Analysis I

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0. INTRODUCTORY MATERIAL

0.1 Syllabus

Real numbers: arithmetic, ordering, suprema, infima; the real numbers as a complete ordered field. The triangle inequality. [3]

Definition of a countable set. The countability of the rational numbers. The reals are uncountable. The complex number system. [1.5]

Sequences of real or complex numbers. Definition of a limit of a sequence of numbers. Limits and inequalities. The algebra of limits. Order notation: O, o. Bounded monotone sequences converge. [4]

Subsequences; a proof that every subsequence of a convergent sequence converges to the same limit. Bolzano–Weierstrass Theorem. Cauchy's convergence criterion. [2]

Series of real or complex numbers. Convergence of series. Simple examples to include geometric progressions and some power series. Absolute convergence, Comparison Test, Ratio Test, Integral Test. Alternating Series Test. [2]

Power series, radius of convergence. Examples to include definition of and relationships between exponential, trigonometric functions and hyperbolic functions. [2.5]

0.2 Reading list

(1) *Introduction to real analysis,* Robert Bartle, Donald Sherbert, Wiley 4th ed 2011
(2) *Real analysis and infinity,* H. Sedghat, OUP 2022
(3) *Guide to analysis*, Mary Hart, Macmillan 2nd ed 2001
(4) *A radical approach to real analysis,* David Bressoud, MAA 2007
(5) *Mathematical analysis: a straightforward approach,* K. G. Binmore, CUP 2nd ed 1982
(6) *Mathematical analysis,* Tom Apostol, Pearson, 2nd ed 1974

0.3 Further Reading

(1) *Understanding analysis*, Stephen Abbott, Springer 2nd ed 2015
(2) *A very short introduction to mathematical analysis,* Richard Earl, OUP 2023
(3) *The real numbers: an introduction to set theory and analysis,* John Stillwell, Springer 2016
(4) *A companion to analysis,* T. W. Körner, AMS 2003

0.4 Introduction and Historical Background

The story of analysis, as a separate subject within mathematics, begins in the nineteenth century. Three mathematicians, who might reasonably share the title of 'fathers of analysis' are Bolzano, Cauchy and Weierstrass, who were working around 1817, 1821 and 1861 respectively. Their names each arise in this *Analysis I* course but, however seminal their work, it was around 150 years later than ideal.

During your degree you will see that analysis has ideas and themes of its own – this will be especially apparent when you meet *metric spaces* and *complex analysis* in the second year. But the biggest need driving a focus on analysis in the nineteenth century was the lack of rigor in mathematics, particularly with regard to calculus. The origins of calculus are typically associated with the work of Leibniz and Newton in the 1680s, though their work built on the work of many others, especially Fermat. Their work was a great leap forward but was definitely not the final word in the development of the calculus: neither Leibniz nor Newton could frame their work without reference to infinitesimals, fluxions or other ill-defined terms. At this time there were still no formal definitions of *convergence* or *limit*. Given the widespread impact of calculus within mathematics and science, there was a desperate need for rigorous foundations; Newton's and Leibniz's methods had been widely applied with success for 100 years or so, but by the nineteenth century the need for more clarity and careful definitions was becoming paramount.

Euler, in his seminal *Introductio in Analysin Infinitorum* of 1748, moved the discussion forward significantly. Euler placed functions – as defined locally by power series – at the centre of his work, and during his life calculated an amazing array of infinite sums. But still there was no formal defintion of a limit. And whilst it was known that the general solution to the wave equation was

$$y(x,t) = f(x+ct) + g(x-ct),$$

where f and g are *arbitrary functions*, what did this phrase mean? Certainly, in the case of a plucked string, something different from Euler's definition of what a function is. The physical and mathematical descriptions just didn't quite match.

Around 1816-17 in Prague, the mathematician and philosopher, Bernard Bolzano, first gave the modern definition of a limit, introduced so-called ε-δ analysis, defined the notion of 'greatest lower bound' and proved the *intermediate value theorem* (HT) and *Bolzano-Weierstrass theorem*. Unfortunately Bolzano's work would go unrecognized during his lifetime and only surface some fifty years later. Instead the next (widely recognized) step forward would be Augustin-Louis Cauchy's *Cours d'Analyse* of 1821. The text implicitly involved ε-δ arguments, and defined continuous functions (HT), but still Cauchy referred to infinitely small quantities and his proof of the Fundamental Theorem of Calculus (TT) is incomplete through a lack of appreciation of uniform continuity (HT).

Finally we come to the work of Karl Weierstrass, lecturing in Berlin around 1861. Weierstrass explicitly employs ε-δ arguments and defines convergence without any reference to infinitesimals. He appreciated the difference between pointwise and uniform convergence of functions (HT), and between continuity and uniform continuity (HT). He proved that a continuous function is bounded on a closed bounded interval and achieves its bounds – which is often referred

to as *Weierstrass' Theorem* (HT) – and he gave an example of a function which is continuous but nowhere differentiable.

As far as this Analysis I course goes, the above describes the relevant history of the topics you will meet (and more). But still there would be much work needed, even within real analysis, especially in the development of theories of integration (TT) (Riemann 1854, Darboux 1875, Lebesgue 1902).

0.5 Motivation and Pedagogy

Rigour is at the heart of the three Prelims real analysis courses. These courses, more than any others, instil a care of exposition and proof from foundational axioms. Given that many of the results proven may seem obvious, some students (not unreasonably) find real analysis rather dry and pedantic. This is also unfair to the broader subject, analysis, as it has many interesting ideas of its own and doesn't merely exist, say, to put calculus on a firmer footing. So why do we teach real analysis in the first year?

The worth of a mathematics degree boils down to two, equally imporant, aims: the learning of, understanding of, and facility with a wide range of concepts and methods, that provide a diverse toolkit for addressing physical and logical problems; *and* the means of exposition to be able to share those ideas, especially when technical language is necessary, to convince others of your bulletproof arguments.

A grounding in real analysis, being able to rigorously reason from precisely stated axioms, and an ability to extend/adapt that thinking to new problems, are key aims of these courses. The mathematical results you will meet in this course are at least a century old, and these results are no longer in any doubt, but the technical communication skills and nuance of thought that you learn will be paramount throughout your degree and beyond. In due course, you will be proving results that don't seem obvious, are subtle or even counter-intuitive; hopefully you will at least appreciate where you originally learnt those skills.

The analysis courses present the axioms of the real numbers – the field, order and completeness axioms – and seek to build a comprehensive theory of real analysis with each result building on what has been previously demonstrated. Early we may be proving that every real square is non-negative, or demonstrating the existence of $\sqrt{2}$, but we will quickly develop an array of theory that will allow us to deal with infinite series and define the elementary functions.

Whilst we will stick to that creed as much as possible, ultimately it is a pedagogical pipe dream and there will be good reasons to step off that path on occasion. For example, to have diverse enough examples of sequences we will early on refer to functions like sine, cosine and logarithm, even though we won't define them until the end of the course; the existence of the real logarithm will rely of results from next term; differentiating power series likewise relies on Hilary results; and we will introduce the integral test at the end of term, but not develop a theory of integration till Trinity. But the alternative – leaving the integral test, which is a test for convergence of infinite series, until Trinity term makes no pedagogical sense.

More disingenuously, the completeness axiom – *the* fundamental axiom to the real numbers – states that every non-empty bounded set of real numbers has a least upper bound. Yet you

won't find in these notes quite what we mean by a set; that is rather brushed under the carpet. You can find out more about set theory in the third year, if you wish, but if we insisted on a treatment of the ZF (Zermelo-Fraenkel) axioms first then the pure mathematics streams would take even longer to develop.

0.6 Notation

IMPORTANT SETS

$\mathbb{N}$ – the set of *natural* numbers $\{0, 1, 2, \ldots\}$.
$\mathbb{Z}$ – the set of *integers* $\{0, \pm 1, \pm 2, \ldots\}$.
$\mathbb{Q}$ – the set of *rational* numbers.
$\mathbb{R}$ – the set of *real* numbers.
$\mathbb{C}$ – the set of *complex* numbers.
$\mathbb{Z}_n$ – the *integers, modulo* $n \geqslant 2$.
$\mathbb{R}^n$ – *n-dimensional real space* – the set of all real n-tuples $(x_1, x_2, \ldots, x_n)$.
$\mathbb{R}[x]$ – the set of *polynomials* in x with real coefficients.
$\varnothing$ – the empty set

For $a, b \in \mathbb{R}$ with $a < b$ we define
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.
$(a, b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$.
$[a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$.
$[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$.

SET THEORETIC NOTATION

$X \cup Y$ – the *union* of X and Y – $\{s \mid s \in X \text{ or } s \in Y\}$.
$X \cap Y$ – the *intersection* of X and Y – $\{s \mid s \in X \text{ and } s \in Y\}$.
$X \times Y$ – the *Cartesian product* of X and Y – $\{(x, y) \mid x \in X \text{ and } y \in Y\}$.
Y^c – the *complement* of a subset Y.
$X \backslash Y$ – the *complement* of Y in X – $\{s \mid s \in X \text{ and } s \notin Y\}$.
$\mathcal{P}(X)$ – the *power set* of a set X, that is the set of subsets of X.
$\in$ – is an *element* of, e.g. $\sqrt{2} \in \mathbb{R}$ and $\pi \notin \mathbb{Q}$.
$\subset, \subseteq$ – is a *subset* of, e.g. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
$f\colon X \to Y$ – f is a *function*, map, mapping from a set X (the domain) to a set Y (the codomain).
$f\colon X \hookrightarrow Y$ – f is a *injective function* from a set X to a set Y.
$f\colon X \twoheadrightarrow Y$ – f is a *surjective function* from a set X to a set Y.
$f(X)$ – the *image* or *range* of the function f – i.e. the set $\{f(x) \mid x \in X\}$.
$g \circ f$ – the *composition* of the maps g and f – apply f first and then g.
$|X|$ – the *cardinality* (size) of the set X.

LOGICAL NOTATION

$\forall$	for all	$\implies$	implies, is sufficient for, only if
$\exists$	there exists	$\impliedby$	is implied by, is necessary for, if
$\exists!$	there exists unique	$\iff$	if and only if, is logically equivalent to
$\neg$	negation, not	: or $\mid$ or s.t.	such that
$\vee$	or	$\square$ or QED	found at the end of a proof
$\wedge$	and		

ANALYTICAL NOTATION

$\operatorname{Re} z$ – the real part of a complex number.
$\operatorname{Im} z$ – the imaginary part of a complex number.
(x_n) , $(x_n)_1^\infty$ – a sequence of elements from a set, usually real or complex numbers
x_k – the kth term of a sequence (x_n)
$x_n \to L$ – the sequence (x_n) converges to L.
$\lim x_n$ – the limit of the convergent sequence (x_n) .
$(\exists\, N)\ (\forall n \geqslant N)$ – eventually, or for a tail of the natural numbers
$(\forall\, N)\ (\exists n \geqslant N)$ – for infinitely many, or for arbitrarily large, natural numbers
AC – absolutely convergent
$\sum x_n$ – the infinite series $x_1 + x_2 + x_3 + \cdots$, whether convergent or not.
$\sum_1^\infty x_n$ – when denoting a (real or complex) number, the infinite sum of the series.
$O(n)$ – large O notation.
$o(n)$ – small o notation.

The Greek Alphabet

A, α	alpha	H, η	eta	N, ν	nu	T, τ	tau
B, β	beta	Θ, θ	theta	Ξ, ξ	xi	Y, υ	upsilon
Γ, γ	gamma	I, ι	iota	O, o	omicron	Φ, ϕ, φ	phi
Δ, δ	delta	K, κ	kappa	Π, π	pi	X, χ	chi
E, ϵ	epsilon	Λ, λ	lambda	P, ρ	rho	Ψ, ψ	psi
Z, ζ	zeta	M, μ	mu	$\Sigma, \sigma, \varsigma$	sigma	$, \omega$	omega

0.7 The Real Number Axioms

The Field Axioms

For every pair of real numbers $a, b \in \mathbb{R}$ there is a unique real number $a + b$, called their 'sum'.
For every pair of real numbers $a, b \in \mathbb{R}$ there is a unique real number $a \times b$, called their 'product'.
For real number $a \in \mathbb{R}$ there is a unique real number $-a$, called its 'negative' or 'additive inverse'.
For real number $a \in \mathbb{R}$, with $a \neq 0$, there is a unique real number a^{-1}, called its 'reciprocal' or 'multiplicative inverse'.
There is a special element $0 \in \mathbb{R}$ called 'zero' or 'the additive identity'.
There is a special element $1 \in \mathbb{R}$ called 'one' or 'the multiplicative identity'.

The following hold for all real numbers a, b, c:

A1	$a + b = b + a$	[+ is commutative]
A2	$a + (b + c) = (a + b) + c$	[+ is associative]
A3	$a + 0 = a$	[additive identity]
A4	$a + (-a) = 0$	[additive inverse]
M1	$a \times b = b \times a$	[× is commutative]
M2	$a \times (b \times c) = (a \times b) \times c$	[× is associative]
M3	$a \times 1 = a$	[multiplicative identity]
M4	If $a \neq 0$ then $a \times a^{-1} = 1$	[multiplicative inverse]
D	$a \times (b + c) = a \times b + a \times c$	[× distributes over +]
Z	$0 \neq 1$	[to avoid total collapse]

Notation: We write $\begin{cases} ab & \text{for} \quad a \times b \\ a - b & \text{for} \quad a + (-b); \\ a/b & \text{for} \quad a \times b^{-1}; \end{cases}$

The Order Axioms

There exists a subset $\mathbb{P}$ of $\mathbb{R}$ called the 'positive numbers' such that for all a, b in $\mathbb{R}$:

P1	If $a \in \mathbb{P}$ and $b \in \mathbb{P}$ then $a + b \in \mathbb{P}$.	[addition and the order]
P2	If $a \in \mathbb{P}$ and $b \in \mathbb{P}$ then $a \times b \in \mathbb{P}$.	[multiplication and the order]
P3	Exactly one of $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$ is true	[trichotomy]

Notation: We write $\begin{cases} a > b & \text{for} \quad a - b \in \mathbb{P}; \\ a < b & \text{for} \quad b - a \in \mathbb{P}; \\ a \geqslant b & \text{for} \quad a - b \in \mathbb{P} \cup \{0\}; \\ a \leqslant b & \text{for} \quad b - a \in \mathbb{P} \cup \{0\}. \end{cases}$

The Completeness Axiom

Suppose that $B \subseteq \mathbb{R}$, and that $k \in B$ is such that $b \geqslant k$ for all $b \in B$. We then say that 'k is a least (or minimal) element of B. By the trichotomy axiom P3 we can prove that if there is a least element, there is only one, which we call 'the least element of B'.
Suppose that $S \subseteq \mathbb{R}$, and that $b \in \mathbb{R}$ is such that $b \geqslant s$ for all $s \in S$. We then say that 'b is an upper bound of S', and that 'S is bounded above.'

C Let S be a non-empty subset of $\mathbb{R}$ which is bounded above; then the set of upper bounds of S has a least element. [completeness]

1. THE REAL NUMBERS

1.1 The Field Axioms

What do we mean by the *real numbers*? Certainly they include the natural numbers, integers and rational numbers. But you are likely aware of the existence of irrational (= not rational) numbers, such as $\sqrt{2}$ or π, which are also real numbers. One answer would be that a real number is a number with a decimal expansion, but then we are getting ahead of ourselves: what is meant by saying $0.333\ldots$ is the decimal expansion of $\frac{1}{3}$? Just what details are hiding behind that ellipsis?

Rather we will simply present a set of *axioms* – statements we will assume to be true of the real numbers. We will base all our arguments on these axioms, and develop theorems from these axioms alone. In Remark 5.34 we will make some brief comments about how the natural numbers, integers, rationals and reals can be constructed from simpler sets, but the details are left to more advanced set-theoretic courses.

Unless otherwise made clear, the quantities a, b, x, etc. discussed in the following will be real numbers. The labelling of these axioms largely follows the convention of Bartle & Sherbert.

Definition 1.1 *The **real numbers** are a set $\mathbb{R}$ together with two binary operations*

- ***addition*** $+ : \mathbb{R}^2 \to \mathbb{R}$,
- ***multiplication*** $\times : \mathbb{R}^2 \to \mathbb{R}$

which satisfy the following axioms

- *the **field axioms** A1-A4, M1-M4, D, Z.*
- *the **order axioms** P1-P3.*
- *the **completeness axiom** C.*

*described below. Recall that, as $+$ and $\times$ are binary operations, associated with any ordered pair (a, b) of real numbers are real numbers $a + b$ and $a \times b$, known as their **sum** and **product** respectively.*

The **addition axioms** A1-A4 require that

(A1) $+$ is **commutative**, that is $a + b = b + a$ for all a, b.

(A2) $+$ is **associative**, that is $a + (b + c) = (a + b) + c$ for all a, b, c.

(A3) there is an **additive identity** 0, called **zero**, that satisfies $a + 0 = a$ for all a.

(A4) for each a there is an **additive inverse** $-a$ such that $a + (-a) = 0$.

Remark 1.2 *Note that the axioms of addition A1-A4 are equivalent to* $(\mathbb{R}, +)$ *being an abelian group.*

Remark 1.3 *Associativity guarantees that the sum of* $a_1, \ldots, a_k$ *is independent of how the calculation is executed. For example, four terms can be summed in five ways:*

$$((a+b)+c)+d, \quad (a+(b+c))+d, \quad (a+b)+(c+d), \quad a+((b+c)+d), \quad a+(b+(c+d)),$$

and more generally there are

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

ways to bracket a sum of n *terms. (*C_n *denotes the* n*th Catalan number.)*

It can be shown, say using strong induction, that these C_n *calculations lead to the same sum when the operation is associative.*

We now prove some basic first results about addition.

Proposition 1.4 *If* $a + x = a$ *for all* a*, then* $x = 0$*. Thus the additive identity* 0 *is unique.*

Proof. As $a + x = a$ is true for all a, then it is in particular true when $a = 0$. So we have

$$\begin{array}{rcll} x & = & x+0 & \text{by definition of 0 (A3)} \\ & = & 0+x & \text{by commutativity (A1)} \\ & = & 0 & \text{by hypothesis with } a = 0. \end{array}$$

As 0 has the given property, by the above it is the only real number with this property. ■

Proposition 1.5 *If* $a + x = a + y$*, for some* a*, then* $x = y$*. Thus additive inverses are unique.*

Proof.

$$\begin{array}{rcll} y & = & y+0 & \text{by definition of 0 (A3)} \\ & = & y+(a+(-a)) & \text{by definition of inverses (A4)} \\ & = & (y+a)+(-a) & \text{by associativity (A2)} \\ & = & (a+y)+(-a) & \text{by commutativity (A1)} \\ & = & (a+x)+(-a) & \text{by hypothesis} \\ & = & (x+a)+(-a) & \text{by commutativity (A1)} \\ & = & x+(a+(-a)) & \text{by associativity (A2)} \\ & = & x+0 & \text{by definition of inverses (A4)} \\ & = & x & \text{by definition of 0 (A3).} \end{array}$$

It follows that $-a$ is the unique additive inverse of a: if x is an additive inverse of a then

$$a + (-a) = 0 = a + x$$

and it follows that $x = -a$. ■

I wouldn't wish to suggest at this point that the previous and following proofs are the only or even the best proofs. The above proof is reasonably slick, showing in one chain of equalities that y equals x and applying one axiom at a time. But there are other proofs, some of which are arguably more natural, and any proof that is logically correct and carefully justified is adequate to the task.

A much more natural chain of thought to get from $a + x = a + y$ to $x = y$ would involve 'subtracting a from both sides'. And we can use the axioms to argue exactly along these lines.

Proof. (Alternative proof of Proposition 1.5) Say

$$a + x = a + y.$$

Then

$$(a + x) + (-a) = (a + y) + (-a)\,.$$

By commutativity (A1) we have

$$(x + a) + (-a) = (y + a) + (-a)$$

and by associativity (A2) we then have

$$x + (a + (-a)) = y + (a + (-a))\,.$$

By A4 we have $x + 0 = y + 0$ and finally by A3 we then have $x = y$. ■

Proposition 1.6 $-(-a) = a$.

Proof.

$$\begin{array}{rcll} (-a) + a & = & a + (-a) & \text{by commutativity (A1)} \\ & = & 0 & \text{by definition of inverses (A4)} \end{array}$$

and also

$$(-a) + (-(-a)) \;=\; 0 \quad \text{by definition of inverses (A4).}$$

Hence $-(-a) = a$ as additive inverses are unique (Proposition 1.5). ■

Proposition 1.7 $-(a + b) = (-a) + (-b)\,.$

Proof. This is left as Sheet 1, Exercise 1(iii). ■

Proposition 1.8 $-0 = 0$.

Proof. By definition $0 + (-0) = 0$ and $0 + 0 = 0$. By the uniqueness of additive inverses $-0 = 0$. ■

The **multiplication axioms** M1-M4 require that

(M1) $\times$ is **commutative**, that is $a \times b = b \times a$ for all a, b.

(M2) $\times$ is **associative**, that is $a \times (b \times c) = (a \times b) \times c$ for all a, b, c.

(M3) there is a **multiplicative identity** 1, called **one**, that satisfies $a \times 1 = a$ for all a.

(M4) for each $a \neq 0$ there is a **multiplicative inverse**, denoted a^{-1}, such that $a \times (a^{-1}) = 1$.

Remark 1.9 *The axioms of multiplication M1-M4 state that* $(\mathbb{R}\backslash\{0\}, \times)$ *is an abelian group.*

There are then similar results for $\times$ to those proved previously for $+$.

Proposition 1.10 *If* $a \times x = a$ *for all* a *then* $x = 1$. *So the multiplicative identity is unique.*

Proposition 1.11 *If* $a \neq 0$ *and* $a \times x = a \times y$ *then* $x = y$. *So multiplicative inverses are unique.*

Proposition 1.12 *If* $a \neq 0$ *then* $(a^{-1})^{-1} = a$.

Proposition 1.13 *If* $a \neq 0 \neq b$ *and* $ab \neq 0$ *and* $(ab)^{-1} = a^{-1} \times b^{-1}$.

These results are left as exercises for the reader. Their proofs are very similar to the corresponding results for addition. Further, we will soon see that if $a \neq 0 \neq b$ then $ab \neq 0$ (Proposition 1.16) so the hypothesis that $ab \neq 0$ is in fact unnecessary in the last proposition.

There are two remaining field axioms to introduce.

(D) The **distributive law** states that $\times$ *distributes* over $+$. That is,

$$a \times (b + c) = (a \times b) + (a \times c)$$

for all a, b, c.

(Z) $0 \neq 1$.

The importance of axiom Z may not be immediately obvious. If it were the case that $1 = 0$, we would have (by M3 and Proposition 1.15 below) that

$$x = x \times 1 = x \times 0 = 0 \qquad \text{for all } x.$$

So the singleton set $\{0\}$ satisfies all the *other* field axioms and we need axiom Z above to make clear we are not discussing this example. We will also find axiom Z, or rather its negation, a useful conclusion for proofs by contradiction – should an initial assumption lead to the conclusion that $0 = 1$, we would know the initial assumption to be incorrect.

Some important consequences of the distributive law appear below.

Proposition 1.14 $(a + b) \times c = a \times c + b \times c$.

Proof.

$$\begin{aligned} (a+b) \times c &= c \times (a+b) && \text{by commutativity (M1)} \\ &= c \times a + c \times b && \text{by distributivity (D)} \\ &= a \times c + b \times c && \text{by commutativity (M1) twice.} \end{aligned}$$

■

Proposition 1.15 $a \times 0 = 0$.

