurate | first-order accurate |
| Throughout | re-arrange | rearrange |
| Throughout | timestep | time step |

I would also standardise the following conventions:

- Use "θ-method" rather than alternating between "θ-method" and "theta-method".
- Use "Crank–Nicolson" consistently, including in figure legends.
- Use "Equation (1)" or "equation (1)" consistently.
- Use British English throughout, as you have already done with "summarise".

**LaTeX and formatting issues**

A few problems appear to come from the LaTeX source rather than the mathematical writing itself.

**Page 3:** The sentence "Recall the definition of the θ-method (??)" contains an unresolved cross-reference. This needs fixing.

**Page 2:** The heading "2.0.1 Truncation Error" looks odd because there is no Section 2.0. I would make this Section 2.1, with *Pointwise Errors* becoming Section 2.2.

**Page 5:** The two solution plots are quite small relative to the page. Increasing their width would make the curves and legends easier to read.

**References:** The second reference is incomplete. I would add the publisher and year and standardise the author-name formatting between the two entries.

## 3. Structure and organisation

For a short revision summary, I think the overall structure is good:

- **Introduction and problem definition**
- **Definition of the θ-method**
- **Truncation-error derivation**
- **Global error estimate**
- **Numerical demonstration and conclusion**

The progression from the method itself, through theoretical error analysis, to numerical verification is logical.

I would make only a few structural adjustments.

**Keep the introduction short.** The paragraph on the heat equation is not really needed for a summary focused on ODEs. You could remove it without losing anything essential.

**Highlight the two main theoretical results.** The truncation-error formula and the global error estimate are the most important results. Displaying them prominently would make the document more useful for revision.

**Consider combining Figures 1 and 2.** The numerical solution plot and reference solution plot communicate very similar information. A single plot comparing the numerical methods with the reference solution would be more economical.

**Make the conclusion more useful as a reminder.** Rather than repeating the document's contents, you could summarise the key properties of the method in a compact table:

| Method | $\theta$ | Global order | Implicit? |
|---|---|---|---|
| Forward Euler | $0$ | 1 | No |
| Crank–Nicolson | $1/2$ | 2 | Yes |
| Backward Euler | $1$ | 1 | Yes |
| General θ-method | Other values in $(0, 1)$ | 1 | Yes |

These are the standard orders for sufficiently smooth solutions.

## 4. Suggested revised introduction

Here is a more concise version that better reflects the content of your document.

Writing

### 1 Introduction

In this report, we summarise the θ-method for solving initial value problems for ordinary differential equations (ODEs). We introduce the method and derive its truncation error, before using this to establish a bound on the pointwise error. Finally, we present a numerical example to demonstrate the expected orders of convergence.

We consider initial value problems of the form

[
\frac{du}{dt}=f(t,u), \qquad t\in(0,T],
]

subject to the initial condition (u(0)=u_0). We assume that (f) satisfies a Lipschitz condition in its second argument and that the exact solution is sufficiently smooth on ([0,T]).

## 5. Overall verdict

Your document achieves its main purpose: it provides a relatively concise account of the θ-method, including both the theoretical justification for its accuracy and a numerical illustration.

The most important changes, in priority order, are:

1. **Correct the remainder term in equation (4)** on page 3.
2. **State the step-size restriction** $L\theta\Delta t < 1$ in the global error argument.
3. **Correct the reference to implicit Euler** being second order on page 6.
4. **Clarify the use of the numerical reference solution**, especially near $N = 10000$.
5. Fix the spelling, grammar and LaTeX referencing errors identified above.

I would not recommend substantially expanding the document. The derivations are already detailed enough for a mathematical revision summary, and the structure is sensible.

**The main mathematical results are sound once the corrections above are made.** In particular, the distinction between first-order accuracy for $\theta \neq \frac{1}{2}$ and second-order accuracy for $\theta = \frac{1}{2}$ is correctly established, subject to the usual smoothness assumptions.