Proof.

$$\begin{array}{rcll} a \times 0 + 0 & = & a \times 0 & \text{by definition of 0 (A3)} \\ & = & a \times (0+0) & \text{by definition of 0 (A3)} \\ & = & a \times 0 + a \times 0 & \text{by distributivity (D).} \end{array}$$

Hence $a \times 0 = 0$ by Proposition 1.5. ■

Proposition 1.16 *If $a \times b = 0$ then either $a = 0$ or $b = 0$ (or both).*

Proof. If $a \neq 0$ then we have

$$\begin{array}{rcll} 0 & = & a^{-1} \times 0 & \text{by Proposition 1.15} \\ & = & a^{-1} \times (a \times b) & \text{by hypothesis} \\ & = & (a^{-1} \times a) \times b & \text{by associativity of } \times \text{ (M2)} \\ & = & 1 \times b & \text{by definition of inverse (M4)} \\ & = & b \times 1 & \text{by commutativity of } \times \text{ (M1)} \\ & = & b & \text{by definition of 1 (M3).} \end{array}$$

(Note the above proof amounts to carefully showing that if $a \neq 0$ then we can divide by a to show $b = 0$.) Thus $b = 0$ if $a \neq 0$, or if $a \neq 0$ does not hold then $a = 0$ as required. ■

Proposition 1.17

$$(-b) \times a = -(b \times a).$$

In particular $(-1) \times a = -a$.

Proof.

$$\begin{array}{rcll} (b \times a) + ((-b) \times a) & = & (b + (-b)) \times a & \text{by Proposition 1.14} \\ & = & 0 \times a & \text{by definition of inverse (A4)} \\ & = & 0 & \text{by Proposition 1.15 and M1} \end{array}$$

and

$$(b \times a) + (-(b \times a)) \;=\; 0 \quad \text{by definition of inverse (A4).}$$

As additive inverses are unique then $(-b) \times a = -(b \times a)$. The final part follows from setting $b = 1$ and applying M3 and M1. ■

Proposition 1.18

$$(-1) \times (-1) = 1.$$

Proof.

$$\begin{array}{rcll} (-1) \times (-1) & = & -(-1) & \text{by Proposition 1.17} \\ & = & 1 & \text{by Proposition 1.6.} \end{array}$$

■

Notation 1.19 *From now on we will instead write*

$$\begin{array}{ccc} ab & for & a \times b \\ a - b & for & a + (-b) \\ a/b \text{ or } \frac{a}{b} & for & a \times b^{-1}. \end{array}$$

Also, we define integer powers for $a \neq 0$ *by*

$$\begin{array}{rcll} a^0 & = & 1 & \\ a^{k+1} & = & a^k \times a & \text{for all } k = 0, 1, 2, 3, \ldots \\ a^{-l} & = & \left(a^l\right)^{-1} & \text{for all } l = 1, 2, 3, \ldots. \end{array}$$

Remark 1.20 *Note that we have only defined integer powers of* a *here. For* $a > 0$ *and rational* $q = m/n$ *we will, in due course (Theorem 1.73 et seq.), define*

$$a^q = \sqrt[n]{a^m}.$$

For general real x *and* $a > 0$*, we will not be able to define*

$$a^x = e^{x \log a}$$

until we meet the exponential (Definition 7.18) and logarithm functions (Definition 7.19).

Remark 1.21 *Other number systems also satisfy A1–A4, M1–M4, D, Z. Such systems are called* ***fields****. Fields are important algebraic structures in mathematics, and all the linear algebra and matrix theory you are meeting in Linear Algebra I extends naturally over any given field. The notion of a field was introduced by Richard Dedekind in 1871.*

The rational numbers $\mathbb{Q}$*, the real numbers* $\mathbb{R}$ *and the complex numbers* $\mathbb{C}$ *are all examples of fields. Other examples include* $\mathbb{Z}_p$*, that is the integers modulo a prime number* p*, and the field with four elements. See extension exercises 7, 8 and 9 on Sheet 1.*

$\mathbb{Z}$ *is not a field as it does not meet M4 (multiplicative inverses) though it does satisfy the remainder of the field axioms.* $\mathbb{N}$ *further fails to meet A4 (additive inverses).*

1.2 The Order Axioms

As there are many systems, *fields*, that satisfy the field axioms, we clearly require some further axioms to fully characterize the real numbers. The real numbers are commonly represented by a number line with the numbers increasing in a left-to-right fashion. So it is clear that the real numbers have other properties which we need to capture, including notions of 'being greater than' or 'being to the right of', together with other geometric notions such as distance.

There are various (ultimately equivalent) ways of introducing the notion of 'greater than'. We will again follow Bartle & Sherbert and address this by introducing axioms for what it is to be *positive*.

Definition 1.22 *(**Order Axioms**) There is a subset $\mathbb{P}$ of $\mathbb{R}$, of **positive** real numbers, that satisfies the following three axioms.*

(P1) If a and b are positive, then their sum $a+b$ is positive.

(P2) If a and b are positive, then their product ab is positive.

(P3) For any a precisely one of the following is true:

$$a \text{ is positive}; \qquad a = 0; \qquad -a \text{ is positive}.$$

So to say 'a is positive' means $a \in \mathbb{P}$. Written as an interval, $\mathbb{P} = (0, \infty)$.
*The third axiom P3 is called the **trichotomy** axiom. By this axiom, 0 is not positive.*

Remark 1.23 *Any structure satisfying the field and order axioms is called an **ordered field**. $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields, and any subfield of $\mathbb{R}$ is an ordered field (see Sheet 1, Exercise 7 for such an example). However there is no subset $\mathbb{P}$ of $\mathbb{C}$ which makes it an ordered field (Proposition 1.80).*

Proposition 1.24 *1 is positive.*

Proof. By P3 precisely one of

$$1 \in \mathbb{P}, \qquad 1 = 0, \qquad -1 \in \mathbb{P}$$

must hold. Axiom Z discounts the second possibility. The third option, $-1 \in \mathbb{P}$, leads to a contradiction as follows:

$$\begin{array}{lll} -1 \in \mathbb{P} & \implies & (-1)(-1) \in \mathbb{P} \qquad \text{by P2} \\ & \implies & 1 \in \mathbb{P} \qquad \text{by Proposition 1.18} \\ & \implies & -1 \in \mathbb{P} \text{ and } 1 \in \mathbb{P} \end{array}$$

which contradicts P3. By elimination, it follows that $1 \in \mathbb{P}$. ■

There are alternative, equivalent, means of introducing order to the real numbers by defining binary relations $\leqslant$ or $<$ with appropriate properties. The equivalence of the two approaches is left to Sheet 1, Exercise 10. For now we introduce the following notation and definitions.

Notation 1.25 *(a) We write*

$$\begin{array}{lll} a > b & \text{for} & a - b \in \mathbb{P}\,. \\ a < b & \text{for} & b - a \in \mathbb{P}\,. \\ a \geqslant b & \text{for} & a - b \in \mathbb{P} \cup \{0\}. \\ a \leqslant b & \text{for} & b - a \in \mathbb{P} \cup \{0\}. \end{array}$$

In this notation the trichotomy axiom, P3, reads as: 'precisely one of $a > 0$, $a = 0$, $a < 0$ holds'.

*(b) Elements of $\mathbb{P} \cup \{0\} = [0, \infty)$ are referred to as **non-negative**, of $\mathbb{P}^c = (-\infty, 0]$ are called **non-positive** and of $(\mathbb{P} \cup \{0\})^c = (-\infty, 0)$ are called **negative**.*

Proposition 1.26 *$a > b$ if and only if $-a < -b$. In particular, $x > 0$ if and only if $-x < 0$.*

Proof.

$$\begin{array}{rrcll} & a & > & b & \\ \iff & a-b & \in & \mathbb{P} & \text{by definition} \\ \iff & -(-a)-b & \in & \mathbb{P} & \text{by Proposition 1.6} \\ \iff & -b-(-a) & \in & \mathbb{P} & \text{by commutativity of } + \text{ (A1)} \\ \iff & -b & > & -a & \text{by definition} \\ \iff & -a & < & -b & \text{by definition.} \end{array}$$

The last claim follows as $-0 = 0$ (Proposition 1.8). ■

Proposition 1.27 *For all a, b, c*

$$a \leqslant a; \tag{1.1}$$

$$a \leqslant b \text{ and } b \leqslant a \implies a = b; \tag{1.2}$$

$$a \leqslant b \text{ and } b \leqslant c \implies a \leqslant c. \tag{1.3}$$

$$\text{either } a \leqslant b \text{ or } b \leqslant a \tag{1.4}$$

Proof. You may recognize (1.1), (1.2), (1.3) as being the reflexivity, anti-symmetry and transitivity properties of a partial order. Combined with (1.4), this means $\leqslant$ is a total order.

(a) By A4 we have $a - a = 0 \in \mathbb{P} \cup \{0\}$ and so $a \leqslant a$.

(b) By definition $a \leqslant b$ and $b \leqslant a$ mean

$$b - a \in \mathbb{P} \cup \{0\} \quad \text{and} \quad a - b = -(b-a) \in \mathbb{P} \cup \{0\}.$$

There are then two cases to consider:

(i) $b - a \in \mathbb{P}$ and $-(b-a) \in \mathbb{P}$. This contradicts trichotomy (P3).

(ii) if $b - a = 0$ then $a = b$; similarly if $a - b = 0$.

(c) If $a = b$ or $b = c$ this is trivial, so we need only consider the case where $b - a \in \mathbb{P}$ and $c - b \in \mathbb{P}$. We then have

$$c - a = (c-b) + (b-a) \in \mathbb{P},$$

by P1 as required.

(d) For $a, b \in \mathbb{R}$, precisely one of the following holds

$$\begin{array}{rcl} b-a \in \mathbb{P} & \implies & a \leqslant b; \\ b-a = 0 & \implies & a \leqslant b \text{ and } b \leqslant a; \\ -(b-a) = a-b \in \mathbb{P} & \implies & b \leqslant a. \end{array}$$

■

Proposition 1.28 *For a, b, c with $a \leqslant b$, then $a + c \leqslant b + c$.*

Proof. Note $(b + c) - (a + c) = b - a \in \mathbb{P} \cup \{0\}$. ■

Proposition 1.29 *For a, b, c with $a \leqslant b$ and $0 \leqslant c$, then $ca \leqslant cb$.*

Proof. If $c = 0$ or $a = b$ then the result holds immediately. Otherwise $b - a \in \mathbb{P}$ and $c \in \mathbb{P}$ so that, by P2, $c(b - a) \in \mathbb{P}$. By D and Proposition 1.17 we see

$$c(b - a) = cb + c(-a) = cb - ca \in \mathbb{P}$$

and so $ca < cb$ as required. ■

Corollary 1.30 *For a, b, c with $a \leqslant b$ and $c \leqslant 0$. Then $ca \geqslant cb$.*

Proof. This is left as Sheet 1, Exercise 1(v). ■

Corollary 1.31 *$a^2 \geqslant 0$ for any real a.*

Proof. By the trichotomy axiom we have $a > 0$ or $a = 0$ or $a < 0$. If $a > 0$ then $a^2 \in \mathbb{P}$ by P1 and so $a^2 \geqslant 0$. If $a < 0$ then $a^2 > a0 = 0$ by Corollary 1.30 and Proposition 1.15. If $a = 0$ then $a^2 = 0 \geqslant 0$ again. ■

Proposition 1.32 *If $a > 0$ then $a^{-1} > 0$.*

Proof. Certainly $a^{-1} \neq 0$ and if $a^{-1} < 0$ then $-a^{-1} > 0$ giving the contradiction

$$-1 = \left(-a^{-1}\right) a > 0$$

by P2. ■

Corollary 1.33 *If $0 < a < b$ then $b^{-1} < a^{-1}$.*

Proof. By the previous proposition and P2

$$a^{-1} - b^{-1} = a^{-1}b^{-1}(b - a) > 0.$$

■

Using the order axioms we may define the **maximum** and **minimum** of two numbers.

Definition 1.34 *Define* $\max\colon \mathbb{R}^2 \to \mathbb{R}$ *and* $\min\colon \mathbb{R}^2 \to \mathbb{R}$ *by*

$$\max(x, y) = \begin{cases} x & \text{if } x \geqslant y; \\ y & \text{if } y > x. \end{cases} \qquad \min(x, y) = \begin{cases} y & \text{if } x \geqslant y; \\ x & \text{if } y > x. \end{cases}$$

By the trichotomy axiom, these are well-defined functions.

We can extend these to functions of finitely many variables. For example, recursively we can define

$$\max(a_1, \ldots, a_n, a_{n+1}) = \max(\max(a_1, \ldots, a_n), a_{n+1}).$$

Example 1.35 $\max(x,y) = -\min(-x,-y)$ *for any* x, y.

Solution. We argue by cases, recalling that if $x < y$ then $-x > -y$ by Proposition 1.26.

	$\max(x,y)$		$-\min(-x,-y)$
$x > y$	x	$-x < -y$	$-(-x) = x$
$x = y$	x	$x = y$	$-(-x) = x$
$x < y$	y	$-y < -x$	$-(-y) = y$

■

Definition 1.36 *We define the* ***modulus*** *function* $|\quad| : \mathbb{R} \to \mathbb{R}$ *by*

$$|x| = \begin{cases} x & \text{if } x \geqslant 0; \\ -x & \text{if } x < 0. \end{cases}$$

These cases are disjoint and cover all possibilities by the trichotomy axiom, so we obtain a well-defined function. $|x|$ *is read as 'mod* x*' or 'the modulus of* x*', and also referred to as 'the absolute value of* x*'.*

Remark 1.37 *As the maximum, minimum and modulus functions are defined in terms of different cases, proofs involving them commonly need to demonstrate the result in a case-by-case manner.*

Proposition 1.38 *For any* x*:*

(a) $0 \leqslant |x|$.
(b) $x \leqslant |x|$.
(c) $|-x| = |x|$.
(d) $|x|^2 = x^2$.

Proof. (a) For $x \geqslant 0$ this is obvious. If $x < 0$ then $|x| = -x > 0$ by definition of $>$.

(b) If $x \geqslant 0$ then $x = |x|$ and so $x \leqslant |x|$. If $x < 0$ then $x < 0 \leqslant |x|$ from (a).

(c) If $x \geqslant 0$ then by Proposition 1.26 we have $-x \leqslant -0 = 0$. So by definition

$$|-x| = -(-x) = x = |x|.$$

If $x < 0$ then $-x > 0$. So $|-x| = -x = |x|$ by definition.

(d) In either case we have $|x|^2 = x^2$ or $|x|^2 = (-x)^2 = x^2$ by Proposition 1.17. ■

Proposition 1.39 ***(Modulus of a product)*** $|ab| = |a|\,|b|$.

Proof. If either $a = 0$ or $b = 0$ then the LHS and RHS are both zero. If $a, b > 0$ then there is nothing to prove. If $a > 0 > b$ then $ab < 0$ and

$$|ab| = -(ab) = a(-b) = |a|\,|b|$$

by Proposition 1.17. Any if $a, b < 0$ then $ab > 0$ and

$$|ab| = ab = (-a)(-b) = |a|\,|b|$$

by Proposition 1.17. ■

Definition 1.40 *Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$. Then f is said to be:*

- ***increasing*** *if $f(x) \leqslant f(y)$ whenever $x \leqslant y$.*
- ***decreasing*** *if $f(x) \geqslant f(y)$ whenever $x \leqslant y$.*
- ***strictly increasing*** *if $f(x) < f(y)$ whenever $x < y$.*
- ***strictly decreasing*** *if $f(x) > f(y)$ whenever $x < y$.*

Proposition 1.41 *The function $f(x) = x^2$ is strictly increasing on $[0, \infty)$.*

Proof. Given $0 \leqslant a < b$,

$$f(b) - f(a) = b^2 - a^2 = (b-a)(b+a).$$

As $b - a \in \mathbb{P}$ and $b + a \in \mathbb{P}$ then $f(b) - f(a) \in \mathbb{P}$ and so $f(b) > f(a)$, showing that f is strictly increasing. ■

Theorem 1.42 *(**Triangle Inequality**) For any real numbers a, b,*

$$|a+b| \leqslant |a| + |b|,$$

with equality if and only if ($a \geqslant 0$ and $b \geqslant 0$) or ($a \leqslant 0$ and $b \leqslant 0$).

Proof. Note that

$$\begin{array}{rcll} |a+b|^2 & = & (a+b)^2 & \text{by Proposition 1.38(d)} \\ & = & a^2 + 2ab + b^2 & \\ & = & |a|^2 + 2ab + |b|^2 & \text{by Proposition 1.38(d)} \\ & \leqslant & |a|^2 + 2\,|ab| + |b|^2 & \text{by Proposition 1.38(b)} \\ & = & |a|^2 + 2\,|a|\,|b| + |b|^2 & \text{by Proposition 1.39} \\ & = & (|a| + |b|)^2\,. & \end{array}$$

As $f(x) = x^2$ is strictly increasing on $[0, \infty)$, then $|a| + |b| < |a+b|$ is impossible and the result follows. The cases when equality holds are left to Sheet 1, Exercise 3(i). ■

See Remark 1.85 for an explanation of the inequality's name.

Corollary 1.43 *(**Reverse triangle inequality**) For any real numbers a, b,*

$$|a-b| \geqslant ||a| - |b||\,.$$

Proof. Note by the triangle inequality that

$$\begin{array}{rcccl} |a-b| + |b| & \geqslant & |a| & \Longrightarrow & |a-b| \geqslant |a| - |b|\,, \\ |b-a| + |a| & \geqslant & |b| & \Longrightarrow & |a-b| \geqslant |b| - |a|\,, \end{array}$$

and as $||a| - |b||$ equals $|a| - |b|$ or $|b| - |a|$ then the reverse triangle inequality follows. ■

We can use the modulus function to define *distance* on the real line (which is the same as *difference*).

Definition 1.44 *Given real numbers x, y, the* ***distance*** *$d(x, y)$ between x and y is defined to be*

$$d(x, y) = |x - y| .$$

The function $d\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the properties of a ***metric****, namely, for any real x, y, z:*

(M1) $d(x, y) \geqslant 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.

(M2) $d(x, y) = d(y, x)$.

(M3) $d(x, z) \leqslant d(x, y) + d(y, z)$.

Property M3 is equivalent to the triangle inequality. Properties M1 and M2 follow readily from properties of the modulus function.

We conclude with two useful inequalities.

Theorem 1.45 ***(Bernoulli's inequality***[1]*)* *Let x be a real number with $x > -1$ and let n be a positive integer. Then*

$$(1 + x)^n \geqslant 1 + nx.$$

Proof. We shall prove the inequality by induction – note that the inequality is immediate when $n = 1$. Suppose that

$$(1 + x)^N \geqslant 1 + Nx$$

holds for all real $x > -1$ and a particular $N \geqslant 1$. Then $1 + x > 0$ and $Nx^2 \geqslant 0$ as $N > 0$ and $x^2 \geqslant 0$. Hence

$$\begin{array}{rcll} (1 + x)^{N+1} & = & (1 + x)(1 + x)^N & \text{by definition} \\ & \geqslant & (1 + x)(1 + Nx) & \text{by hypothesis and Proposition 1.29} \\ & = & 1 + (N + 1)x + Nx^2 & \text{by A1–A4} \\ & \geqslant & 1 + (N + 1)x & \text{by Proposition 1.28.} \end{array}$$

The result follows by induction. ■

Proposition 1.46 ***(Powers dominate polynomials)*** *Let a be a real number with $a > 1$, and k be a positive integer. Then there exists $c > 0$ such that*

$$a^n \geqslant cn^k \quad \textit{for} \quad n = 1, 2, 3, \ldots$$

Proof. Let $a = 1 + b$, so that $b > 0$, and take $n > k$. Then

$$n \geqslant k + 1 \quad \Longrightarrow \quad \frac{n - k}{n} = 1 - \frac{k}{n} \geqslant 1 - \frac{k}{k + 1} = \frac{1}{k + 1}.$$

[1] Named after Jacob Bernoulli (1655-1705) who applied the inequality frequently in a text of 1689.

By the binomial theorem,

$$\begin{aligned} a^n &= 1 + \binom{n}{1} b + \binom{n}{2} b^2 + \cdots + \binom{n}{k} b^k + \cdots + b^n \\ &\geqslant \binom{n}{k} b^k \quad \text{[as all other terms are positive]} \\ &= \frac{n\,(n-1)\cdots(n-k+1)}{k!} b^k \\ &> \frac{(n-k)^k}{k!} b^k \\ &= \frac{b^k}{k!} \left(\frac{n-k}{n} \right)^k n^k \\ &\geqslant \left(\frac{b^k}{k!\,(k+1)^k} \right) n^k, \end{aligned}$$

with the last line following from the previous inequalities. We have thus

$$c = \frac{b^k}{k!\,(k+1)^k}$$

is such that $a^n/n^k \geqslant c$ for $n > k$. If instead we set

$$c = \min \left\{ a, \frac{a^2}{2^k}, \ldots, \frac{a^k}{k^k}, \frac{b^k}{k!\,(k+1)^k} \right\} > 0$$

then $a^n/n^k \geqslant c$ holds for $n \geqslant 1$. (It's important to note that the above choice of c is independent of n, depending only on a and k.) ■

Remark 1.47 *In due course we will meet the exponential and logarithm and will have a much simpler proof of the above result. Indeed we will be able to easily prove $a^x \geqslant cx^k$ for all $x > 0$, not just integers. This is because*

$$a^x = e^{x \log a} > \frac{(\log a)^{k+1}}{(k+1)!} x^{k+1}$$

for $x > 0$.

Example 1.48 *In each of the cases (a) and (b),*

$$\begin{aligned} &(a) \quad x(n) = \frac{3^n + n^3}{2^n}, \quad y(n) = n^4 + 3n^2; \\ &(b) \quad x(n) = \frac{2^n + n^5}{n^2}, \quad y(n) = \frac{(-3)^n}{n^3 + 1}, \end{aligned}$$

determine
(i) whether there exists $c > 0$ such that $x(n) \geqslant cy(n)$ for all n;
(ii) whether there exists $c > 0$ such that $x(n) \leqslant cy(n)$ for all n;
(iii) whether neither (i) nor (ii) applies.

Thoughts: This example now has genuine flavours of analysis; we are no longer simply applying axioms. First thoughts are often qualitative, appreciating which terms in the sequences are significant and which are relatively negligible. For example, in answering (a) we will ultimately ignore the n^3 term in the numerator; it is helpful, in the sense its presence increases $x(n)$, but it is inconsequential by comparison with 3^n, so we simply do not need to make any use of it. It's also important to note that the example requires us to find a positive c, if it exists, and not in any sense a minimum such c. The existence of any such c is sufficient to answer the question; see Sheet 1, Exercise 11 for further comment.

Solution. (a) Note that $n^3/2^n > 0$ for all n. By Proposition 1.46, there exist $c_1, c_2 > 0$ such that

$$\left(\frac{3}{2}\right)^n \geqslant c_1 n^4 \quad \text{and} \quad \left(\frac{3}{2}\right)^n \geqslant c_2 n^2$$

for all n. Hence

$$\begin{aligned} \frac{3^n + n^3}{2^n} &\geqslant \left(\frac{3}{2}\right)^n \\ &= \frac{1}{2}\left(\frac{3}{2}\right)^n + \frac{1}{2}\left(\frac{3}{2}\right)^n \\ &\geqslant \frac{c_1}{2} n^4 + \frac{c_2}{6}\left(3n^2\right) \\ &\geqslant \min\left(\frac{c_1}{2}, \frac{c_2}{6}\right)\left(n^4 + 3n^2\right). \end{aligned}$$

Hence (i) holds in this case by setting $c = \min\left(c_1/2, c_2/6\right)$.

(b) Note immediately that (ii) cannot hold as $x(n)$ is always positive and $y(n)$ is negative when n is odd. So (i) can only apply if $x(n)$ exceeds some $cy(n)$ for all even n. For even n the inequality $x(n) \geqslant cy(n)$ is equivalent to

$$\left(n^3 + 1\right)\left(\frac{2}{3}\right)^n + \left(\frac{n^8 + n^5}{3^n}\right) \geqslant cn^2.$$

By Proposition 1.46, there exists a number K which exceeds the LHS for all n, whilst the RHS increases without bound as n increases. So no such c exists. To appreciate this in detail, note that:

- There exists $c_1 > 0$ such that $(3/2)^n \geqslant c_1 n^3$ and hence $n^3 (2/3)^n \leqslant 1/c_1$.
- $(2/3)^n < 1$.
- There exists $c_2 > 0$ such that $3^n \geqslant c_2 n^8$ and hence $n^8/3^n \leqslant 1/c_2$.
- There exists $c_3 > 0$ such that $3^n \geqslant c_3 n^5$ and hence $n^5/3^n \leqslant 1/c_3$.

Thus we require there to be $c > 0$ such that

$$\frac{1}{c}\left(\frac{1}{c_1} + 1 + \frac{1}{c_2} + \frac{1}{c_3}\right) \geqslant n^2 \geqslant n$$

for all positive integers n. The LHS would then be an upper bound for the unbounded $\mathbb{N}$ which does not exist. (See Corollary 1.56.) ■

1.3 The Completeness Axiom

At this stage we can surely persuade ourselves that we could write down proofs of all the usual *algebraic* properties of $\mathbb{R}$, and all the usual order properties of $\leqslant$.

Many structures share these properties – the ordered fields – and in particular both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields. So why won't $\mathbb{Q}$ suffice? Why do mathematicians not settle for working with this rather nice field of easily understood ratios of integers; countable, too, so that we can list the elements?

The ancient Greeks had at least one reason – in $\mathbb{Q}$ we can't find an element to measure the length of the hypotenuse of a right-angled isosceles triangle with two short sides of length 1. Here is the proof of that fact.

Theorem 1.49 *There is no element $\alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$*

Proof. If there were such an α, then we could write $\alpha = m/n$ for some $m, n \in \mathbb{Z}, n \neq 0$. Further we could assume that this fraction is in lowest terms, so that m and n are coprime. Then $2n^2 = m^2$. As m^2 is even then m is also even as a product of odd numbers is odd. We can then write $m = 2k$ and hence $n^2 = 2k^2$. But then n, too, is even by the same reasoning and m/n wasn't in lowest terms after all. This is the required contradiction. ■

So $\mathbb{Q}$ is lacking in some ways, certainly if we wish to discuss distances, and we look to describe the way(s) $\mathbb{R}$ is different from $\mathbb{Q}$.

Definition 1.50 *Let $B \subseteq \mathbb{R}$.*

*We say that b_1 is a **least element** or **minimum** of B if (i) $b_1 \in B$ and (ii) $b_1 \leqslant b$ for all $b \in B$. In this case we write $b_1 = \min B$.*

*We say that b_2 is a **greatest element** or **maximum** of B if (i) $b_2 \in B$ and (ii) $b \leqslant b_2$ for all $b \in B$. In this case we write $b_2 = \max B$.*

Example 1.51 *1 is the minimum of $[1, 2)$ but there is no maximum for this set. If $x \in [1, 2)$ were a maximum then $x < 2$ and so $1 + x/2$ is a greater element of the set.*

Proposition 1.52 *A maximum (if it exists) is unique. Similarly a minimum is unique.*

Proof. Suppose that b and c are both maxima of B. Then as $b \in B$ and c is a maximum, $b \leqslant c$; as $c \in B$ and b is a maximum then $c \leqslant b$. By anti-symmetry $b = c$. Similarly minima are unique if defined. ■

Proposition 1.53 *Every non-empty subset of $\mathbb{N}$ has a minimum.*

Proof. Suppose, for a contradiction, that S is a non-empty subset of $\mathbb{N}$ with no minimum and define

$$S^* = \{n \in \mathbb{N} \mid \text{none of } 0, 1, \ldots, n \text{ is in } S\}.$$

We shall show that $S^* = \mathbb{N}$ and conclude that S is empty, a contradiction.

Note that $0 \in S^*$. If not then 0 is in S and S has a minimum (namely 0). Now suppose that n is in S^*. This means that none of $0, 1, \ldots, n$ is in S. It follows that $n + 1$ is not in S, or else $n + 1$ would be the minimum of S. Hence none of $0, 1, \ldots, n, n + 1$ is in S or equivalently $n + 1$ is in S^*. By induction $S^* = \mathbb{N}$ and so S is empty. ■

Definition 1.54 *Let $B \subseteq \mathbb{R}$.*

*We say that l is a **lower bound** of B if $l \leqslant b$ for all $b \in B$ and that B is **bounded below**.*

*We say that u is an **upper bound** of B if $b \leqslant u$ for all $b \in B$ and that B is **bounded above**.*

We say that $B \subseteq \mathbb{R}$ is bounded if it is bounded above and below.

Example 1.55 *(a) 23 and π are both upper bounds of $[1, 2)$. And 1 is a lower bound as is -37. The set of upper bounds is $[2, \infty)$ and the set of lower bounds is $(-\infty, 1]$.*

(b) $\mathbb{Q}$ is neither bounded above nor below, $\mathbb{N}$ is bounded below, $(-\infty, e]$ is bounded above.

(c) $\varnothing$ is bounded. The set of upper bounds for $\varnothing$ is $\mathbb{R}$ as is the set of lower bounds.

The below results follow from Proposition 1.53 or can be similarly proved.

Corollary 1.56 *(a) A non-empty subset of $\mathbb{Z}$ which is bounded below has a minimum.*

(b) A non-empty subset of $\mathbb{Z}$ or $\mathbb{N}$ which is bounded above has a maximum.

(c) $\mathbb{N}$ has no maximal element.

We are now ready to give our final axiom which characterises the real numbers.

Definition 1.57 ***(Completeness Axiom)***

(C) Let $S \subseteq \mathbb{R}$ be a non-empty set which is bounded above. Then the set of upper bounds of S has a least element.

Remark 1.58 ***(Equivalent axioms)*** *There are various alternative axioms that are equivalent to the completeness axiom as stated in C. One such is:*

- *Let A and B be non-empty bounded sets such that $a \leqslant b$ for all $a \in A$ and $b \in B$. Then there exists c such that $a \leqslant c \leqslant b$ for all $a \in A$ and $b \in B$.*

This is shown to be equivalent to C in Sheet 2, Exercise 10.

In this course we will meet further equivalent assumptions:

- *Bounded, monotone sequences converge (Theorem 5.3, Sheet 4, Exercise 6).*
- *Cauchy completeness (Theorem 5.29) and the Archimedean property.*
- *The Nested Intervals Theorem (Theorem 5.7) and the Archimedean property.*
- *The Bolzano-Weierstrass Theorem (Theorem 5.20).*

Another famous equivalent axiom is Dedekind completeness (which is off-syllabus):

- *A Dedekind cut is a set $\varnothing \neq A \subsetneq \mathbb{Q}$ satisfying (i) if $x < y, x \in \mathbb{Q}$, $y \in A$ then $x \in A$ and (ii) if $x \in A$ then there exists $y \in A$ with $y > x$. Dedekind completeness states that $A = (-\infty, z) \cap \mathbb{Q}$ for a unique real number z.*

Remark 1.59 *(**Uniqueness of the real numbers**) The completeness axiom is the last axiom we will introduce in defining the real numbers. Two natural questions arise: do the real numbers exist and is the set of real numbers unique?*

The first question is not meant ontologically. But, rather than just assuming that there is a set $\mathbb{R}$ which satisfies the field axioms, the order axiom and completeness axiom, can such a set be constructed with those properties from more concrete sets such as $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$? We will address this aspect of the existence of $\mathbb{R}$ when we meet Cauchy sequences (Remark 5.34).

The second question – the uniqueness of $\mathbb{R}$ – is also a subtle one. What does uniqueness mean here? It is true that, up to order isomorphism, there is a unique complete ordered field. For those wishing to understand the details of this statement, see Körner pp.359-360.

Definition 1.60 *We call this least element the **least upper bound** or **supremum** of S, written as $\sup S$. Note that we can refer to $\sup E$ as **the** least upper bound as we have already shown in Proposition 1.52 that minima are unique.*

Example 1.61 *2 is the supremum of $[1, 2)$.*

Proof. For all $x \in [1, 2)$, $1 \leqslant x < 2$ by definition, so clearly 2 is an upper bound. Now suppose that there was a smaller upper bound, t. So $t < 2$, and as t is an upper bound, $t \geqslant 1$. Then $\frac{3}{2} \leqslant \frac{t+2}{2} < 2$. So $\frac{t+2}{2} \in [1, 2)$ but $t < \frac{t+2}{2}$ contradicting the fact that t was an upper bound. ■

Example 1.62 *The set of upper bounds of $\varnothing$ is $\mathbb{R}$ which has no minimum element.*

Proposition 1.63 *If $S \subseteq \mathbb{R}$ has a maximum then $\max S = \sup S$.*

Proof. Note $\max S \geqslant x$ for all $x \in S$ by definition of being a maximum. Further if u is an upper bound for S then $u \geqslant \max S$ by virtue of $\max S$ being an element of S. Hence $\max S$ is the least upper bound. ■

Proposition 1.64 *(**The Approximation Property**) Let S be bounded above and non-empty and let $\varepsilon > 0$. Then there exists $s \in S$ such that*

$$\sup S - \varepsilon < s \leqslant \sup S.$$

Proof. If this were not the case, then $\sup S - \varepsilon$ is an upper bound of S less than the least upper bound, which is a contradiction. ■

Remark 1.65 *To prove that a real number M is the supremum of a set $S \subseteq \mathbb{R}$ it is enough to show that (i) $s \leqslant M$ for all $s \in S$ and either (ii) for any upper bound M' of S we have $M \leqslant M'$ or alternatively (ii)' for any $\varepsilon > 0$ there is $s \in S$ such that $M - \varepsilon < s \leqslant M$.*

Corollary 1.66 *Let S be bounded above and non-empty. There is a function $a\colon \mathbb{N} \to \mathbb{R}$, such that*

$$\sup S - \frac{1}{n} < a(n) \leqslant \sup S \qquad \text{for all } n \geqslant 1.$$

In due course, we will see that this means there is a sequence $(a(n))$ in S which converges to $\sup S$.

We would like to make the symmetric definition for the maximum (if it exists) of the lower bounds of a set which is bounded below. One way would be to introduce yet another axiom guaranteeing its existence. But we don't need to do this; we now have sufficient axioms to prove this as a theorem.

Theorem 1.67 *Let T be a non-empty set which is bounded below. Then the set of lower bounds of T has a greatest element.*

Proof. Let $S = \{-t \mid t \in T\}$. As T is non-empty then S is also non-empty.

In Proposition 1.26 we showed that $x \leqslant y \iff -y \leqslant -x$. Let l be a lower bound of T. Then $l \leqslant t$ for all $x \in T$ and so $-t \leqslant -l$ for all $t \in T$. That is $s \leqslant -l$ for all $s \in S$. Hence S is bounded above, and non-empty, so by the completeness axiom, $\sup S$ exists.

We shall prove (i) $-\sup S$ is a lower bound of T, (ii) if l is a lower bound of T then $l \leqslant -\sup S$.

(i) If $t \in T$ then $-t \in S$ and so $-t \leqslant \sup S$. Hence $t \geqslant -\sup S$ and we see $-\sup S$ is a lower bound of T.

(ii) If $l \leqslant t$ for all $t \in T$ then $-l \geqslant -t$ for all $t \in T$. Hence $-l \geqslant \sup S$ by virtue of $\sup S$ being the least upper bound of S. Finally $l \leqslant -\sup S$. ■

Definition 1.68 *This element is known as* ***the greatest lower bound*** *or* ***infimum*** *of T and is written* $\inf T$.

- By an argument similar to Proposition 1.52 we can show easily show that infima are unique.
- Note if T has a minimum element then $\min T = \inf T$.

Example 1.69 $\sup[1, 2) = 2$ *and* $\inf[1, 2) = 1$. *Also* $\min[1, 2) = 1$ *whilst* $\max[1, 2)$ *does not exist.*

Corollary 1.70 ***(The Approximation Property for infima)*** *Let T be bounded below and non-empty and let $\varepsilon > 0$. Then there exists $t \in T$ such that*

$$\inf T \leqslant t < \inf T + \varepsilon.$$

Corollary 1.71 *Let T be bounded below and non-empty. There is a function $a\colon \mathbb{N} \to \mathbb{R}$, such that for all n we have*

$$\inf T \leqslant a(n) < \inf T + \frac{1}{n}.$$

Again we will shortly see that this means there is a sequence $(a(n))$ in T which converges to $\inf T$.

Example 1.72 *Let S be a bounded subset of $\mathbb{R}$. Let $c < 0$ and set $cS = \{cs \mid s \in S\}$. Show that cS bounded above, and that* $\sup(cS) = c \inf S$.

Solution. (i) As S is bounded below then $s \geqslant \inf S$ for all $s \in S$. As $c < 0$ then $cs \leqslant c \inf S$ for all $s \in S$ and hence $c \inf S$ is an upper bound for cS. We now have to show that $c \inf S$ is the *least* upper bound of cS and there are two ways of showing this:

(ii) Suppose that u is an upper bound of cS. Then $cs \leqslant u$ for all $s \in S$ and hence $s \geqslant u/c$ for all $s \in S$. This means that u/c is a lower bound of S and in particular $\inf S \geqslant u/c$ as $\inf S$ is the greatest lower bound of S. Then $c \inf S \leqslant u$ and we finally see that $c \inf S$ is less than or equal to any other upper bound of cS.

(ii)' Alternatively let $\varepsilon > 0$ and then, as $\inf S$ is the greatest lower bound for S, there is $s \in S$ such that

$$\inf S \leqslant s < \inf S - \varepsilon/c \qquad \Longrightarrow \qquad c \inf S - \varepsilon < cs \leqslant c \inf S,$$

verifying the supremum approximation property. ■

Theorem 1.73 *($\sqrt{2}$ **exists**) There exists a unique positive number α such that $\alpha^2 = 2$.*

Proof. Let $S = \{x \in \mathbb{R} \mid x^2 < 2\}$. Note that $1^2 = 1 < 2$, so that $1 \in S$ and in particular S is non-empty. Further if $x > 2$ then

$$x^2 = xx > 2x > 4 > 2.$$

Hence 2 is an upper bound for S and so we may define

$$\alpha = \sup S.$$

Note further that $\alpha \geqslant 1 > 0$ is positive. We split the remainder of the proof into showing that $\alpha^2 < 2$ and $\alpha^2 > 2$ both lead to contradictions. By the trichotomy axiom it then follows that $\alpha^2 = 2$.

- Suppose for a contradiction that $\alpha^2 < 2$. Our aim is to show that $(\alpha + h)^2 < 2$ for some $h > 0$ which would give a contradiction. Let $0 < h < 1$ so that $h^2 < h$ in particular. Then
$$(\alpha + h)^2 = \alpha^2 + 2h\alpha + h^2 < \alpha^2 + 2h\alpha + h = \alpha^2 + (2\alpha + 1)\, h.$$
We will have $\alpha^2 + (2\alpha + 1)\, h < 2$, and hence $(\alpha + h)^2 < 2$, if further
$$h < \frac{2 - \alpha^2}{2\alpha + 1},$$
noting the RHS is positive. So we set
$$0 < h < \min\left(1, \frac{2 - \alpha^2}{2\alpha + 1}\right),$$
then $\alpha + h \in S$. But this contradicts $\alpha = \sup S$.

- Suppose instead that $\alpha^2 > 2$. Our aim is to show that $(\alpha - h)^2 > 2$ for some $h > 0$ which again gives a contradiction. Note

$$(\alpha - h)^2 = \alpha^2 - 2h\alpha + h^2 > \alpha^2 - 2h\alpha.$$

So if further

$$0 < h < \frac{\alpha^2 - 2}{2\alpha},$$

then $(\alpha - h)^2 > 2$. As $x \mapsto x^2$ is an increasing function for $x > 0$ then no element of S lies in the interval $(\alpha - h, \alpha)$ which contradicts the approximation property.

Finally, by trichotomy, $\alpha^2 = 2$ is the only remaining possibility. To show uniqueness of α suppose that β is a positive number such that $\beta^2 = 2$. Then

$$\alpha^2 = \beta^2 \quad \Longrightarrow \quad \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 0.$$

It follows that $\beta = \alpha$ or $\beta = -\alpha$. As $\alpha > 0$ then $-\alpha < 0$ and so α is the only positive solution of $x^2 = 2$.

■

Remark 1.74 *Note that this result shows that $\mathbb{Q}$ does not satisfy the completeness axiom as the set $\{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum in $\mathbb{Q}$.*

Notation 1.75 *We write $\sqrt{2}$ for α.*

Theorem 1.76 *Let a be any positive real number. Then there exists a unique real number – denoted by $\sqrt{a}$ – whose square is a.*

Proof. This just involves a refining of the previous argument. See also Example 5.9. ■

We've already noted (Corollary 1.56) that $\mathbb{N}$ has no maximal element. This is something that can be proved within the axioms for $\mathbb{N}$. Situating $\mathbb{N}$ within $\mathbb{R}$ we now note:

Theorem 1.77 *(**Archimedean Property**) $\mathbb{N}$ is not bounded within $\mathbb{R}$. That is, for any $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof. If not, $\mathbb{N}$ is bounded above and not empty. Set $\alpha = \sup \mathbb{N}$. Then so $\alpha - 1 < k$ for some $k \in \mathbb{N}$ by the approximation property. But as $k + 1 \in \mathbb{N}$ and $\alpha < k + 1$, contradicting α being an upper bound for $\mathbb{N}$. ■

Corollary 1.78 *Let $\varepsilon > 0$. Then $0 < \frac{1}{n} < \varepsilon$ for some $n \in \mathbb{N}$.*

Proof. Apply the Archimedean Property to $1/\varepsilon$. ■

Corollary 1.79 *Given reals a, b with $a < b$ then there exists a rational number q and an irrational number r such that $a < q < b$ and $a < r < b$.*

Proof. Left as Sheet 2, Exercises 5ii and 5iii. ■

1.4 Complex numbers

[This section is by way of recap from the *Introduction to Complex Numbers* course. and will not be lectured.]

We can define $\mathbb{C}$ from $\mathbb{R}$ by taking the set $\mathbb{C}$ to be $\mathbb{R}^2$, the set of real ordered pairs and defining addition $+$ and multiplication $\times$ by

$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2)\\ (a_1, b_1) \times (a_2, b_2) &= (a_1 a_2 - b_1 b_2, a_2 b_1 + a_1 b_2)\,.\end{aligned}$$

So that, for example

$$(0,1)^2 = (0,1) \times (0,1) = \left(0^2 - 1^2, 0 \times 1 + 0 \times 1\right) = (-1, 0)\,.$$

We more commonly write i for $(0,1)$ and write $a + bi$ for (a, b), so that the above equation states $i^2 = -1$. Further we identify each real number r with $r + 0i$ and so can think of the reals as a subset of the complex numbers.

It is not hard to check that the field axioms A1-A4, M1-M4, D, Z are all true of the complex numbers.

Proposition 1.80 *There is no subset $\mathbb{P}$ of $\mathbb{C}$ which satisfies the order axioms P1–P3.*

Proof. Suppose for a contradiction that such $\mathbb{P} \subseteq \mathbb{C}$ exists. By P3, precisely one of the following holds:

$$i \in \mathbb{P}; \qquad i = 0; \qquad -i \in \mathbb{P}.$$

If $i = 0$ then $1 = i^4 = 0^4 = 0$ which contradicts Z. If $i \in \mathbb{P}$ then by P2 we have $-i = i^3 \in \mathbb{P}$, but then $\pm i \in \mathbb{P}$ which contradicts P3. Assuming $-i \in \mathbb{P}$ leads to the same contradiction. Hence not one of the the possibilities required by P3 holds and no subset $\mathbb{P} \subseteq \mathbb{C}$ exists with the requisite properties. ■

So the complex numbers cannot be made into an ordered field. There is, though, the complex modulus function, that we can use to determine the 'size' of complex numbers. Let $z = x + yi$, where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, throughout the following.

Definition 1.81 *The **modulus** of z, written $|z|$, is given by*

$$|z| = \sqrt{x^2 + y^2}$$

This makes sense as $x^2 + y^2 \geqslant 0$.

Definition 1.82 *The **conjugate** of z, written $\bar{z}$ (or z^* in some texts), is given by*

$$\bar{z} = x - yi.$$

None of the following properties is at all difficult to prove – they are algebra, not analysis.

1. If z is real (i.e. $z = x + 0i$) then $|z| = |x|$; that is the definitions of real and complex modulus agree where applicable.
2. $|z| = |\bar{z}|$.
3. $|z|^2 = z\bar{z}$.
4. $|\operatorname{Re} z| \leqslant |z|$, $|\operatorname{Im} z| \leqslant |z|$.
5. $z + \bar{z} = 2 \operatorname{Re} z$.
6. $z - \bar{z} = 2i \operatorname{Im} z$.
7. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
8. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

Theorem 1.83 *For $z, w \in \mathbb{C}$,*

$$|zw| = |z|\,|w|.$$

Proof. Let $z = x + yi$ and $w = u + vi$. Then all we need is the factorisation

$$(xv + yu)^2 + (xu - yv)^2 = (x^2 + y^2)(u^2 + v^2)$$

and the existence of unique non-negative square roots. ■

Theorem 1.84 *(**Triangle Inequality**) For complex numbers z, w,*

$$|z + w| \leqslant |z| + |w|.$$

Proof. Using the above properties.

$$\begin{aligned} |z+w|^2 &= (z+w)\,(\overline{z+w}) \\ &= (z+w)\,(\bar{z}+\bar{w}) \\ &= z\bar{z} + (z\overline{w} + \bar{z}w) + w\overline{w} \\ &= z\bar{z} + 2\operatorname{Re}(z\overline{w}) + w\overline{w} \\ &\leqslant z\bar{z} + 2|z\overline{w}| + w\overline{w} \\ &= |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

■

Remark 1.85 *The name 'triangle inequality' is clearer to explain in $\mathbb{C}$ rather than in $\mathbb{R}$. Consider the triangle in $\mathbb{C}$ with vertices $A = 0$, $B = z$ and $C = z + w$. Then $|z + w| = |AC|$, $|z| = |AB|$ and $|BC| = |(z + w) - z| = |w|$. So the triangle inequality states that the length of one edge is less than the sum of the lengths of the other two edges.*

2. COUNTABILITY

In this section we introduce some of the simplest ideas about the size or cardinality of a set. You should note, but not be too concerned, that we have not rigorously defined what a set is. Most (but by no means all) mathematicians agree on what a set is, but you will have to wait till the third year B1.2 option *Set Theory* to find out what the current consensus is.

Almost all the results in this section are due to the German mathematician Georg Cantor (1845-1918). The lectures will focus mainly on the notion of *countability* and in particular that the real numbers are *uncountable*. The cardinality of finite sets was discussed in the *Introduction to University Mathematics* course.

Definition 2.1 *Let A and B be sets. We say A and B are **equinumerous**, and write $A \approx B$, if there is a bijection $f\colon A \to B$.*

Note that for any sets A, B, C,

$$\begin{aligned} A &\approx A; \\ A \approx B &\iff B \approx A; \\ A \approx B,\ B \approx C &\implies A \approx C. \end{aligned}$$

These properties rely on the identity map being a bijection, bijections being invertible and the composition of two bijections being a bijection.

Example 2.2 *The sets $A = \{0, 1, 2, 3, \ldots\}$ and $B = \{1, 2, 3, 4, \ldots\}$ are equinumerous despite B being a proper subset of A; we can see this by considering the bijection $f\colon A \to B$ given by $f(n) = n + 1$.*

Definition 2.3 *A set A is called **finite** if either $A = \varnothing$ or we have that $A \approx \{1, 2, \ldots, k\}$ for some non-zero natural number k. In the former case we say that A has cardinality 0, in the latter has cardinality k. We denote the **cardinality** of A by $|A|$.*

Remark 2.4 *Note that the cardinality of a finite set is well-defined. There would be issues with the above definition if it were possible to find a set A and distinct k, l such that*

$$A \approx \{1, 2, \ldots, k\}, \quad \text{and} \quad A \approx \{1, 2, \ldots, l\}.$$

To sketch a 'least criminal' proof, consider the smallest k for which $A \approx \{1, 2, \ldots, k\}$ and distinct l with $A \approx \{1, 2, \ldots, l\}$. We could then construct a bijection f from $\{1, \ldots, k\}$ to $\{1, \ldots, l\}$. Remove k and $f(k)$, and (with some adjustment) we'd get two equinumerous sets of sizes $k - 1$ and $l - 1$. But $k - 1 \neq l - 1$ which contradicts the minimality of k.

Exercise 2.5 *How would you prove the following for finite sets A and B?*

- If $A \subseteq B$ then $|A| \leqslant |B|$.
- If $f\colon A \to B$ is a 1-1 map then $|A| \leqslant |B|$.
- If $g\colon A \to B$ is an onto map then $|A| \geqslant |B|$.

Remark 2.6 *(Off-syllabus) More generally given two (possibly infinite) sets A and B we write $|A| \leqslant |B|$ if there is a 1-1 map from A to B. The **Cantor-Bernstein-Schröder Theorem** states that*

$$\text{if } |A| \leqslant |B| \text{ and } |B| \leqslant |A| \text{ then } A \approx B,$$

i.e. if there is a 1-1 maps $A \to B$ and $B \to A$ then there is a bijection from A to B.

Cantor was the first to publish this result, without proof, in 1887. A later proof by Cantor relied on the Axiom of Choice, which is a non-stand axiom of set theory, and unnecessary to this theorem. In 1887 Dedekind proved the theorem, without reference to the Axiom of Choice but did not publish his result. In 1897 Bernstein and Schröder independently published proofs.

Definition 2.7 *A set which is not finite is called **infinite**.*

Remark 2.8 *An equivalent definition for a set to be infinite is that the set has an equinumerous proper subset.*

Example 2.9 *The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all infinite.*

Somewhat surprisingly, we will see that the above sets are not all equinumerous.

Definition 2.10 *A set A is called **countably infinite** (or denumerable) if $\mathbb{N} \approx A$. We say A is **countable** if A is finite or countably infinite. A set which is not countable is called **uncountable.** (Note some authors use 'countable' to mean 'countably infinite'.)*

We then have:

Proposition 2.11 *A set A is countable if and only if there is a 1-1 map $f\colon A \to \mathbb{N}$.*

Corollary 2.12 *If $B \subseteq A$ and A is countable then B is also countable. Equivalently if B is uncountable then A is uncountable.*

Example 2.13 *The set of integers is countably infinite.*

Solution. A bijection from $\mathbb{N} = \{0, 1, 2, \ldots\}$ to $\mathbb{Z}$ can be described using the list

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

or more formally by setting

$$f(n) = \begin{cases} (n+1)/2 & n \text{ is odd} \\ -n/2 & n \text{ is even} \end{cases}$$

■

We can generalise this approach to show:

Proposition 2.14 *Suppose that A_1 and A_2 are disjoint and countably infinite. Then $A_1 \cup A_2$ is countably infinite.*

Proof. As A_i is countably infinite then there is a bijection $f_i \colon \mathbb{N} \to A_i$. We define the map $g \colon \mathbb{N} \to A_1 \cup A_2$ by

$$g(n) = \begin{cases} f_1\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ f_2\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

This map can be readily checked to be a bijection onto $A_1 \cup A_2$. ■

Remark 2.15 *The above proposition still holds even if A_1 and A_2 are not disjoint; essentially the same g can be used to list A_1 and A_2 but skipping over any repetitions as they occur.*

Proposition 2.16 *Suppose that A and B are countably infinite. Then the Cartesian product $A \times B$ is countably infinite.*

Proof. As both sets are countably infinite then they can be listed as

$$a_0, a_1, a_2, \ldots \qquad b_0, b_1, b_2, \ldots$$

The elements (a_i, b_j) of $A \times B$ can be put into a grid as below

$$\begin{array}{ccccccccc}
(a_0,b_0) & \to & (a_1,b_0) & & (a_2,b_0) & \to & (a_3,b_0) & & (a_4,b_0) \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
(a_0,b_1) & & (a_1,b_1) & & (a_2,b_1) & & (a_3,b_1) & & (a_4,b_1) \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
(a_0,b_2) & & (a_1,b_2) & & (a_2,b_2) & & (a_3,b_2) & & (a_4,b_2) \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
(a_0,b_3) & & (a_1,b_3) & & (a_2,b_3) & & (a_3,b_3) & & (a_4,b_3) \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
(a_0,b_4) & & (a_1,b_4) & & (a_2,b_4) & & (a_3,b_4) & & (a_4,b_4)
\end{array}$$

and then can themselves be listed, in accordance with the arrows, as

$$(a_0,b_0)\,,\ (a_1,b_0)\,,\ (a_0,b_1)\,,\ (a_0,b_2)\,,\ (a_1,b_1)\,,\ (a_2,b_0)\,,\ (a_3,b_0)\,,\ldots$$

■

Corollary 2.17 *If $A_0, A_1, A_2, \ldots$ are countable sets then so is their union $\bigcup_{i=0}^{\infty} A_i$.*

Proof. As each set A_i is countable then it can be listed as

$$a_{i0}, a_{i1}, a_{i2}, \ldots$$

By placing the a_{ij} into a square grid as in the previous proof then these can be counted in a similar fashion, omitting any repetitions of elements that arise. ■

Remark 2.18 *For those with a particular interest in set theory, note that the above proof relies on the Axiom of Choice in a subtle way. In listing each set A_i we are effectively choosing a bijection $f_i \colon \mathbb{N} \to A_i$ and to do so for each i requires the Axiom of Choice.*

Notation 2.19 *The symbol $\aleph_0$ is used to denote the cardinality $|\mathbb{N}|$ of $\mathbb{N}$, or any countably infinite set. $\aleph$ is aleph, the first letter of the Hebrew alphabet and $\aleph_0$ is read as 'aleph-null' or 'aleph-nought'.*

Example 2.20 *The set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable as we have seen. An explicit example of an injection $f\colon \mathbb{N}^2 \to \mathbb{N}$ is the map*

$$f(m,n) = 2^m 3^n.$$

Example 2.21 *The set $\mathbb{Q}^+$ of positive rationals is countable as the map taking a rational m/n in its lowest form to (m,n) is an injection from $\mathbb{Q}^+$ to $\mathbb{N}^2 \approx \mathbb{N}$. So $\mathbb{Q} = \{0\} \cup \mathbb{Q}^+ \cup \{-q \mid q \in \mathbb{Q}^+\}$ is also countable.*

However it turns out that not all infinite sets are countable. In particular it is a fact of considerable importance that $\mathbb{R}$ is uncountable. This was first shown in 1874 by Cantor, producing a second more intuitive proof using his *diagonal argument* in 1891. Below we give two proofs. The second is Cantor's diagonal argument which makes use of decimal expansions – something we are yet to define and construct (see Example 5.12) – whilst the first proof uses results we have so far demonstrated.

Theorem 2.22 *$\mathbb{R}$ is uncountable.*

Proof. Proof 1: If $\mathbb{R}$ were countable, then so too would be $[0,1]$. Clearly $[0,1]$ is not finite as it contains all $\frac{1}{k}$ where $k \geqslant 1$. We proceed now with a proof by contradiction to show that $[0,1]$ is not countably infinite. Suppose $f : \mathbb{N} \to [0,1]$ is a bijection and we write $x_k = f(k)$.

- We choose distinct a_0, b_0 so that $x_0 \notin [a_0, b_0]$. If $x_0 \neq 1$ then we can find a_0 and b_0 such that $x_0 < a_0 < b_0 < 1$ and if $x_0 = 1$ then we can take the interval $[0, 1/2]$.
- Having chosen a_0, b_0 we then select a_1, b_1 so that $a_0 < a_1 < b_1 < b_0$ and $x_1 \notin [a_1, b_1]$. In a similar fashion to the above if $x_1 < b_0$ we can find a_1 and b_1 so that

  $$\max(a_0, x_1) < a_1 < b_1 < b_0$$

  and if $x_1 \geqslant b_0$ then we can take the interval $[(2a_0 + b_0)/3, (a_0 + 2b_0)/3]$, i.e. the middle third of the previous interval.
- We repeat this process producing reals a_i and b_j such that

  $$0 \leqslant a_0 < a_1 < a_2 < \cdots < b_2 < b_1 < b_0 \leqslant 1$$

  and $x_i \notin [a_i, b_i]$ for each i.

Now set $S = \{a_j \mid j \in \mathbb{N}\}$ which is bounded above by 1 and $T = \{b_j \mid j \in \mathbb{N}\}$ is bounded below by 0. So we may define

$$\lambda = \sup S \quad \text{and} \quad \mu = \inf T.$$

For all m, n we have $a_m \leqslant b_n$. In particular, each b_n is an upper bound of S and so $\lambda \leqslant b_n$ for all n as λ is the least upper bound of S. So λ is a lower bound of T which means $\lambda \leqslant \mu$ as μ is the greatest lower bound of T. Then

$$a_n \leqslant \lambda \leqslant \mu \leqslant b_n \text{ for all } n.$$

For all n we have $\lambda \in [a_n, b_n]$ and $x_n \notin [a_n, b_n]$ and so $\lambda \neq x_n$ for all n which contradicts the fact that f is a bijection. ■

Proof. Proof 2 (diagonal argument): We will prove $\mathbb{R}$ is uncountable by showing that the interval $(0, 1]$ is uncountable. To each x in this interval corresponds a unique decimal expansion $0.a_1a_2a_3\ldots$ which does not end in a string of zeros.

Suppose for a contradiction that $f\colon \mathbb{N} \to (0, 1]$ is a bijection. Then we may uniquely write out the decimal expansions of $f(1), f(2), \ldots$. Say:

$$\begin{aligned} f(1) &= 0.r_{11}r_{12}r_{13}r_{14}\ldots \\ f(2) &= 0.r_{21}r_{22}r_{23}r_{24}\ldots \\ f(3) &= 0.r_{31}r_{32}r_{33}r_{34}\ldots \end{aligned}$$

Cantor then created a real α not on the list by setting

$$\alpha = 0.a_1a_2a_3\ldots$$

where

$$a_k = \begin{cases} 6 & \text{if } r_{kk} \neq 6 \\ 7 & \text{if } r_{kk} = 6 \end{cases}$$

The decimal expansion of α is allowed (in that it doesn't conclude in a string of 0s) and we see, for any k, that $\alpha \neq f(k)$ as α and $f(k)$ disagree in the kth decimal position. This contradicts the surjectivity of f. ■

Notation 2.23 *The symbol* $\mathfrak{c}$*, which stands for 'continuum' (an old name for the real line), denotes the cardinality of* $\mathbb{R}$*.*

Corollary 2.24 $\mathbb{C}$ *is uncountable. (In fact,* $\mathbb{C} \approx \mathbb{R}$*, which can be proved using the Cantor-Bernstein-Schröder theorem)*

The following result, known as Cantor's Theorem. It shows that any set has more subsets than elements. It further proves that there are ever increasingly large sets that can be formed.

Theorem 2.25 ***(Cantor's Theorem**, 1891) Let* A *be a set, and let* $\mathcal{P}(A)$ *be the* ***power set*** *of* A*, that is the set of subsets of* A*. Then*

$$|A| < |\mathcal{P}(A)|.$$

This means there is an injection from A *to* $\mathcal{P}(A)$ *but there is no bijection from* A *to* $\mathcal{P}(A)$*.*

Proof. The map $A \to \mathcal{P}(A)$ given by $a \mapsto \{a\}$ is an injection.

Suppose we have a map $f\colon A \to \mathcal{P}(A)$. We show that f cannot be a surjection, and so cannot be a bijection. We consider the set

$$X = \{a \in A \mid a \notin f(a)\},$$

and will show that $X \notin f(A)$. Hence f is not onto.

Suppose to the contrary that $X = f(x)$ for some $x \in A$. Then either $x \in X$ or $x \notin X$. From the definition of X, we know that $x \in X$ if and only if $x \notin f(x) = X$. This is the required contradiction and so $X \notin f(A)$, as claimed. ■

Example 2.26 *Let $A = \{1, 2, 3\}$ and define $f\colon A \to \mathcal{P}(A)$ by*

$$f(1) = \varnothing, \qquad f(2) = A, \qquad f(3) = \{1, 2\}.$$

Find the set $X \subseteq A$ guaranteed by Cantor's theorem not to be in $f(A)$.

Solution. As $1 \notin f(1)$, $2 \in f(2)$, $3 \notin f(3)$ then $X = \{1, 3\}$. ■

Example 2.27 $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$. *(This can be proved using the Cantor-Bernstein-Schröder theorem.)*

Remark 2.28 *In the remainder of the course there will be very few explicit references to the uncountability of the real numbers. Having said that, it is the uncountability of $\mathbb{R}$ that characterises how we describe real numbers and impacts the nature of analysis.*

*The integers, rational numbers, algebraic numbers (Sheet 2, Exercise 6) are all countable sets. Further the **computable numbers** can be shown to be countable.*

A real number is said to be computable if there is a finite length computer program, written in a finite alphabet, that can (in principle) calculate that real number to any required accuracy. Essentially the set of computable numbers comprises all real numbers that can be described by finite means. However Cantor's proofs can be readily adapted to show that there are countably many such programs and so countably many computable numbers. This means, to describe the uncountably many real numbers, some infinite description is necessary – such as infinite decimal expansions.

Quite what this means is somewhat contentious. In this course we will consider arbitrary decimal expansions involving $0, 1, \ldots, 9$, but some logicians and mathematical philosophers take issue with this. In particular, 'intuitionists' would be content only with a decimal expansion that is defined constructively.

3. SEQUENCES AND CONVERGENCE

How do we handle a specific real number in practice? One option is to look at successive approximations. For example, we could have the following approximations for $\sqrt{2}$:

$$1,\ \frac{14}{10},\ \frac{141}{100},\ \frac{1414}{1000},\ \ldots$$

– namely the truncated decimal expansions for $\sqrt{2}$ – or we could approximate π with the sequence

$$3,\ \frac{22}{7},\ \frac{333}{106},\ \frac{103993}{33102},\ \ldots$$

which are the 'continued fraction convergents' of π. Our first task is to make precise the idea that these approximations approach the real numbers that they represent.

Definition 3.1 *A **sequence of real numbers** or, more simply, a **real sequence** is a function* $a\colon \mathbb{N} \to \mathbb{R}$.

Definition 3.2 *A **sequence of complex numbers** or, more simply, a **complex sequence** is a function* $a\colon \mathbb{N} \to \mathbb{C}$.

In these definitions we can take $\mathbb{N}$ to be either $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$.

Definition 3.3 *Given a natural number n, the n**th term** of the sequence a is $a(n)$ and we denote this a_n.*

Example 3.4 *Here are some sequences:*

- $n \mapsto \alpha(n) = (-1)^n$,
- $n \mapsto \beta(n) = 0$,
- $n \mapsto \gamma(n) = n$.

Note, in practice, we often just give the sequence's values, and say 'the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$' if it is clear what the function 'must be'. Or we may be more explicit and write 'the sequence $(a_n)_{n=1}^{\infty}$' or 'the sequence (a_n)' where a_n is a formula in n.

Note also that although n determines the nth value of a sequence, the nth value does not determine n because the defining function need not be injective. Consider the sequences α and β above for example.

The space of real (or complex) sequences is naturally a vector space; in fact it is naturally an algebra where elements can be multiplied. Suppose that (a_n) and (b_n) are sequences of real (or complex) numbers and $c \in \mathbb{R}$ (or $\mathbb{C}$). We define the sequences

$$(a_n + b_n), \qquad (ca_n), \qquad (a_n b_n), \qquad (a_n / b_n)$$

in the obvious, termwise way. All are well defined except possibly the quotient, where we must insist on all the terms of (b_n) being non-zero.

Example 3.5 *$a_n = (-1)^n$ and $b_n = 1$ for all $n \geqslant 0$.*

$$(a_n + b_n) = (0, 2, 0, 2, 0, 2, 0, \ldots)\,; \qquad (-a_n) = (-1)^{n+1}\,;$$
$$(a_n b_n) = (a_n)\,; \qquad (a_n^2) = (b_n)\,.$$

3.1 Convergence

Definition 3.6 *Let (a_n) be a real sequence and $L \in \mathbb{R}$. We say that (a_n) **converges to** L if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

*We also say that (a_n) **tends** to L. We write this as*

$$(a_n) \to L \qquad or \qquad a_n \to L \ \ as \ \ n \to \infty \qquad or\ just \qquad a_n \to L.$$

Note than N can, and typically will, depend on ε. The smaller ε is, the larger N will typically need to be.

Definition 3.7 *If $(a_n) \to L$ then we say that L is a **limit** of (a_n) and we write*

$$L = \lim_{n\to\infty} a_n \qquad or\ just \qquad L = \lim a_n.$$

Definition 3.8 *We say that (a_n) **converges** or is **convergent** if there exists $L \in \mathbb{R}$ such that $(a_n) \to L$. In full*

$$(a_n)\ converges \quad \Longleftrightarrow \quad \exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

Definition 3.9 *We say that (a_n) **diverges** or is **divergent** if it does not converge. In full*

$$(a_n)\ diverges \quad \Longleftrightarrow \quad \forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geqslant N \quad |a_n - L| \geqslant \varepsilon.$$

Remark 3.10 *In the above, ε is an arbitrary positive number though instinctively we usually think of ε as being very small. The smaller the value of ε the further into the sequence we will usually have to look to find a value of N that will suffice.*

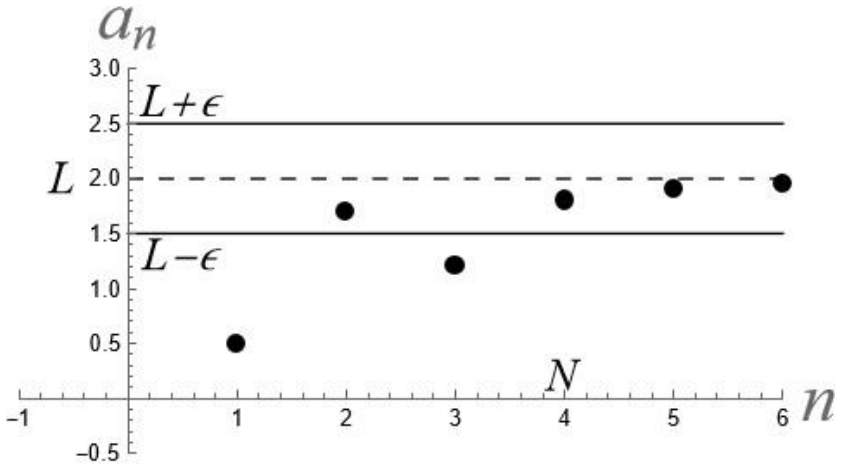

Fig. 3.1 – Graphing a Real Sequence

In Figure 3.1 we have $L = 2$ and $\varepsilon = 0.5$. Note that a_2 lies in the range $(L - \varepsilon, L + \varepsilon)$ though 2 cannot act as N here as a_3 is not in the required range. It seems, from the figure, that $N = 4$ would suffice as each of $x_4, x_5, x_6, \ldots$ appears to lie in $(L - \varepsilon, L + \varepsilon)$. In fact any $N \geqslant 4$ would be satisfactory, it doesn't have to be first such N. The smaller ε is, the larger N will need to be.

Looking then at the definition of $a_n \to L$, we need to find some N, not necessarily the smallest, such that $a_N, a_{N+1}, a_{N+2}, \ldots$ lies in $(L - \varepsilon, L + \varepsilon)$ and we need to be able to do this for all $\varepsilon > 0$.

The definition of '(a_n) converges' makes no specific mention of the limit L, and so to demonstrate this the first task is to determine the candidate limit L and then to show $a_n \to L$.

Remark 3.11 *We also note from the above that showing*

$$\exists L \in \mathbb{R} \quad \forall \varepsilon \in (0, 1) \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon$$

is sufficient to show convergence. That is, WLOG, we can assume $0 < \varepsilon < 1$. Previously we had to concern ourselves with, say, finding the sequence eventually in $(L - 2, L + 2)$. But as we can find the sequence eventually in $(L - 0.5, L + 0.5)$ then the sequence is eventually in $(L - 2, L + 2)$ as well.

And there's nothing special about assuming $\varepsilon < 1$ here. If it suited us we could assume $0 < \varepsilon < \varepsilon_0$ for any positive ε_0.

Definition 3.12 *(**Tails and Neighbourhoods**) Let (a_n) be a sequence, and let k be a natural number. Then the k**th tail** of (a_n) is the sequence $n \mapsto a_{n+k}$ i.e. it equals the sequence*

$$(a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$$

which we will also write as $(a_{n+k})_{n=0}^{\infty}$ or $(a_n)_k^{\infty}$.

*For $L \in \mathbb{R}$ and $\varepsilon > 0$, we refer to the set $(L - \varepsilon, L + \varepsilon)$ as a **neighbourhood** of L (or sometimes a **basic neighbourhood** of L).*

So we can rephrase '(a_n) converges to L' as 'any neighbourhood of L contains a tail of (a_n).'

In practice, we will not be interested in a specific kth tail so much as in some (unspecified) tail or all tails past a certain point in the sequence. The tails give a way of focusing on the *long-term behaviour* of a sequence ignoring any short-term aberrant behaviour at the start of a sequence. Whether or not a sequence converges purely depends on the long-term behaviour of the sequence as we see in the next proposition.

Proposition 3.13 *Let (a_n) be a real sequence and let $L \in \mathbb{R}$. Then the following three statements are equivalent.*

(a) (a_n) converges to L;

(b) some tail of (a_n) converges to L;

(c) all tails of (a_n) converge to L.

Proof. We shall demonstrate the implications as (a) implies (c), (c) implies (b) and (b) implies (a).

(a) $\implies$ (c): Suppose that (a_n) converges to L and let $k \in \mathbb{N}$, $\varepsilon > 0$. As $(a_n) \to L$ then there exists N such that

$$\forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

For such n, we have $n + k \geqslant n \geqslant N$ and so

$$\forall n \geqslant N \quad |a_{n+k} - L| < \varepsilon.$$

Hence, for any $k \in \mathbb{N}$, the kth tail of (a_n) converges to L.

(c) $\implies$ (b): (c) clearly implies (b).

(b) $\implies$ (a) : Suppose that the kth tail of (a_n) converges to L. Let $\varepsilon > 0$. Then there exists N such that

$$\forall n \geqslant N \quad |a_{n+k} - L| < \varepsilon.$$

Hence

$$\forall n \geqslant N + k \quad |a_n - L| < \varepsilon$$

and we see that (a_n) converges to L. ■

Remark 3.14 *(Intersection of tails) We will often find ourselves in a situation where we know something is true of a sequence (a_n) for $n \geqslant N_1$ and a second thing is true for $n \geqslant N_2$. Note that both facts will apply for the tails' intersection, which is when $n \geqslant \max(N_1, N_2)$, which is itself a tail of the sequence.*

This argument can be applied finitely many times, but only finitely many. The intersection of infinitely many tails can be empty – e.g. when $N_k = k^2$ say.

Before giving some examples, we show that a limit, if it exists, is unique. So we are justified in the use of the language 'the limit'.

Theorem 3.15 ***(Uniqueness of Limits)*** *Let (a_n) be a real sequence such that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Then $L_1 = L_2$.*

Thoughts: Proofs of uniqueness usually begin by assuming non-uniqueness and obtaining a contradiction or assuming there are two such elements and showing they're equal. Our proof is by contradiction. If there were two limits, $L_1 \neq L_2$, then there would be tails of the sequence in a neighbourhood of each. Provided these neighbourhoods are disjoint, there is nowhere for the tails' intersection to be.

Proof. Suppose not and set $\varepsilon = |L_1 - L_2| > 0$. Then $\varepsilon/2 > 0$ and there exists N_1 such that

$$n \geqslant N_1 \quad \implies \quad |a_n - L_1| < \varepsilon/2$$

Likewise there exists N_2 such that

$$n \geqslant N_2 \quad \implies \quad |a_n - L_2| < \varepsilon/2.$$

Then for $n \geqslant \max(N_1, N_2)$ both inequalities hold and

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leqslant |L_1 - a_n| + |a_n - L_2| \qquad \text{[by the triangle law]} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= |L_1 - L_2| \end{aligned}$$

which is the required contradiction. ■

Example 3.16 *Let*

$$a_n = \frac{2^n - 1}{2^n} \qquad \text{for } n \geqslant 1.$$

Then $(a_n) \to 1$.

Thoughts: Here the limit is given, namely $L = 1$, so we don't have to put any thought into deciding what the limit is (as in the next example). The statement of $(a_n) \to 1$ is

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

To address the first quantifier all we need do is introduce a positive ε. Given this ε, our task is to find a suitable N. In the example below we include the necessary 'back of the envelope' calculation as part of the proof.

Solution. Note

$$|a_n - 1| = \left|1 - 2^{-n} - 1\right| = 2^{-n}.$$

Let $\varepsilon > 0$. We need to find N such that

$$n \geqslant N \quad \Longrightarrow \quad 2^{-n} < \varepsilon.$$

But note that $2^n > n$ for all $n \in \mathbb{N}$ and so if $N > 1/\varepsilon$ (which we know to exist by the Archimedean Property) and $n \geqslant N$ we have

$$|a_n - 1| = 2^{-n} = \frac{1}{2^n} < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon.$$

■

Example 3.17 *The sequence*

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4} \qquad (n \geqslant 1)$$

is convergent.

Thoughts: Because the statement for convergence is

$$\exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon$$

our first work is in deciding what the limit L is. Note the limit was given to us in the previous example. Our 'back of the envelop' argument might go: l for large positive n,

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{4}{n^2}} \approx \frac{1}{3}$$

We give no exact definition of $\approx$ (approximately equal to) but none of the above is part of our, rather informal first thoughts. So $\frac{1}{3}$ seems the obvious candidate for our limit. To begin the proof then, looking at the quantifiers we need to address:

Solution. Let $\varepsilon > 0$. Note

$$\begin{aligned}\left|a_n - \frac{1}{3}\right| &= \left|\frac{n^2+n+1}{3n^2+4} - \frac{1}{3}\right| \\ &= \frac{3n-1}{3(3n^2+4)} \\ &\leqslant \frac{3n}{3(3n^2+4)} \\ &\leqslant \frac{3n}{3\times 3n^2} \\ &< \frac{1}{n}.\end{aligned}$$

By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then, for any $n \geqslant N$, we have

$$\left|a_n - \frac{1}{3}\right| < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$$

to complete the proof. ■

Example 3.18 *Let*

$$a_n = \frac{(-1)^n n^2}{n^2+1} \qquad (n \geqslant 1).$$

Then (a_n) *diverges.*

Thoughts: The quantified definition of divergence is

$$\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geqslant N \quad |a_n - L| \geqslant \varepsilon,$$

so we need to show that any limit real L cannot be a limit.

Looking at the sequence we can see that for large even n

$$a_n = \frac{(-1)^n n^2}{n^2+1} = \frac{1}{1+n^{-2}} \approx 1,$$

whilst for large odd n we have

$$a_n = \frac{(-1)^n n^2}{n^2+1} = \frac{-1}{1+n^{-2}} \approx -1.$$

A natural way forward seems to be to suppose, for a contradiction, that a limit exists and argue (carefully!) that this limit would need to be both near 1 and -1; this would be the required contradiction. In fact, if we look in more detail at the sequence we see that $a_{2n} \geqslant \frac{4}{5}$ for all n and $a_{2n-1} \leqslant -\frac{1}{2}$, so we will take ε in such a way that $2\varepsilon < \frac{4}{5} + \frac{1}{2} = \frac{13}{10}$ which is the closest the even and odd terms get. Our proof thus begins:

Solution. Suppose, for a contradiction, that a limit L exists and set $\varepsilon = \frac{1}{2}$. Then there exists N such that for $n \geqslant N$

$$\left|\frac{(-1)^n n^2}{n^2+1} - L\right| < \frac{1}{2}.$$

In particular, for even $n \geqslant N$ we have

$$\frac{n^2}{n^2+1} - L < \frac{1}{2},$$
$$\implies L > \frac{1}{1+n^{-2}} - \frac{1}{2} \geqslant \frac{1}{5/4} - \frac{1}{2} = \frac{3}{10}.$$

Similarly, for odd $n \geqslant N$ we have

$$L + \frac{n^2}{n^2+1} < \frac{1}{2},$$
$$\implies L < \frac{1}{2} - \frac{1}{1+n^{-2}} \leqslant \frac{1}{2} - \frac{1}{2} = 0.$$

The necessary inequalities $L > \frac{3}{10}$ and $L < 0$ give us our required contradiction. ■

Corollary 3.19 *Let a be a real number with $a > 1$, and k be a positive integer. Then*

$$\frac{n^k}{a^n} \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. This is a corollary to Proposition 1.46. There we showed that there is some $c > 0$ such that

$$\frac{a^n}{n^k} \geqslant c$$

for all $n \geqslant 1$. Replacing k with $k+1$ there exists $c > 0$ such that $a^n/n^{k+1} \geqslant c$ for all $n \geqslant 1$; hence

$$0 < \frac{n^k}{a^n} \leqslant \frac{1}{cn}.$$

Given $\varepsilon > 0$, by the Archimedean property there exists N such that $\left|n^k/a^n\right| < \varepsilon$ for all $n \geqslant N$. That is $n^k/a^n \to 0$ as $n \to \infty$. ■

Proposition 3.20 *(**Convergent sequences are bounded**) Let (a_n) be a real convergent sequence. Then (a_n) is bounded.*

Thoughts: Pick any neighbourhood of the limit and a tail of the sequence of the sequence will be in that neighbourhood. Only finitely many terms of the sequence aren't in that tail.

Proof. This is left as Sheet 3, Exercise 5(iii). ■

3.2 Complex Sequences

Definition 3.21 *Let (z_n) be a sequence of complex numbers and let $w \in \mathbb{C}$. We say that (z_n) converges to L and write $(z_n) \to L$ or $\lim z_n = L$ if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |z_n - L| < \varepsilon.$$

That is $|z_n - L| \to 0$ as $n \to \infty$.

Corollary 3.22 *(**Uniqueness of Limits**) Let (a_n) be a complex sequence and suppose that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Then $L_1 = L_2$.*

Proof. The proof of uniqueness is identical to the previous proof for real sequences. ■

Corollary 3.23 *Let (a_n) be a convergent complex sequence. Then the sequence is bounded.*

Proof. The proof of boundedness is identical to the previous proof for real sequences. ■

Remark 3.24 *(**Graphing complex sequences**) We can represent the behaviour of complex sequences in $\mathbb{C}$ by plotting the terms in the Argand diagram. In Figure 3.2 below, the sequence's limit is $L = 1+i$ and $\varepsilon = 0.3$. Rather than an open interval $(L-\varepsilon, L+\varepsilon)$, the region $|z-L| < \varepsilon$ is an open disc with centre L and radius ε. That is the (basic) neighbourhoods of L are discs centred at L. Again $z_n \to L$ if every neighbourhood of L contains a tail of (z_n). In the figure it appears that any tail of the sequence from $N \geqslant 5$ is inside the sketched disc.*

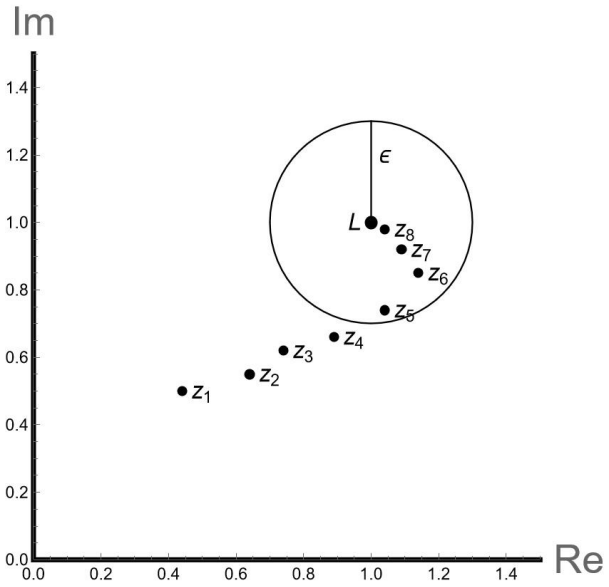

Fig 3.2 – graphing convergence in $\mathbb{C}$

Theorem 3.25 *Let $z_n = x_n + iy_n$. Then (z_n) converges if and only if (x_n) and (y_n) both converge.*

Thoughts: Visually this result is not surprising. For z_n to be within a radius ε disc of $L_1 + iL_2$ means $x_n + iy_n$ is within a square of side 2ε. For a tail of x_n to be within $\varepsilon/2$ of L_1 and a tail of y_n to be within $\varepsilon/2$ of L_2 means the tails' interection of $x_n + iy_n$ is within a quare of side ε which itself is within a radius ε disc of $L_1 + iL_2$.

Proof. Suppose that x_n and y_n both converge and that $\varepsilon > 0$. Set $x = \lim x_n$, $y = \lim y_n$ and $L = x + iy$. By the Triangle Inequality

$$|z_n - L| = |(x_n - x) + i\,(y_n - y)| \leqslant |x_n - x| + |y_n - y|\,.$$

As $x_n \to x$ and $y_n \to y$ then we can find N_1 and N_2 such that

$$\begin{aligned} |x_n - x| &< \varepsilon/2 \text{ whenever } n \geqslant N_1, \\ |y_n - y| &< \varepsilon/2 \text{ whenever } n \geqslant N_2. \end{aligned}$$

So if $n \geqslant \max(N_1, N_2)$ we have $|z_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and we see that $z_n \to L$.

Conversely suppose that z_n converges to L and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|z_n - L| < \varepsilon$ whenever $n \geqslant N$. As $|\mathrm{Re}\, w| \leqslant |w|$ and $|\mathrm{Im}\, w| \leqslant |w|$ for any $w \in \mathbb{C}$ then

$$\begin{aligned} |x_n - x| &= |\mathrm{Re}\,(z_n - L)| \leqslant |z_n - L| < \varepsilon \text{ whenever } n \geqslant N, \\ |y_n - y| &= |\mathrm{Im}\,(z_n - L)| \leqslant |z_n - L| < \varepsilon \text{ whenever } n \geqslant N. \end{aligned}$$

Hence $x_n \to x$ and $y_n \to y$ as required. ■

Example 3.26 *Let*

$$z_n = \left(\frac{1}{1+i}\right)^n.$$

Then $z_n \to 0$.

Proof. Let $\varepsilon > 0$. Note

$$|z_n - 0| = \left|\left(\frac{1}{1+i}\right)^n\right| = \left|\frac{1}{1+i}\right|^n = \frac{1}{|1+i|^n} = \left(\frac{1}{\sqrt{2}}\right)^n.$$

We have already shown that $2^{-k} < \varepsilon$ for $k > 1/\varepsilon$ and so $\left(\sqrt{2}\right)^n < \varepsilon$ for $n > 2/\varepsilon$. ■

Remark 3.27 *Note that in the above example Theorem 3.25 is not particularly useful. It is often simpler to work with a complex sequence as such rather than as a sequence made up of its real and complex parts. By De Moivre's Theorem, the real and imaginary parts of* z_n *are*

$$\begin{aligned} x_n &= \mathrm{Re}\left(\frac{\cos(\pi/4) - i\sin(\pi/4)}{\sqrt{2}}\right)^n = \frac{1}{2^{n/2}}\cos\left(\frac{n\pi}{4}\right); \\ y_n &= \mathrm{Im}\left(\frac{\cos(\pi/4) - i\sin(\pi/4)}{\sqrt{2}}\right)^n = \frac{(-1)^n}{2^{n/2}}\sin\left(\frac{n\pi}{4}\right), \end{aligned}$$

and it only makes for unnecessary work to show that both of these tend to 0.

Notation 3.28 *(**Asymptotic notation**) Let* a_n, b_n *be sequences.*

(a) We write $a_n = O(b_n)$ *if there exist* c *such that for some* N

$$n \geqslant N \quad \Longrightarrow \quad |a_n| < cb_n.$$

This is referred to as ***big O****.*

(b) We write $a_n = o(b_n)$ *if* a_n/b_n *is defined and*

$$\frac{a_n}{b_n} \to 0.$$

This is referred to as ***little o****.*

(c) We write

$$a_n \sim b_n$$

if $a_n/b_n \to 1$ *as* $n \to \infty$. *We say that* a_n *and* b_n *are* ***asymptotically equal****.*

Example 3.29 *As examples*

- $n = O(n^2)$
- $n = o(n^2)$
- $\sin n = O(1)$
- $\sin n = o(n)$
- $(n+1)^2 = n^2 + O(n)$
- $(n+1)^2 \sim n^2$.

3.3 Infinity

Definition 3.30 ***(Real Infinities)*** *Let a_n be a sequence of real numbers. We say 'a_n tends to infinity' and write $a_n \to \infty$ as $n \to \infty$ to mean*

$$\forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad a_n > M.$$

Similarly we write $b_n \to -\infty$ if

$$\forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad b_n < M.$$

(Here we tend to think of M as being a very large positive/negative number.)

Definition 3.31 ***(Complex Infinity)*** *Let z_n be a complex sequence. We say that $z_n \to \infty$ if*

$$\forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |z_n| > M.$$

That is $|z_n| \to \infty$ as a real sequence.

- **Note that the real infinities $\pm\infty$ are not real numbers and complex infinity ∞ is not a complex number** and should not be treated as such.
- Certainly you should **never** be writing anything like the following:

$$\lim_{n\to\infty} \frac{n}{n+1} = \frac{\infty}{\infty} = 1, \qquad \text{or} \qquad \lim_{n\to\infty} \frac{n}{n^2} = \frac{\infty}{\infty} = 1.$$

  The first limit, by a fluke, is correct and the second is false; if properly argued it would be seen that the second limit exists and equals 0.

Remark 3.32 *(**Indeterminate forms**) If $a_n \to \infty$ and $b_n \to \infty$ then there is nothing that can be said about the long term behaviour of a_n/b_n as seen from the examples above. This can be expressed as '$\frac{\infty}{\infty}$ is an indeterminate form'. Other indeterminate forms are*

$$\frac{\infty}{\infty}, \qquad \frac{0}{0}, \qquad 0 \times \infty, \qquad \infty - \infty, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0.$$

It can be useful to talk about '$\frac{\infty}{\infty}$' type limits but this is only informal, shorthand language to describe a family of such sequences. For a specific example, a limit might be found using careful analysis, but there is no single answer for limits of such sequences.

Note that $\infty + \infty$ and $\infty \times \infty$ are not on the list of indeterminates because if $a_n \to \infty$ and $b_n \to \infty$ then $a_n + b_n \to \infty$ and $a_n b_n \to \infty$.

Below is a list of examples to show that the other examples above are indeed indeterminates.

Type	a_n	b_n	*form*	*long term*	*Type*	a_n	b_n	*form*	*long term*
$\frac{0}{0}$	$\frac{1}{n}$	$\frac{1}{n}$	1	$\to 1$	0^0	$\frac{1}{n}$	$\frac{1}{n}$	$1/\sqrt[n]{n}$	$\to 1$
$\frac{0}{0}$	$\frac{1}{n}$	$\frac{(-1)^n}{n}$	$(-1)^n$	*no limit*	0^0	2^{-n}	$\frac{1}{n}$	$1/2$	$\to \frac{1}{2}$
$0 \times \infty$	$\frac{1}{n}$	n	1	$\to 1$	1^∞	$1 + \frac{1}{n}$	n	$\left(1 + \frac{1}{n}\right)^n$	$\to e$
$0 \times \infty$	$\frac{1}{n}$	n^2	n	$\to \infty$	1^∞	$1 + \frac{1}{n^2}$	n	$\left(1 + \frac{1}{n^2}\right)^n$	$\to 1$
$0 \times \infty$	$(-2)^{-n}$	2^n	$(-1)^n$	*no limit*	1^∞	$1 + \frac{1}{n}$	n^2	$\left(1 + \frac{1}{n}\right)^{n^2}$	$\to \infty$
$\infty - \infty$	n	$2n$	$-n$	$\to -\infty$	∞^0	n	$\frac{1}{n}$	$\sqrt[n]{n}$	$\to 1$
$\infty - \infty$	n	$n + \sin n$	$-\sin n$	*no limit*	∞^0	2^n	$\frac{1}{n}$	2	$\to 2$

Remark 3.33 *(**Hyperreals** – **off-syllabus**) There are ways to formally treat infinities and infinitesimals. One such approach is the **hyperreals** which were first studied by Edwin Hewitt in 1948 and greatly extended by Abraham Robinson in 1966. The use of hyperreals is called 'non-standard analysis'. For more see Sheet 3, Exercises 10 and 11.*

Remark 3.34 *(**Neighbourhoods of Infinity** – **off-syllabus**) Note that a real sequence (a_n) converges to $L \in \mathbb{R}$ if every $(L - \varepsilon, L + \varepsilon)$ contains a tail of (a_n). The interval $(L - \varepsilon, L + \varepsilon)$ is called a **neighbourhood** of L.*

Now $(a_n) \to \infty$ if every interval (M, ∞) contains a tail of (a_n) and we call (M, ∞) a neighbourhood of ∞. Note that $a_n \neq \infty$ for all n as (a_n) is a sequence of real numbers.

By comparison, when we have a real sequence (a_n) in the interval $(-\infty, r]$ with $a_n \neq r$ for all n, then $(a_n) \to r$ if every interval (M, r) contains a tail of (a_n).

So when we include ∞ and $-\infty$ to make an 'extended real line' then we essentially make the closed interval $[-\infty, \infty]$.

The situation is rather different with the 'extended complex plane'. There is only one complex infinity which is 'out there' in all directions. A neighbourhood of ∞ is a set

$$\{z \in \mathbb{C} \mid |z| > M\}.$$

*Effectively we are wrapping up the complex plane with a single point at infinity and, topologically, this creates a sphere, commonly known as the **Riemann sphere**. There are actually quite*

detailed connections between the geometry of the sphere and that of the extended complex plane, which can be made explicit via ***stereographic projection****.*

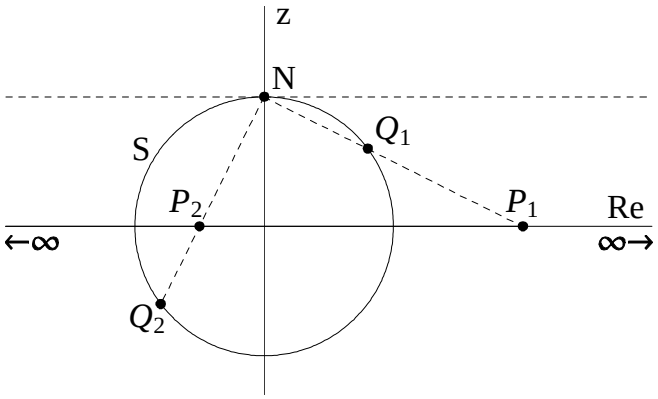

Fig. 3.3 – Stereographic Projection

Let S denote the unit sphere in $\mathbb{R}^3$. Thinking of $\mathbb{C}$ as the xy-plane, every complex number $P = X + YI$ can be identified with a point Q on S by drawing a line from $(X, Y, 0)$ in the xy-plane to the sphere's north pole $N = (0, 0, 1)$; this line intersects the sphere at two points Q and N. We define a map π from the sphere $S\backslash\{N\}$ to $\mathbb{C}$ by setting $\pi(Q)) = P$. The points Q that are near N are mapped to P with large moduli, so it makes sense to extend π by setting $\pi(N) = \infty$ and then we have a bijection from S to $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ which is called stereographic projection.

Specifically this defines

$$\pi(x, y, z) = \begin{cases} \frac{x+yi}{1-z} & z \neq 1 \\ \infty & z = 1 \end{cases}$$

with inverse

$$\pi^{-1}(X + iY) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{X^2 + Y^2 - 1}{1 + X^2 + Y^2} \right).$$

But π is much more than a simple bijection. It has important geometric properties.

- *Under stereographic projection, circles on S which pass through N correspond to lines in $\mathbb{C}$, and circles on S which don't pass through N correspond to circles in $\mathbb{C}$.*
- *The map π is conformal, meaning it is angle-preserving.*
- *The Möbius transformations $z \to (az + b)/(cz + d)$, where $ad \neq bc$, are bijections of $\mathbb{C}_\infty$, which correspond to the conformal bijections of the sphere.*

Returning now to real and complex sequences:

Proposition 3.35 *(a) Let (a_n) be a sequence of positive real numbers. The following are equivalent:*
(i) $a_n \to \infty$ as $n \to \infty$;
(ii) $1/a_n \to 0$ as $n \to \infty$.

The equivalence fails if the (a_n) are simply non-zero.
(b) Let (a_n) be a sequence of non-zero complex numbers. The following are equivalent:
(i) $a_n \to \infty$ as $n \to \infty$;
(ii) $1/a_n \to 0$ as $n \to \infty$.

Proof. (a): (i) $\implies$ (ii) Let $\varepsilon > 0$ and set $M = 1/\varepsilon$. As $a_n \to \infty$ then there exists N such that $a_n > M$ for all $n \geqslant N$. But then $0 < 1/a_n < \varepsilon$ for all $n \geqslant N$ and $1/a_n \to 0$.

(a): (ii) $\implies$ (i): Let $M > 0$ and $\varepsilon = 1/M$. As $1/a_n \to 0$ then there exists N such that $1/a_n < \varepsilon$ for all $n \geqslant N$. But then $a_n > 1/\varepsilon = M$ for all $n \geqslant N$ and $a_n \to \infty$.

(a): If we set $a_n = (-1)^n n$ then $1/a_n = (-1)^n/n \to 0$ but $a_n \nrightarrow \infty$ as $a_{2n} \to \infty$ yet $a_{2n+1} \to -\infty$.

(b) : (i) $\implies$ (ii) Let $\varepsilon > 0$ and set $M = 1/\varepsilon$. As $a_n \to \infty$ then there exists N such that $|a_n| > M$ for all $n \geqslant N$. But then $|1/a_n| < \varepsilon$ for all $n \geqslant N$ and $1/a_n \to 0$.

(b): (ii) $\implies$ (i): Let $M > 0$ and $\varepsilon = 1/M$. As $1/a_n \to 0$ then there exists N such that $|1/a_n| < \varepsilon$ for all $n \geqslant N$. But then $|a_n| > 1/\varepsilon = M$ for all $n \geqslant N$ and $a_n \to \infty$. ■

Example 3.36 *Let (a_n) be a real sequence such that $a_n \to \infty$ as $n \to \infty$. Prove, or disprove with a counter-example, each of the following statements.*
(a) If (b_n) is a bounded, non-zero sequence then $a_n/b_n \to \infty$.
(b) If (b_n) is a bounded, positive sequence then $a_n/b_n \to \infty$.
(c) If b_n is a non-zero sequence which converges to $L > 0$ then $a_n/b_n \to \infty$.

Solution. (a) False. We can see this by taking $a_n = n$ and $b_n = (-1)^n$. Then $a_n/b_n = (-1)^n n$ does not tend to ∞. [Note that part (a) is a trivial consequence of (b) and so it would have made for an odd question if (a) had been true.]

(b) True. As b_n is bounded then there exists $K > 0$ such that $0 < b_n < K$ for all n. Let $M \in \mathbb{R}$. As $a_n \to \infty$ then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $a_n > MK$. So for all $n \geqslant N$ we have $a_n/b_n > (MK)/K = M$ and we see $a_n/b_n \to \infty$.

(c) True. Taking $\varepsilon = L/2 > 0$ we see that there exists N with $|b_n - L| < L/2$ for all $n \geqslant N$. In particular, $0 < L/2 < b_n < 3L/2$ for $n \geqslant N$. By the previous part, the tail of $(a_n/b_n)_N^\infty$ tends to ∞ and hence so does the whole sequence (a_n/b_n). ■

Example 3.37 *Let (a_n) be a real sequence.*
(a) If $a_n \to \infty$ as a real sequence, need $a_n \to \infty$ as a complex sequence?
(b) If $a_n \to \infty$ as a complex sequence, need $a_n \to \infty$ as a real sequence?

Solution. (a) *True: As $a_n \to \infty$* then $|a_n| \to \infty$ which is equivalent to $a_n \to \infty$ as a complex sequence.

(b) *False*: A counter-example is $a_n = (-1)^n n$. ■

Example 3.38 ***(Harmonic numbers)*** *The nth harmonic number is*

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

where $n \geqslant 1$. Show that $H_n \to \infty$ as $n \to \infty$.

Solution. Note that

$$\begin{aligned} H_{2^k} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\cdots+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{k-1}}+\cdots+\frac{1}{2^k}\right) \\ &> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^k}+\cdots+\frac{1}{2^k}\right) \\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} \\ &= 1+\frac{k}{2}. \end{aligned}$$

Given $M > 0$ there is a positive integer k such that $H_{2^k} > 1 + k/2 \geqslant M$. Hence $H_n > M$ for all $n \geqslant 2^k$ as H_n is increasing. ■

Proposition 3.39 *Let p_n, q_n be integers with $q_n > 0$ such that $p_n/q_n \to \alpha$ is irrational. Then $q_n \to \infty$.*

Proof. SFAC that $q_n \nrightarrow \infty$. This means

$$\exists M \qquad \forall N \qquad \exists n \geqslant N \qquad q_n \leqslant M.$$

WLOG we may assume that M is a positive integer. This means that $q_n \leqslant M$ for infinitely many n As α is irrational, x lies in some open interval

$$\left(\frac{k}{M!}, \frac{k+1}{M!}\right),$$

and so is some positive distance d from the closest end-point. For all n where $q_n \leqslant M$, note p_n/q_n is some integer multiple of $1/M!$. For these infinitely many n and any $0 < \varepsilon < d$, we have

$$\left|\frac{p_n}{q_n} - \alpha\right| \geqslant d > \varepsilon.$$

This contradicts $p_n/q_n \to \alpha$. The result follows. ■

4. THE ALGEBRA OF LIMITS

Reassuringly limits respect important relations and algebraic operations that mean we can don't need to go back to first principle definitions of convergence and divergence to analyze more complicated sequences.

Theorem 4.1 *(**Limits respect weak inequalities**) Let (a_n) and (b_n) be real sequences such that $(a_n) \to L$ and $(b_n) \to M$. If $a_n \leqslant b_n$ for all n, then $L \leqslant M$.*

Thoughts: A proof by contradiction. If $L > M$ then there would be a tail of the a_n in a neighbourhood of L and a tail of the b_n near M. If these neighbourhoods are small enough to be disjoint, then $a_n > b_n$ in the tails' intersection. Note ε is chosen in the proof below so that $(L-\varepsilon, L+\varepsilon)$ is disjoint from and to the right of $(M-\varepsilon, M+\varepsilon)$.

Proof. Suppose, for a contradiction, that $L > M$. Set $\varepsilon = (L-M)/2 > 0$.

$$\begin{aligned} &\text{As } a_n \to L \text{ then there exists } N_1 \text{ such that } n \geqslant N_1 \implies |a_n - L| < \varepsilon; \\ &\text{as } b_n \to M \text{ then there exists } N_2 \text{ such that } n \geqslant N_2 \implies |b_n - M| < \varepsilon. \end{aligned}$$

So

$$\begin{aligned} n \geqslant N_1 &\implies \frac{L+M}{2} = L - \varepsilon < a_n \\ n \geqslant N_2 &\implies b_n < M + \varepsilon = \frac{L+M}{2}. \end{aligned}$$

Hence for $n \geqslant \max(N_1, N_2)$ we have

$$a_n > \frac{L+M}{2} > b_n$$

which contradicts $a_n \leqslant b_n$ for all n. ■

Remark 4.2 *Note* lim *does not respect strict inequalities: e.g. $\frac{1}{n} > 0$ for all $n \geqslant 1$ but $0 = \lim \frac{1}{n} > \lim 0 = 0$ is false.*

Note in the above proof that $n \geqslant \max(N_1, N_2)$ is the intersection of both tails, so both inequalities hold there.

A second important result, that helps us ignore or bound unimportant expressions in a sequence, is the following. This result is also referred to as the 'squeeze theorem'.

Theorem 4.3 *(**Sandwich Rule**) Suppose that $x_n \leqslant a_n \leqslant y_n$ for all n and that*

$$L = \lim x_n = \lim y_n.$$

Then $a_n \to L$ as $n \to \infty$.

Thoughts: Given any neighbourhood of L, there will be tails of (x_n) and (y_n) in that neighbourhood. These tails bound a tail of (a_n).

Proof. Let $\varepsilon > 0$. Then there exist N_1 and N_2 such that

$$\begin{aligned} x_n - L &> -\varepsilon \quad \text{for all } n \geqslant N_1, \\ y_n - L &< \varepsilon \quad \text{for all } n \geqslant N_2. \end{aligned}$$

So for $n \geqslant \max(N_1, N_2)$ we have

$$-\varepsilon < x_n - L \leqslant a_n - L \leqslant y_n - L < \varepsilon,$$

which shows that $a_n \to L$ also. ■

Note how the sandwich rule helps bound the effects of the trigonometric terms in the following example, and focus on the important large terms.

Example 4.4 *Show that the sequence*

$$a_n = \frac{2n + \cos(n^2)}{3n^2 - \sin(n^3)}$$

converges.

Solution. We note for all $n \geqslant 1$,

$$\frac{1}{3n} = \frac{n}{3n^2} \leqslant \frac{2n-1}{3n^2+1} \leqslant a_n = \frac{2n + \cos(n^2)}{3n^2 - \sin(n^3)} \leqslant \frac{2n+1}{3n^2-1} \leqslant \frac{3n}{2n^2} = \frac{3}{2n}.$$

As the LHS and RHS both tend to 0 then $a_n \to 0$ by sandwiching. ■

Most sequences can be built up from simpler ones using addition, multiplication, etc. The algebra of limits (AOL) tells us how the corresponding limits behave. Throughout the following (a_n) and (b_n) denote real or complex sequences.

Proposition 4.5 ***(AOL: Constants)*** *If $a_n = a$ for all n, then $a_n \to a$.*

Proof. For any $\varepsilon > 0$, take $N = 1$; $n \geqslant N \Longrightarrow |a_n - a| = 0 < \varepsilon$. ■

Proposition 4.6 ***(AOL: Sums)*** *If $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$.*

Thoughts: We need to show that $|(a_n + b_n) - (a + b)|$ is eventually small given that $|a_n - a|$ and $|b_n - b|$ are each eventually small. The triangle inequality helps here by noting

$$|(a_n + b_n) - (a + b)| \leqslant |a_n - a| + |b_n - b|$$

In the following proof we use two standard techniques of analysis. We know two facts which hold in two tails of a sequence, so we take the tails' intersection where both are true – we've employed this idea before. The second issue is that we need a final inequality to hold within a margin of ε. But the final inequality relies on two previous inequalities. The idea is to achieve

each of the first two inequalities with margins of $\varepsilon/2$ and then the triangle inequality, within the tails' intersection, shows the final inequality holds with a margin of $\varepsilon/2 + \varepsilon/2 = \varepsilon$.

Proof. Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$ and so

$$\begin{array}{llllll} \exists N_1 & n & \geqslant & N_1 & \Longrightarrow & |a_n - a| < \varepsilon/2, \\ \exists N_2 & n & \geqslant & N_2 & \Longrightarrow & |b_n - b| < \varepsilon/2. \end{array}$$

Put $N_3 = \max(N_1, N_2)$. Then

$$\begin{array}{lrcll} n \geqslant N_3 \Longrightarrow & |(a_n + b_n) & - & (a + b)| & \\ \leqslant & |a_n - a| & + & |b_n - b| & \text{by the } \Delta \text{ law} \\ < & \varepsilon/2 & + & \varepsilon/2 & \\ = & \varepsilon & & & \end{array}$$

■

Proposition 4.7 *(**AOL: Scalar Products**) If $a_n \to a$ as $n \to \infty$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) then $\lambda a_n \to \lambda a$.*

Proof. Let $\varepsilon > 0$. Then $\varepsilon/(|\lambda| + 1) > 0$ and so there exists N such that $|a_n - a| < \varepsilon/(|\lambda| + 1)$ for all $n \geqslant N$. Hence

$$|\lambda a_n - \lambda a| = |\lambda|\,|a_n - a| \leqslant \frac{|\lambda|\,\varepsilon}{|\lambda| + 1} < \varepsilon$$

for all $n \geqslant N$. (Note that we use $\varepsilon/(|\lambda| + 1)$ rather than $\varepsilon/|\lambda|$ to avoid the possibility of dividing by zero.) ■

Corollary 4.8 *(**AOL: Differences**) If $a_n \to a$ and $b_n \to b$ then $a_n - b_n \to a - b$.*

Corollary 4.9 *(**AOL: Translations**) If $a_n \to a$ and $c \in \mathbb{R}$ (or $\mathbb{C}$) then $a_n + c \to a + c$.*

Lemma 4.10 *If $x_n \to 0$ and $y_n \to 0$ then $x_n y_n \to 0$.*

Proof. Let $\varepsilon > 0$. By Remark 3.11, WLOG we can further assume that $\varepsilon < 1$. Then

$$\begin{array}{llllll} \exists N_1 & n & \geqslant & N_1 & \Longrightarrow & |x_n| < \varepsilon_1, \\ \exists N_2 & n & \geqslant & N_2 & \Longrightarrow & |y_n| < \varepsilon_1. \end{array}$$

So if $n \geqslant \max(N_1, N_2)$ we have

$$|x_n y_n| \leqslant |x_n|\,|y_n| < \varepsilon^2 < \varepsilon,$$

which completes the proof. ■

Proposition 4.11 *(**AOL: Products**) If $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.*

Proof. Note that

$$a_n b_n - ab = (a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b),$$

that $(a_n - a)(b_n - b) \to 0$ by the previous lemma, that $b(a_n - a) \to 0$ and $a(b_n - b) \to 0$ by Proposition 4.7. Hence $a_n b_n \to ab$ by Proposition 4.6. ■

Proposition 4.12 ***(AOL: Reciprocals)*** *If $a_n \to a \neq 0$ and $a_n \neq 0$ for all n, then $1/a_n \to 1/a$.*

Thoughts: Our aim is to show

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|}$$

is arbitrary small in a tail, and we know $|a_n - a|$ is small. The $|a|$ in the denominator is non-zero and constant and so is not problematic. At first glance though, whilst $|a_n|$ is non-zero it might be arbitrarily small, which would be problematic. But remembering $a_n \to a \neq 0$ then we can focus on a tail of a_n suitably close to a. If a_n is within $|a|/2$ of a, then a_n will be at least $|a|/2$ away from zero.

Proof. Let $\varepsilon > 0$. As $a \neq 0$ then $|a|/2 > 0$. So there exists N_1 such that for $n \geqslant N_1$ we have $|a_n - a| < |a|/2$. By the triangle inequality

$$|a| \leqslant |a_n| + |a - a_n| = |a_n| + |a_n - a|$$

and so $|a_n| > |a|/2$ and $|1/a_n| < 2/|a|$.

Further, as $|a|^2 \varepsilon/2 > 0$ then there exists N_2 such that for $n \geqslant N_2$

$$|a_n - a| < |a|^2 \frac{\varepsilon}{2}.$$

For $n \geqslant \max(N_1, N_2)$ we have

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|} < \left(|a|^2 \frac{\varepsilon}{2}\right) \frac{2}{|a|} \frac{1}{|a|} = \varepsilon.$$

■

Corollary 4.13 ***(AOL: Quotients)*** *If $a_n \to a$, $b_n \to b$, and $b_n \neq 0$ for all n and $b \neq 0$, then $a_n/b_n \to a/b$.*

Proof. This follows from Propositions 4.11 and 4.12. ■

Proposition 4.14 ***(AOL: Modulus)*** *If $a_n \to a$ then $|a_n| \to |a|$.*

Proof. By the reverse triangle inequality

$$0 \leqslant ||a_n| - |a|| \leqslant |a_n - a| \to 0$$

So $||a_n| - |a|| \to 0$ by sandwiching. ■

Example 4.15 *Show*

$$a_n = \frac{n^2+n+1}{3n^2+4} \to \frac{1}{3}.$$

Solution. We write

$$\frac{n^2+n+1}{3n^2+4} = \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3+4\frac{1}{n^2}} \to \frac{1+0+0}{3+0} = \frac{1}{3}$$

by the algebra of limits, specifically noting

- $\frac{1}{n} \to 0$ by the Archimedean property;
- $\frac{1}{n^2} \to 0$ by Proposition 4.11;
- $1 \to 1$ by Proposition 4.5;
- $1 + \frac{1}{n} + \frac{1}{n^2} \to 1$ by Proposition 4.6;
- $3 + \frac{4}{n^2} \to 3$ by Proposition 4.6;
- $\frac{1}{3+\frac{4}{n^2}} \to \frac{1}{3}$ by Corollary 4.13;
- $a_n \to \frac{1}{3}$ by Proposition 4.11.

■

Example 4.16 ***(Fibonacci numbers)*** *Suppose $F_1 = 1$, $F_2 = 1$, and we recursively define*

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for } n \geqslant 1.$$

It is easy to prove by induction on n that there is then a unique sequence of natural numbers satisfying these requirements. They are called the Fibonacci numbers.

Proposition 4.17 *F_{n+1}/F_n is convergent.*

Proof. By induction, $F_n \geqslant 1$ for all n. So for $n \geqslant 1$

$$\left(\frac{F_{n+2}}{F_{n+1}}\right) = 1 + \left(\frac{F_{n+1}}{F_n}\right)^{-1}.$$

Write $x_n = F_{n+1}/F_n$ for $n \geqslant 1$. Note that $F_n > 0$ for all n. Then

$$x_1 = 2 \text{ and } x_{n+1} = 1 + 1/x_n.$$

Suppose that we did have convergence and that $x_n \to L$ so that $x_{n+1} \to L$. Note $L \geqslant 1 > 0$ as $F_{n+1} > F_n$ and so $1 + \frac{1}{x_n} \to 1 + \frac{1}{L}$ by AOL. So

$$L = 1 + \frac{1}{L}$$

by the uniqueness of limits. Hence $L^2 - L - 1 = 0$ giving $L = \frac{1\pm\sqrt{5}}{2}$. But $L \geqslant 1$ giving

$$L = \frac{1+\sqrt{5}}{2} > 1.$$

All the above was based on the assumption that x_n converged. We will show that x_n is convergent to $\frac{1+\sqrt{5}}{2}$, which we will denote φ, and is called the **golden ratio**.

$$x_{n+1} - \varphi = 1 + \frac{1}{x_n} - \varphi = 1 + \frac{1}{x_n} - 1 - \frac{1}{\varphi} = \frac{1}{x_n} - \frac{1}{\varphi} = \frac{x_n - \varphi}{x_n\varphi}$$

as $\varphi^2 = \varphi + 1$. So

$$\left|\frac{x_{n+1} - \varphi}{x_n - \varphi}\right| = \frac{1}{|x_n||\varphi|} = \frac{1}{\varphi x_n} \leqslant \frac{1}{\varphi}$$

as $x_n > 1$ for all n. By induction we get

$$-\frac{1}{\varphi^n} \leqslant x_n - \varphi \leqslant \frac{1}{\varphi^n}$$

and are done by the sandwich rule, since $\varphi > 1$ and so $\pm\dfrac{1}{\varphi^n} \to 0$. ■

Example 4.18 *Which of the following statements are true of the given non-zero real or complex sequence (a_n)? Provide a proof or a counter-example.*

(a) If (a_n) converges then $a_{n+1} - a_n \to 0$.
(b) If $a_{n+1} - a_n \to 0$ then (a_n) converges.
(c) If (a_n) converges then $a_{n+1}/a_n \to 1$.
(d) If $a_n \to L \neq 0$ then $a_{n+1}/a_n \to 1$.
(e) If $a_{n+1}/a_n \to 1$ then (a_n) converges.
(f) If $a_{n+1}/a_n \to 1$ and (a_n) is bounded then (a_n) converges.

Solution. Let H_n denote the nth harmonic number.

(a) *True*: If $a_n \to L$ then by the algebra of limits $a_{n+1} - a_n \to L - L = 0$.
(b) *False*: Let $a_n = H_n$. Then $a_{n+1} - a_n = (n+1)^{-1} \to 0$ yet $H_n \to \infty$ (see Example 3.38).
(c) *False*: Let $a_n = (-1)^n/n$ so that $a_n \to 0$. However $a_{n+1}/a_n = -\frac{n}{n+1} \to -1$.
(d) *True*: If $a_n \to L \neq 0$ then by the algebra of limits $a_{n+1}/a_n \to L/L = 1$.
(e) *False*: Let $a_n = n$. Then $a_{n+1}/a_n = 1 + n^{-1} \to 1$ but $a_n \to \infty$.
(f) *False*: Let $a_n = e^{iH_n}$. Then

$$a_{n+1}/a_n = e^{i(H_{n+1}-H_n)} = e^{i/(n+1)} \to e^0 = 1,$$

as $n \to \infty$ but e^{iH_n} does not converge as $H_n \to \infty$. ■

Remark 4.19 *The necessary AOL properties to justify the answer to (f) won't be proven until Analysis II in Hilary Term. The notion of a continuous function will be defined there and we will see that if $a_n \to L$ and f is continuous then $f(a_n) \to f(L)$. In fact, this property is an alternative definition of f being continuous.*

As was commented in Remark 3.32, there are several indeterminate forms including ∞, so we cannot expect any AOL results re

$$\infty - \infty, \qquad \frac{\infty}{\infty}, \qquad 0 \times \infty.$$

But there are some cases where AOL-like results are true.

Proposition 4.20 *(**AOL: Infinity**) Let (a_n) and (b_n) be real sequences.*
(a) If $a_n \to \infty$ and $b_n \to \infty$ then $a_n + b_n \to \infty$.
(b) If $a_n \to \infty$ and $b_n \to \infty$ then $a_n b_n \to \infty$.
(c) If $a_n \to \infty$ and $b_n \to -\infty$ then $a_n b_n \to -\infty$.
(d) If $a_n \to \infty$ and (b_n) is bounded then $a_n + b_n \to \infty$.
(e) If $a_n \to \infty$ and and (b_n) is bounded then $b_n/a_n \to 0$.
(f) If $a_n \to \infty$ and $b_n \to L > 0$ then $a_n b_n \to \infty$.

Solution. These are left as exercises. ■

Proposition 4.21 *(**AOL: Asymptotics**) Let (a_n), (α_n), (b_n) and (β_n) be real sequences.*
(a) If $a_n = O(\alpha_n)$ and $b_n = O(\beta_n)$ then $a_n b_n = O(\alpha_n \beta_n)$.
(b) If $a_n = O(\alpha_n)$ and $b_n = O(\beta_n)$ then $a_n + b_n = O(\max(|\alpha_n|, |\beta_n|))$.
(c) If $a_n \sim \alpha_n$ and $b_n \sim \beta_n$ then $a_n b_n \sim \alpha_n \beta_n$.
(d) If $a_n \sim \alpha_n$ and $b_n \sim \beta_n$ then $a_n/b_n \sim \alpha_n/\beta_n$.

Solution. These are left as exercises. ■

Remark 4.22 *(**The Relative Orders of Terms**) Our first thoughts, when considering the long term behaviour of a sequence which has various components to it, should be on which terms dictate the sequence's behaviour in the long term. Usually, for this, we need to appreciate the relative magnitudes of the terms as n becomes large. As a rule of thumb, when it comes to the long term behaviour of functions*

bounded trig functions and constants $<$ logarithms $<$ polynomials $<$ exponentials.

More precisely:

- *$|\cos n| \leqslant 1$ and $|\sin n| \leqslant 1$ for all n.*
- *For any rational $q > 0$, $\log n/n^q \to 0$ as $n \to \infty$.*
- *For any $a > 1$ and polynomial p then $p(n)/a^n \to 0$ as $n \to \infty$.*

The third bullet point is a consequence of Corollary 3.19. The second bullet point is essentially the same result. If we write $n = e^t$ then

$$\lim_{n\to\infty} \frac{\log n}{n^q} = \lim_{t\to\infty} \frac{t}{\left(e^q\right)^t} = 0$$

as $e^q > 1$.

Example 4.23 *Qualitatively describe the long-term behaviour of the following sequences.*

- $$(-1)^n \left(\frac{n^6 + 7n^2}{2^n} \right)$$

  This will tend to 0 *(albeit in an oscillatory way) as the dominant term is* 2^n*.*

- $$\left(\frac{2n+3}{3n+8} \right) \cos n.$$

  At first glance the polynomial terms seem dominant. But being of the same degree, and working to counter one another, we see $(2n+3)/(3n+8) \to 2/3$*. So actually it is the oscillating behaviour of* $\cos n$ *which stops the sequence from converging.*

- $$\frac{\log n}{\sqrt{n}} \cos \left(\frac{2^n - n}{n^2 + 3n - 6} \right).$$

  As $|\cos \theta| \leqslant 1$ *for all* θ *then the cosine takes the sting out of the term* $(2^n - n)/(n^2 + 3n - 6)$ *which is just a red herring. In the long term* $\sqrt{n}$ *dominates* $\log n$ *and* $\log n / \sqrt{n} \to 0$*. The messy cosine term has no crucial effect on this behaviour.*

- *How would you make these first thoughts into rigourous proofs using the algebra of limits, sandwich rule, etc.?*

5. MORE ON SEQUENCES

5.1 Monotone Sequences

We now turn to a crucially important kind of sequence.

Definition 5.1 *Let (a_n) be a real sequence.*
*We say (a_n) is **increasing** if $a_n \leqslant a_m$ whenever $n < m$.*
*We say (a_n) is **decreasing** if $a_n \geqslant a_m$ whenever $n < m$.*
*We say (a_n) is **strictly increasing** if $a_n < a_m$ whenever $n < m$.*
*We say (a_n) is **strictly decreasing** if $a_n > a_m$ whenever $n < m$.*
*We say (a_n) is **monotone** if it is either decreasing or increasing.*

Example 5.2 *Let $a_n = n$. Then (a_n) is increasing. So is $a_n = (2n+1)^2$.*
The sequence $a_n = (-1)^n$ is not monotone as $a_1 < a_2$ and $a_2 > a_3$.

Theorem 5.3 *Let (a_n) be an increasing, bounded above sequence. Then (a_n) converges.*

Proof. Let $L = \sup\{a_n \mid n \in \mathbb{N}\}$; this exists by the completeness axiom as the set is bounded above and non-empty. Let $\varepsilon > 0$. By the approximation property there exists $N \in \mathbb{N}$ such that

$$L - \varepsilon < a_N \leqslant L.$$

As the sequence is increasing then for any $n \geqslant N$

$$L - \varepsilon < a_N \leqslant a_n \leqslant L,$$

and so

$$\forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

That is $a_n \to L$. ■

Corollary 5.4 *An increasing real sequence either converges or tends to infinity.*

Proof. Let (a_n) be an increasing real sequence. If it is bounded above, then (a_n) converges. Otherwise for any $M > 0$ then M is not an upper bound to (a_n). Hence there exists $N \in \mathbb{N}$ such that $a_N \geqslant M$. Now as (a_n) is increasing $a_n \geqslant M$ for all $n \geqslant N$. That is $a_n \to \infty$. ■

Corollary 5.5 *Let (a_n) be a decreasing, bounded below sequence. Then (a_n) converges.*

Proof. $(-a_n)$ is increasing and bounded above so $-a_n \to L$ by the previous result. Hence $a_n \to -L$ by AOL. ■

Remark 5.6 *Theorem 5.3 is in fact equivalent to the completeness axiom. That is, the axioms of an ordered field together with Theorem 5.3 characterize the real numbers. Details are left to Sheet 4, Exercise 6.*

The next theorem, the Nested Intervals Theorem, together with the Archimedean property, form another alternative to the completeness axiom.

Theorem 5.7 ***(Nested Intervals Theorem – Cantor, 1872)*** *Let $I_n = [a_n, b_n]$ be a nested sequence of closed bounded intervals. (That is $I_{n+1} \subseteq I_n$ for all $n \geqslant 1$.)*

(a) Then $\bigcap_1^\infty I_n \neq \varnothing$.

(b) If $l(I_n) = b_n - a_n \to 0$ as $n \to \infty$ then $\bigcap_1^\infty I_n$ is a singleton.

(c) Note the theorem need not hold if the intervals are bounded but not closed, e.g. $I_n = (0, 1/n)$, or closed but not bounded, e.g. $I_n = [n, \infty)$.

Proof. (a) As $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ then $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$. So (a_n) is an increasing sequence which is bounded above, and (b_n) is a decreasing sequence bounded below. This means both sequences converge and set $\alpha = \lim a_n$ and $\beta = \lim b_n$.

As $a_m \leqslant \alpha \leqslant \beta \leqslant b_n$ for all m, n, then $[\alpha, \beta] \subseteq I_n$ for all n and so $\bigcap_1^\infty I_n \neq \varnothing$.

(b) As $b_n - a_n \geqslant \beta - \alpha$ for each n and $b_n - a_n \to 0$ then $\alpha = \beta$. Certainly $\alpha \in \bigcap_1^\infty I_n$. And if $x, y \in \bigcap_1^\infty I_n$ with $x < y$ then

$$0 < y - x < b_n - a_n$$

for all n, a contradiction as $b_n - a_n \to 0$. Hence $\bigcap_1^\infty I_n = \{\alpha\}$. ■

Example 5.8 *(a) Let $x \in \mathbb{R}$. Show that $x^n/n! \to 0$.*
(b) Deduce that $z^n/n! \to 0$ for $z \in \mathbb{C}$.

Solution. (a) If $x = 0$ this is clear. Otherwise set $a_n = |x|^n / n!$ and note

$$\frac{a_{n+1}}{a_n} = \frac{n!\,|x|^{n+1}}{(n+1)!\,|x|^n} = \frac{|x|}{n+1} \to 0 \quad \text{as } n \to \infty.$$

So in some tail $a_{n+1}/a_n < 1$ and (a_n) is eventually decreasing and bounded below by 0. Hence a_n converges to some limit L.

We have

$$a_{n+1} = \left(\frac{|x|}{n+1}\right) a_n.$$

Letting $n \to \infty$ and applying AOL, we have

$$L = 0 \times L = 0,$$

as required.

(b) Now take $z \in \mathbb{C}$. By (a) $|z^n/n!| = |z|^n / n! \to 0$ and hence $z^n/n! \to 0$. ■

Example 5.9 *(a) Let $a \geqslant 1$. By considering the iteration*

$$x_0 = a, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \qquad \text{for } n \geqslant 0,$$

show the existence and uniqueness of $\sqrt{a}$.

(b) Deduce the existence and uniqueness of $\sqrt{a}$ for $0 \leqslant a < 1$.

Remark 5.10 *This iteration was known to the Babylonians for finding square roots. From a modern perspective it is an instance of the* ***Newton-Raphson method*** *applied to the function* $f(x) = x^2 - a$.

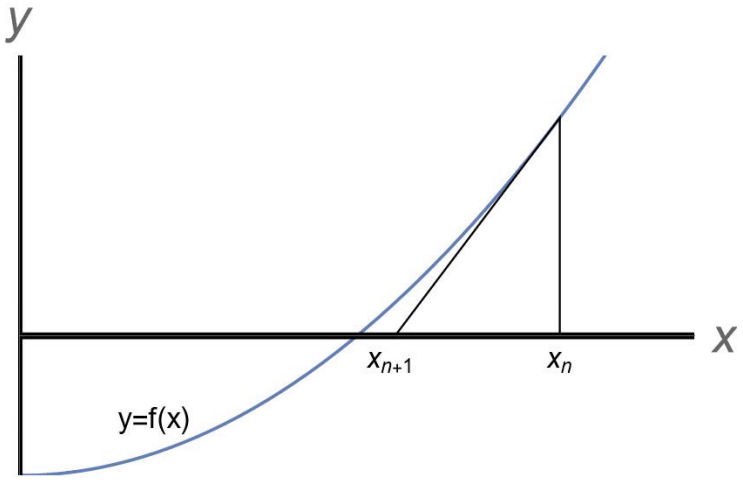

Fig. 5.1 – Newton-Raphson method

The Newton-Raphson iteration seeks to solve an equation $f(x) = 0$. *It takes an estimate* x_n *for a root and replaces it with*

$$x_{n+1} = x_n - \frac{f(x_n)}{f\,'(x_n)}.$$

This estimate x_{n+1} *is achieved (as in Figure 5.1) by drawing the tangent to the curve* $y = f(x)$ *at the point* $(x_n, f(x_n))$ *and intersecting it with the* x*-axis.*

In this particular case $f(x) = x^2 - a$ *and so*

$$x_{n+1} = x_n - \left(\frac{x_n^2 - a}{2x_n}\right) = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Solution. (a) I claim the following to be true of the sequence (x_n):

(i) $a \leqslant x_n^2$ for all n;

(ii) (x_n) is decreasing;

(iii) $L = \lim x_n$ satisfies $L^2 = a$.

(i) As $x_0 = a$ then (i) is true for $n = 0$ as $a^2 - a = a\,(a-1) \geqslant 0$. If $a \leqslant x_n^2$ then

$$\begin{aligned} x_{n+1}^2 - a &= \left[\frac{1}{2}\left(x_n + \frac{a}{x_n}\right)\right]^2 - a \\ &= \frac{1}{4x_n^2}\left[\left(x_n^2 + a\right)^2 - 4ax_n^2\right] \\ &= \frac{1}{4x_n^2}\left[x_n^4 - 2ax_n^2 + a^2\right] \\ &= \frac{1}{4x_n^2}\left[x_n^2 - a\right]^2 \geqslant 0 \end{aligned}$$

Hence (i) follows by induction.

(ii) Note that

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = \frac{x_n^2 - a}{2x_n} \geqslant 0$$

by (i).

(iii) So (x_n) is decreasing and bounded below and therefore converges. Let $L = \lim x_n$. Letting $n \to \infty$ in the iteration

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

we get

$$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$$

by AOL and the uniqueness of limits. This rearranges to $L^2 = a$. As $x_n \geqslant 0$ for all n then $L = \sqrt{a}$ (as opposed to $-\sqrt{a}$).

Now L is a root of $x^2 = a$. As $x^2 - a = (x - L)(x + L)$ then we see that the two roots of $x^2 - a$ are $\pm L$. From this we also see that the two square roots of a are $\pm\sqrt{a}$, showing there is a unique positive square root of a.

(b) Clearly 0 is the only square root of 0. Say now that $0 < a < 1$ so that $a^{-1} > 1$. By (a)

$$x^2 = a \quad \Longleftrightarrow \quad \left(\frac{1}{x}\right)^2 = a^{-1} \quad \Longleftrightarrow \quad \frac{1}{x} = \pm\sqrt{a^{-1}},$$

only one root of which is positive. Hence $\sqrt{a}$ is uniquely defined with $\sqrt{a} = \left(\sqrt{a^{-1}}\right)^{-1}$. ■

Remark 5.11 ***(Cobwebbing)*** *The previous iteration can also be achieved via cobwebbing which aims to solve equations of the form $x = f(x)$. The previous Newton-Raphson iteration took the form*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

which, if it converges, leads to a solution of $x = f(x)$ where

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

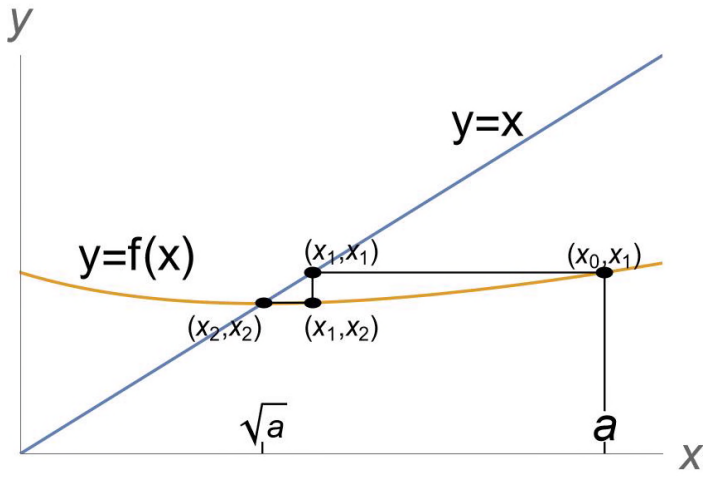

Fig. 5.2 – Cobwebbing

We sketch $y = x$ and $y = f(x)$ on the same axes. Given an initial estimate x_0 we draw a vertical line to the curve to get to (x_0, x_1) and then move horizontally to (x_1, x_1) and so on to $(x_1, x_2), (x_2, x_2), (x_2, x_3), \ldots$. If the sequence (x_n) converges to α, say, then α is a fixed point. That is $\alpha = f(\alpha)$; this essentially follows by AOL.

*We can see from Figure 5.2 how any sequence beginning with $x_0 \geqslant \sqrt{a}$ will monotonically decrease to $\sqrt{a}$. Any sequence beginning with $0 < x_0 < \sqrt{a}$ will jump to $x_1 > \sqrt{a}$ and then decrease again to $\sqrt{a}$. Of course, the figure itself **proves** nothing but provides useful qualitative information for what needs proving.*

In this particular case the iteration converges quickly. As $f(\alpha) = \alpha$ then

$$x_{n+1} - \alpha \approx f'(\alpha)(x_n - \alpha),$$

and for this particular iteration $-1 < f'(\alpha) < 1$ as

$$f'\left(\sqrt{a}\right) = \frac{1}{2}\left(1 - \frac{a}{\left(\sqrt{a}\right)^2}\right) = 0.$$

*When $|f'(\alpha)| < 1$ the the fixed point α is said to be an **attracting** fixed point.*

*The convergence will be monotonic if $0 < f'(\alpha) < 1$ and will be oscillatory if $-1 < f'(\alpha) < 0$. When $|f'(\alpha)| > 1$ the fixed point is called **repelling** and the iteration will not generally converge.*

We conclude this section by defining the decimal expansion (and more generally base expansions) for a real number. For uniqueness we do this in such a way that the truncated decimal expansions form a strictly convergent sequence converging to the real number in question.

Example 5.12 *(**Decimal Expansions**) Let $0 < x \leqslant 1$. Then there is a unique sequence of integers $a_1, a_2, a_3, \ldots$ such that*

(a) $0 \leqslant a_n \leqslant 9$ for each n;

(b) for each n,

$$x - \frac{1}{10^n} \leqslant \sum_{k=1}^{n} \frac{a_k}{10^k} < x;$$

(c)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k} = x.$$

Solution. We will proceed inductively. The integer a_1 needs to satisfy

$$x - \frac{1}{10} \leqslant \frac{a_1}{10} < x \quad \Longrightarrow \quad 10x - 1 \leqslant a_1 < 10x.$$

The interval $[10x - 1, 10x)$ contains a unique integer a_1 and further, as

$$-1 < 10x - 1 \leqslant a_1 < 10x \leqslant 10$$

then $0 \leqslant a_1 \leqslant 9$.

Suppose now, as our inductive hypothesis, that $a_1, a_2, \ldots, a_N$ have been uniquely found satisfying (i) and (ii). Then

$$x - \frac{1}{10^{N+1}} \leqslant \sum_{k=1}^{N+1} \frac{a_k}{10^k} < x$$

which rearranges to

$$x - \frac{1}{10^{N+1}} - \sum_{k=1}^{N} \frac{a_k}{10^k} \leqslant \frac{a_{N+1}}{10^{N+1}} < x - \sum_{k=1}^{N} \frac{a_k}{10^k}$$

and then to

$$\left(10^{N+1}x - \sum_{k=1}^{N} 10^{N+1-k}a_k\right) - 1 \leqslant a_{N+1} < \left(10^{N+1}x - \sum_{k=1}^{N} 10^{N+1-k}a_k\right).$$

There is a unique integer in this range, and we set a_{N+1} to be this integer. Further, by hypothesis,

$$\begin{aligned} a_{N+1} &\geqslant 10^{N+1}\left(x - \textstyle\sum_{k=1}^{N} 10^{-k}a_k\right) - 1 > -1, \\ a_{N+1} &< 10^{N+1}\left(x - \textstyle\sum_{k=1}^{N} 10^{-k}a_k\right) \leqslant 10^{N+1} \times \frac{1}{10^N} = 10. \end{aligned}$$

So $0 \leqslant a_{N+1} \leqslant 9$ as required. Finally, letting $n \to \infty$ and applying the sandwich rule to

$$x - \frac{1}{10^n} \leqslant \sum_{k=1}^{n} \frac{a_k}{10^k} < x,$$

we find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k} = x.$$

This sequence is called the **decimal expansion** of x and we write

$$x = 0.a_1a_2a_3\ldots.$$

■

Remark 5.13 *In the sense of the above example* $\frac{1}{5}$ *would have decimal expansion* $0.1999\ldots$ *rather than* $0.200\ldots$ *To avoid any ambiguity for those reals with two different decimal expansions (in the usual sense) the above example chooses decimal expansions whose terminating decimal expansions never equal the real in question.*

A similar argument to that above shows the uniqueness for any base $b \geqslant 2$ *expansions. As with the example of* $\frac{1}{5}$ *in decimal, in binary,* $b = 2$*, we would have* $\frac{1}{2} = 0.0111\ldots_2$ *rather than* $\frac{1}{2} = 0.1_2$*.*

5.2 Subsequences

Example 5.14 *Let* $a_n = \frac{1}{n^2}$ *so that*

$$(a_n)_1^\infty = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots\right).$$

We can get new sequences by selectively looking at

everything after second place	$\left(\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots\right)$
all odd terms	$\left(1, \frac{1}{9}, \frac{1}{25}, \dots\right)$
all prime terms	$\left(\frac{1}{4}, \frac{1}{9}, \frac{1}{25}, \frac{1}{49}, \dots\right)$

etc.. These are examples of subsequences of (a_n).

Definition 5.15 *Let* (a_n) *be a sequence. We say that a sequence* (b_n) *is a* ***subsequence*** *of* (a_n) *if there is a strictly increasing sequence of natural numbers* $(f(n))$ *that* $(b_n) = \left(a_{f(n)}\right)$. *Often we write* n_r *for* $f(r)$ *and write a subsequence as* (a_{n_r}) *or* $(a_{n_r})_{r=1}^\infty$.

Example 5.16 *In the previous example* $n_r = r + 2$, $n_r = 2r - 1$ *and* $n_r = p_r$ *(the* r*th prime) respectively.*

Example 5.17 *Let*

$$\begin{aligned} (a_n) &= \left(n^2\right) = (1, 4, 9, 16, \dots); \qquad (b_n) = (0) = (0, 0, 0, 0, \dots); \\ (f(n)) &= (2n) = (2, 4, 6, 8, \dots); \qquad (g(n)) = (2n - 1) = (1, 3, 5, 7, \dots). \end{aligned}$$

Then

$$\begin{aligned} \left(a_{f(n)}\right) &= (a_{2n}) = (4, 16, 36, 64, \dots); \qquad \left(a_{g(n)}\right) = (a_{2n+1}) = (1, 9, 25, 49, \dots); \\ \left(b_{f(n)}\right) &= (b_{2n}) = (0, 0, 0, 0, \dots); \qquad \left(b_{g(n)}\right) = (b_{2n+1}) = (0, 0, 0, 0, \dots). \end{aligned}$$

Proposition 5.18 *Suppose that the sequence* (a_n) *converges to* L. *Then every subsequence* (a_{n_r}) *also converges to* L.

Proof. Let $\varepsilon > 0$. Then there exist N such that

$$n \geqslant N \quad \Longrightarrow \quad |a_n - L| < \varepsilon$$

As $r \mapsto n_r$ is strictly increasing then $n_r \geqslant r$ for all r and so

$$r \geqslant N \quad \Longrightarrow \quad n_r \geqslant N \quad \Longrightarrow \quad |a_{n_r} - L| < \varepsilon$$

and hence $a_{n_r} \to L$ as $r \to \infty$. ■

The converse in the form 'if all subsequences of (a_n) converge to L then $(a_n) \to L$' is true because the whole sequence is a subsequence of itself. However, just one subsequence converging is clearly not enough to guarantee convergence of the whole sequence. For example $a_n = (-1)^n$ which is divergent despite $a_{2n} \to 1$.

Theorem 5.19 *Let (a_n) be a real sequence. Then (a_n) has a monotone subsequence.*

Proof. We consider the set

$$V = \{k \in \mathbb{N} \mid m > k \Longrightarrow a_m < a_k\}.$$

This is the set of 'scenic viewpoints' – were we to plot the points (k, a_k) in $\mathbb{R}^2$ then from a scenic viewpoint we could see all the way to ∞ with no greater a_n obstructing our view. There are two cases to consider: the set V is either finite or infinite.

- V is infinite. Listing the elements of V in increasing order: $k_1 < k_2 < \ldots$ we see (a_{k_r}) is a subsequence with
$$r > s \quad \Longrightarrow \quad k_r > k_s \quad \Longrightarrow \quad a_{k_r} < a_{k_s}$$
That is (a_{k_r}) is strictly decreasing.
- V is finite. Let m_1 be the last viewpoint and consider a_{m_1+1}.
As $m_1 + 1$ is not a viewpoint then there exists $m_2 > m_1 + 1$ such that $a_{m_2} \geqslant a_{m_1}$.

1. As m_2 is not a viewpoint then there exists $m_3 > m_2$ such that $a_{m_3} \geqslant a_{m_2}$.
...
Continuing in this way and we can generate an increasing sequence (a_{m_k}).

■

Theorem 5.20 ***(Bolzano-Weierstrass Theorem, Bolzano 1817, Weierstrass c. 1861)*** *Let (a_n) be a real bounded sequence. Then (a_n) has a convergent subsequence.*

Proof. By the previous theorem (a_n) has a monotone subsequence which is also bounded. By Theorem 5.3 this subsequence converges. ■

Theorem 5.21 ***(Bolzano-Weierstrass Theorem in $\mathbb{C}$)*** *A bounded sequence in $\mathbb{C}$ has a convergent subsequence.*

Proof. Let (z_n) be a bounded sequence in $\mathbb{C}$. If we write $z_n = x_n + iy_n$ then we also have that (x_n) and (y_n) are bounded sequences. By the Bolzano-Weierstrass Theorem (x_n) has a convergent subsequence (x_{n_k}) which converges to L_1, say. As (y_{n_k}) is also bounded then it in turn has a convergent subsequence $\left(y_{n_{k_r}}\right)$ which converges to L_2, say.

As $\left(x_{n_{k_r}}\right)$ is a subsequence of (x_{n_k}) then it too converges to L_1 by Proposition 5.18. We then have that $\left(z_{n_{k_r}}\right)$ converges to $L_1 + iL_2$ as its real and imaginary parts converge (Theorem 3.25). ■

Here is alternative way of phrasing the Bolzano-Weierstrass Theorem.

Definition 5.22 *Let $S \subseteq \mathbb{R}$ We say that x is a **limit point** or **accumulation point** of S if for every $\varepsilon > 0$ there exists $y \in S$, such that*

$$0 < |y - x| < \varepsilon.$$

Note that x itself need not be in the set. The set of limit points of S is denoted S'.

Example 5.23 *The set of limit points of* $(0,1)$ *is* $[0,1]$
The set of limit points of $\mathbb{Q}$ *is* $\mathbb{R}$.
The set of limit points of $\mathbb{Z}$ *is* $\varnothing$.
The set of limit points of $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ *is* $\{0\}$.

Remark 5.24 *The Bolzano-Weierstrass Theorem can be rephrased as: 'An infinite bounded subset of* $\mathbb{R}$ *or* $\mathbb{C}$ *has a limit point'. Given such a set,* S*, then we can select a sequence* (x_n) *of points of* S *and by the Bolzano-Weierstrass Theorem this sequence has a subsequence* (x_{n_r}) *which converges to a limit* L*. It is not hard to show that* L *is then a limit point of the set* $\{x_{n_1}, x_{n_2}, x_{n_2}, \ldots\} \subseteq S$ *and so of the set* S*.*

5.3 The Cauchy Convergence Criterion

A first difficulty in proving that a sequence converges is in investigating the limit. Cauchy saw that a (real or complex) sequence would converge if and only if the sequence's terms got sufficiently close. This makes it possible to demonstrate convergence without knowing the limit. Further, Cauchy's insight can be used to *construct* the reals from the rationals so that we could show the existence of a complete ordered field rather than assuming that a field satisfying all our axioms exists (see Remark 5.34).

Definition 5.25 *Let* (a_n) *be a real or complex sequence. We say that* (a_n) *is a* ***Cauchy sequence****, or simply is* ***Cauchy,*** *if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n \geqslant N \quad |a_m - a_n| < \varepsilon.$$

Note that the definition makes no mention of a limit, but we shall see that this criterion is in fact equivalent to convergence in $\mathbb{R}$ or $\mathbb{C}$ (but not in $\mathbb{Q}$!).

Proposition 5.26 *A convergent sequence is Cauchy.*

Proof. Let (a_n) be a convergent sequence with limit L, and let $\varepsilon > 0$. Then there exists a natural number N such that

$$|a_k - L| < \varepsilon/2 \text{ for all } k \geqslant N.$$

So for all $m, n \geqslant N$,

$$|a_m - a_n| \leqslant |a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

by the triangle inequality and hence (a_n) is Cauchy. ■

Proposition 5.27 *A (real or complex) Cauchy sequence is bounded.*

Proof. Let (a_n) be a real or complex Cauchy sequence. Taking $\varepsilon = 1$, we know there exists N such that

$$|a_n - a_N| < 1 \quad \text{whenever} \quad n \geqslant N.$$

Hence, by the triangle inequality

$$|a_n| < |a_N| + 1 \quad \text{for all } n \geqslant N.$$

The above inequality bounds all but finitely many terms. So for all m we have

$$|a_m| \leqslant \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$$

and we see that the sequence is bounded. ■

Lemma 5.28 *If (a_n) is a real or complex Cauchy sequence such that a subsequence (a_{n_k}) converges to L, then (a_n) converges to L.*

Proof. Let $\varepsilon > 0$. So there exists $K \in \mathbb{N}$ such that

$$|a_{n_k} - L| < \varepsilon/2 \quad \text{whenever} \quad k \geqslant K.$$

As the sequence (a_n) is Cauchy then there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon/2 \quad \text{whenever } m, n \geqslant N.$$

If we select take $k \geqslant \max(K, N)$ so that $n_k \geqslant N$ then we have, by the triangle inequality

$$|a_n - L| \leqslant |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } n \geqslant N$$

and the proof is complete. ■

Theorem 5.29 *(Cauchy, 1821) A real or complex Cauchy sequence is convergent.*

Proof. Let (a_n) be a real or complex Cauchy sequence. By Proposition 5.27 (a_n) is bounded, and so by the Bolzano-Weierstrass Theorem (a_n) has a convergent subsequence (a_{n_k}). By the previous lemma (a_n) converges to the same limit. ■

We have then established the **Cauchy Convergence Criterion** for real and complex sequences:

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is Cauchy.}$$

Remark 5.30 *The Cauchy convergence criterion, together with the Archimedean property, is equivalent to the completeness axiom.*

Example 5.31 *The terminating decimal expansions of $\sqrt{2}$, namely the sequence (q_n):*

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \ldots$$

is a sequence of rational numbers which is Cauchy (for example, because it is a convergent real sequence) but it is not convergent in the rationals – that is, it does not satisfy

$$\exists L \in \mathbb{Q} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |q_n - L| < \varepsilon.$$

Example 5.32 *(**Mercator's series**) For $n \in \mathbb{N}$ let*

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n+1}\frac{1}{n}.$$

Then with $m \geqslant n > 0$, and $m - n$ even we have

$$\begin{aligned} |s_m - s_n| &= \left| \overbrace{\frac{1}{n+1} - \frac{1}{n+2}}^{>0} + \overbrace{\frac{1}{n+3} - \frac{1}{n+4}}^{>0} + \cdots + \overbrace{\frac{1}{m-1} - \frac{1}{m}}^{>0} \right| \\ &= \underbrace{\frac{1}{n+1}}_{} \underbrace{- \frac{1}{n+2} + \frac{1}{n+3}}_{<0} - \cdots \underbrace{- \frac{1}{m-2} + \frac{1}{m-1}}_{<0} \underbrace{- \frac{1}{m}}_{<0} \\ &\leqslant \frac{1}{n+1}. \end{aligned}$$

If $m - n$ is odd, we write

$$\begin{aligned} |s_m - s_n| &= \left| \overbrace{\frac{1}{n+1} - \frac{1}{n+2}}^{>0} + \overbrace{\frac{1}{n+3} - \frac{1}{n+4}}^{>0} + \cdots + \overbrace{\frac{1}{m-2} - \frac{1}{m-1}}^{>0} + \frac{1}{m} \right| \\ &= \underbrace{\frac{1}{n+1}}_{} \underbrace{- \frac{1}{n+2} + \frac{1}{n+3}}_{<0} - \cdots \underbrace{- \frac{1}{m-1} + \frac{1}{m}}_{<0} \\ &\leqslant \frac{1}{n+1}. \end{aligned}$$

Let $\varepsilon > 0$ and take $N > \frac{1}{\varepsilon}$. Then $|s_n - s_m| < \varepsilon$ whenever $m, n \geqslant N$ and we see that (s_n) is Cauchy. This shows that the sequence is convergent even though we currently have no idea of its limit. In due course we shall see that the limit is $\log 2$ (Sheet 6, Exercise 6). The sum was first published by Mercator in 1668.

Remark 5.33 *(**Double sequences**) A (real) double sequence is a map $x\colon \mathbb{N}^2 \to \mathbb{R}$ and we write $x_{m,n}$ for $x(m,n)$. We write that*

$$\lim_{m,n\to\infty} x_{m,n} = L$$

if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n \geqslant N \quad |x_{m,n} - L| < \varepsilon.$$

So we may rewrite the Cauchy convergence criterion as

$$(a_n) \quad \textit{is Cauchy if} \quad |a_n - a_m| \to 0 \quad \textit{as } m, n \to \infty.$$

Given a double sequence $(x_{m,n})$ the limits

$$\lim_{m,n\to\infty} x_{m,n} \qquad \lim_{m\to\infty} \left(\lim_{n\to\infty} x_{m,n} \right) \qquad \lim_{n\to\infty} \left(\lim_{m\to\infty} x_{m,n} \right)$$

are different notions and may independently exist or not as seen in Sheet 5, Exercise 8.

Remark 5.34 ***(Construction of the real numbers)*** *(Off-syllabus)*

We mentioned in Remark 1.59 the matter of existence and uniqueness of the real numbers. These issues were posed in the sense of 'can the real numbers be constructed from more concrete sets such as $\mathbb{N}$*,* $\mathbb{Z}$ *or* $\mathbb{Q}$*?'*

Construction of the natural numbers.

One approach to define the natural numbers is due to Peano from 1889. Peano's description essentially states:

$\mathbb{N}$ *is the smallest set such that (i)* $0 \in \mathbb{N}$*, (ii) if* $n \in \mathbb{N}$ *then* $n+1 \in \mathbb{N}$*.*

A later model, in the style of the Zermelo-Fraenkel axioms for set theory (1908,1922), was Von Neumann's model from 1923 where he identified 0 *with* $\varnothing$*,* 1 *with* $\{\varnothing\}$*,* 2 *with* $\{\varnothing, \{\varnothing\}\}$ *and in general* n *with* $\{0, 1, \ldots, n-1\}$*. as a collection of sets meeting Peano's axioms.*

Construction of the integers.

From the set $\mathbb{N}$ *we can define the set of integers* $\mathbb{Z}$ *from* $\mathbb{N}^2$*. We define the equivalence relation* $\sim$ *on* $\mathbb{N}^2$ *by* $(m_1, m_2) \sim (n_1, n_2)$ *iff* $m_1 + n_2 = n_1 + m_2$*. Then* $\mathbb{Z} = \mathbb{N}^2/\sim$*. Essentially we are identifying an integer with pairs of natural numbers that differ by that integer.*

Construction of the rational numbers.

Having defined $\mathbb{Z}$ *we can define* $\mathbb{Q}$ *as a set of equivalence classes of* $\mathbb{Z} \times (\mathbb{N} \setminus \{0\})$*. We set* $(m_1, n_1) \sim (m_2, n_2)$ *iff* $m_1 n_2 = n_1 m_2$*. Then* $\mathbb{Q} = \mathbb{Z} \times (\mathbb{N} \setminus \{0\})/\sim$*. Essentially we are identifying a rational with all fractions* $\frac{m}{n}$ *which represent that rational.*

Construction of the real numbers. *Having defined* $\mathbb{Q}$ *we set*

$$S = \{(a_n) \mid (a_n) \text{ is a rational Cauchy sequence}\}.$$

At this point we have yet to define the real numbers, but we know that a rational Cauchy sequence converges to some real limit. These limits are what we want as our model of the real numbers but we can't refer to such limits, irrational ones in particular, whilst only being able to refer to the rational numbers. Also many sequences in S *will converge to the same limit so at this stage each real is overrepresented.*

We can deal at least with this last point within the context of real numbers: for $(a_n), (b_n) \in S$ *we set*

$$(a_n) \sim (b_n) \quad \sim \quad a_n - b_n \to 0.$$

As we see in Sheet 5, Exercise 2, for $(a_n), (b_n) \in S$ *and* $c \in \mathbb{R}$ *then*

$$(a_n \pm b_n), \quad (ca_n), \quad (a_n b_n)$$

are in S *and if* $a_n \nrightarrow 0$ *then* $1/a_n \in S$*.*

Further these operation are well-defined in $S/\sim$*. So if* $(a_n) \sim (\alpha_n)$ *and* $(a_n) \sim (\beta_n)$ *then*

$$(a_n \pm b_n) \sim (\alpha_n \pm \beta_n), \quad (ca_n) \sim (c\alpha_n), \quad (a_n b_n) \sim (\alpha_n \beta_n),$$

and if $a_n \nrightarrow 0$ *and* $\alpha_n \nrightarrow 0$ *then* $(1/a_n) \sim (1/\alpha_n)$*. All these results follow bt AOL.*

Regarding order we define $(a_n) \leqslant (b_n)$ *if* $b_n - a_n \geqslant 0$ *in some tail.*

All this gives $\mathbb{R} = S/\sim$ *the structure of an ordered field. It can further be shown that any non-empty bounded subset of* $S/\sim$ *has a least upper bound; this result is not particularly difficult but is non-trivial (Körner pp. 352-353).*

Construction of the complex numbers.

We showed in Section 1.4 how $\mathbb{C}$ can be constructed from $\mathbb{R}$ by identifying a complex number with an ordered pair of real numbers and defining addition and multiplication as one would expect of complex numbers.

6. SERIES

6.1 Infinite Series

Looking back at the field axioms, given any pair of real numbers a, b we can form their sum $a + b$. By induction, we can form any finite sum $\sum_1^n a_k$. The associative law means we don't have to worry about the order is which the necessary additions are executed.

What our axioms *don't* do is licence us to start writing down **infinite sums**, and behaving as though the mere act of writing down similar looking signs ($\sum_1^\infty$, say) entitles us to assume that all the properties of finite sums still hold. In fact, we will see that there are *conditionally convergent* series that give different sums depending on the order in which the terms are added. (See Sheet 6, Exercise 8 for the *Cauchy Root Test* and *Dirichlet's Test*.)

Definition 6.1 *Let $(a_n)_1^\infty$ be a sequence of (real or complex) numbers. For $n \geqslant 1$, the n**th partial sum** of (a_n) is the finite sum*

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

*By the **series***

$$\sum_{k=1}^{\infty} a_k \quad \textit{or just} \quad \sum a_k,$$

we mean the sequence of partial sums (s_n).

Example 6.2 *(a) **The geometric series.** Let $x \in \mathbb{C}$, and let $a_n = x^n$ Then $\sum x^n$ is*

$$(1,\ 1+x,\ 1+x+x^2,\ \ldots\ ,\ 1+x+x^2+\cdots+x^n,\ \ldots\ ).$$

*(b) **The harmonic series.** Let $a_n = \frac{1}{n}$. Then $\sum \frac{1}{n}$ is*

$$\left(1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{3},\ \ldots\ \right).$$

*(c) **The exponential series.** Let $x \in \mathbb{C}$ and let $a_n = x^n/n!$. Then $\sum x^n/n!$ is*

$$\left(1,\ 1+x,\ 1+x+\frac{x^2}{2!},\ \ldots\ \right).$$

*(d) **The cosine series.** Let $x \in \mathbb{C}$ and set*

$$a_n = \begin{cases} \frac{x^{2m}}{(2m)!}(-1)^m & \textit{if } n = 2m \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\sum a_n$ *is*

$$\left(1,\ 1,\ 1-\frac{x^2}{2!},\ 1-\frac{x^2}{2!},\ 1-\frac{x^2}{2!}+\frac{x^4}{4!},\ \dots\right).$$

Definition 6.3 *Let* (a_n) *be a (real or complex) sequence. We say that the series* $\sum_1^\infty a_k$ ***converges*** *(resp.* ***diverges****) if the sequence* (s_n) *of partial sums converges (resp. diverges). If* $s_n \to L$ *as* $n \to \infty$ *then we write*

$$\sum_{k=1}^{\infty} a_k = L.$$

We refer to L *as the* ***sum*** *(or* ***infinite sum****) of the series.*

Remark 6.4 *Our earlier results regarding the tails of sequences still apply – it follows that* $\sum_1^\infty a_k$ *converges if and only* $\sum_K^\infty a_k$ *converges for some* K *(Proposition 3.13). Consequently it makes sense to discuss the convergence (or otherwise) of* $\sum a_k$ *without needing to identify the initial term. But to determine the sum of a convergent series exactly we do need to specify the initial term.*

Proposition 6.5 *Say that* $\sum a_n$ *is convergent. Then* $a_n \to 0$ ***but the converse is not true.***

Proof. Let s_n denote the nth partial sum; then $s_n \to L$ for some sum L. By AOL

$$a_n = s_n - s_{n-1} \to L - L = 0.$$

But recall from Example 3.38 that $\sum \frac{1}{n}$ is divergent, yet $a_n = \frac{1}{n} \to 0$. ■

Example 6.6 *Let* $a_n = x^n$ *for* $n \geqslant 0$ *where* $x \in \mathbb{C}$.

(a) If $x \neq 1$ *then*

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(b) If $|x| < 1$ *then* $\sum x^n$ *is convergent noting* $x^n \to 0$ *and using the algebra of limits.*

(c) If $|x| \geqslant 1$ *then* $\sum x^n$ *is divergent as* $a_n = x^n \nrightarrow 0$.

Example 6.7 *Let* $a_n = \frac{1}{n^2}$. *Then* $\sum \frac{1}{n^2}$ *is convergent.*

Proof. Clearly the partial sums form an increasing sequence. By comparison with a telescoping sum we note

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} \leqslant 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{n} \left\{ \frac{1}{k-1} - \frac{1}{k} \right\} = 1 + \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} = 1 + 1 - \frac{1}{n} \leqslant 2.$$

Hence (s_n) is a bounded increasing sequence and so convergent. [In due course we will meet, with the Integral Test, a systematic way of dealing with such series and won't have to resort to such algebraic tricks.] ■

Remark 6.8 *The exact sum* $\sum_1^\infty \frac{1}{n^2}$ *is known to be* $\pi^2/6$. *This sum was first found by Euler in 1734 and is known as the* ***Basel problem****, Basel being Euler's hometown.*

Applying Cauchy's criterion for convergence for sequences to series (which, recall, is just a sequence of partial sums) we have:

Theorem 6.9 *(**Cauchy's Criterion for Series**) The series $\sum_0^\infty a_k$ converges if and only if for all $\varepsilon > 0$ there exists N such that for all $m, n \geqslant N$ we have*

$$|s_n - s_m| = \left| \sum_{m+1}^{n} a_k \right| < \varepsilon.$$

Definition 6.10 *Let (a_n) be a real or complex sequence. Then we say that $\sum a_n$ is **absolutely convergent** or **AC** if the series $\sum |a_n|$ converges. A series which is convergent, but not absolutely convergent, is called **conditionally convergent**.*

Theorem 6.11 *An AC (real or complex) series is convergent*

Proof. Suppose that $\sum a_n$ is AC and let $\varepsilon > 0$. By Cauchy's criterion there exists N such that

$$l > k \geqslant N \Longrightarrow \left| \sum_{k+1}^{l} |a_n| \right| < \varepsilon.$$

By the triangle inequality

$$l > k \geqslant N \Longrightarrow \left| \sum_{k+1}^{l} a_n \right| \leqslant \sum_{k+1}^{l} |a_n| = \left| \sum_{k+1}^{l} |a_n| \right| < \varepsilon,$$

and hence $\sum a_n$ is Cauchy and so converges. ■

Example 6.12 *(a) $\sum_0^\infty x^n$ is AC if $|x| < 1$ and diverges for $|x| \geqslant 1$.*
(b) $\sum_1^\infty \frac{(-1)^n}{n^2}$ is AC.
(c) $\sum_1^\infty \frac{\sin n}{n^3}$ is AC.
(d) $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Solution. (a) See Example 6.6.
(b) See Example 6.7.
(c) Note that the partial sums

$$s_n = \sum_1^n \frac{|\sin k|}{k^3} \leqslant \sum_1^n \frac{1}{k^3} \leqslant \sum_1^n \frac{1}{k^2} \leqslant \sum_1^\infty \frac{1}{k^2}$$

form an increasing bounded sequence. Hence they converge.
(d) See Examples 5.32 and 3.38 ■

Definition 6.13 *Let $p\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection and set $b_n = a_{p(n)}$. Then $\sum b_n$ is called a **rearrangement** of the series $\sum a_n$.*

Example 6.14 *(See also Sheet 6, Exercise 6) If we rearrange the* $\log 2$ *series from Example 5.32 then we can change the sum:–*

$$\begin{aligned} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots &= \log 2 \\ 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots &= \frac{3}{2}\log 2. \end{aligned}$$

Theorem 6.15 ***(Dirichlet, 1837)*** *(Off-syllabus) If* $\sum a_n$ *is AC then* $\sum a_{p(n)}$ *is AC for any rearrangement* p *and*

$$\sum_1^\infty a_n = \sum_1^\infty a_{p(n)}.$$

So a (real or complex) series is absolutely convergent iff it is unconditionally convergent.

Theorem 6.16 ***(Riemann Rearrangement Theorem, 1853)*** *(Off-syllabus) If* $\sum_1^\infty a_n$ *is a real conditionally convergent series and* $-\infty \leqslant L \leqslant \infty$ *then there exists a bijection* $p\colon \mathbb{N} \to \mathbb{N}$ *such that*

$$\sum_1^\infty a_{p(n)} = L.$$

Hence a real series is AC if and only if it unconditionally convergent.

Theorem 6.17 ***(Cauchy Multiplication of Series, 1821)*** *(Off-syllabus) Suppose* $\sum_0^\infty a_n$ *and* $\sum_0^\infty b_n$ *are AC. For each* $n \in \mathbb{N}$ *we set*

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Then $\sum_1^\infty c_n$ *is AC and*

$$\sum_0^\infty c_n = \left(\sum_0^\infty a_n\right)\left(\sum_0^\infty b_n\right)$$

Proof. See Sheet 6, Exercise 7. ■

Remark 6.18 *Mertens, in 1875, showed that if just one of* $\sum_0^\infty a_n$ *and* $\sum_0^\infty b_n$ *is AC and the other convergent, then* $\sum_0^\infty c_n$ *converges. (See Apostol, Theorem 12-46.)*

Example 6.19 *For* $x, y \in \mathbb{C}$

$$\left(\sum_0^\infty \frac{x^n}{n!}\right)\left(\sum_0^\infty \frac{y^n}{n!}\right) = \sum \frac{(x+y)^n}{n!}$$

Proof. Let $a_n = \frac{x^n}{n!}$, $b_n = \frac{y^n}{n!}$. Then the series $\sum a_n$ and $\sum b_n$ are absolutely convergent (see Example 6.26). Then

$$c_n = \sum_{r+s=n} \frac{x^r}{r!}\frac{y^s}{s!} = \frac{1}{n!}\sum_{r=0}^n \binom{n}{r} x^r y^{n-r} = \frac{(x+y)^n}{n!}$$

by the binomial theorem. ■

6.2 Some Tests for Convergence

Here we discuss some classic tests for convergence and divergence. The idea that there are 'tests' is very attractive, but in practice (for problems arising from real-word situations) these tests may not apply. However the tests *do* give us clues, suggest ways of thinking about series, what sort of estimates need to be made, and a sense of the relative magnitude of terms.

Proposition 6.20 ***(A Simple Test for Divergence)*** *If $\sum a_n$ converges then $a_n \to 0$. The converse is not true.*

As the converse is not true then, in practice, the contrapositive is used more: if a_n does not tend to 0 then $\sum a_n$ diverges.

Proof. We already noted this in Proposition 6.5. ■

Theorem 6.21 ***(The Comparison Test)*** *Let (a_n), (b_n) be real sequences with $0 \leqslant a_n \leqslant b_n$. Then*

- *$\sum b_n$ is convergent $\Longrightarrow \sum a_n$ is convergent;*
- *$\sum a_n$ is divergent $\Longrightarrow \sum b_n$ is divergent.*

Proof. Note that the second statement is just the contrapositive of the first, and so it is enough to just prove the first. Suppose that $\sum b_k$ converges. Then the partial sums $\sum_1^n a_k$ satisfy

$$\sum_1^n a_k \leqslant \sum_1^n b_k \leqslant \sum_1^\infty b_k$$

and hence form an increasing bounded sequence which converges. ■

Remark 6.22 *At first glance, the comparison test seems limited as it only applies to non-negative terms. In practice, however, it is often used to show a series is AC and hence convergent. (See Example 6.23 (d).)*

And as with the sandwich test for sequences, the comparison test can be used to take care of expresssions that are awkward without being impactful. For example, the term $(2+\cos n)^{-1}$ in Example 6.23 (b) lies between $1/3$ and 1.

Example 6.23 *The following sequences*

$$(a) \quad \sum_1^\infty n^{-5/2}, \quad (b) \quad \sum_1^\infty \frac{1}{n\,(n+1)\,(2+\cos n)},$$

$$(c) \quad \sum_1^\infty \frac{x^n}{n} \ \text{ where } |x|<1, \quad (d) \quad \sum_1^\infty \frac{\sin n}{n^2+1},$$

all converge.

Solution. (a) This converges by comparison with $\sum n^{-2}$.

(b) This converges by comparison with $\sum n^{-2}$.

(c) Even though the terms are not non-negative, this is AC by comparison with $\sum |x|^n$ and hence is convergent.

(d) Even though the terms are not non-negative, this is AC by comparison with $\sum n^{-2}$ and hence is convergent. ■

Theorem 6.24 ***(The Ratio Test)*** *Let (a_n) be a real or complex sequence with $a_n \neq 0$ for all n. Suppose that*

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

exists.

- *If $L < 1$ then $\sum a_n$ converges absolutely;*
- *If $L > 1$ then $\sum a_n$ diverges;*
- *If $L = 1$ then $\sum a_n$ may converge or diverge (that is, the test is inconclusive).*

Proof. (a) Choose K such that $|L| < K < 1$. As $\varepsilon = K - |L| > 0$ there exists N such that

$$n \geqslant N \quad \Longrightarrow \quad \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \varepsilon,$$

so that for $n \geqslant N$

$$\left|\frac{a_{n+1}}{a_n}\right| \leqslant \varepsilon + |L| = K.$$

So for $k \geqslant 0$

$$|a_{N+k}| = \left|\frac{a_{N+k}}{a_{N+k-1}}\right| \times \left|\frac{a_{N+k-1}}{a_{N+k-2}}\right| \times \cdots \times \left|\frac{a_{N+1}}{a_N}\right| \times |a_N| \leqslant |a_N|\, K^k.$$

Now $\sum K^k$ is a convergent geometric series, and so the tail $\sum_N^\infty a_n$ is AC by the comparison test. Hence $\sum a_n$ is AC as it has an AC tail.

(b) Choose K such that $1 < K < |L|$. Then there exists N such that

$$n \geqslant N \quad \Longrightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| > K.$$

Arguing as in (a), $|a_{N+k}| \geqslant K^k\, |a_N|$ and hence we see a_n does not tend to 0. So $\sum a_n$ is divergent by Proposition 6.20.

(c) For each of the series $\sum n^{-1}$ and $\sum n^{-2}$ we have $L = 1$ yet the former diverges and the latter converges. ■

Remark 6.25 *If $a_n > 0$ for all n and $\sum a_n$ converges, this does not mean that $\lim |a_{n+1}/a_n|$ exists; for example*

$$1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{9} + \frac{1}{4} + \frac{1}{27} + \frac{1}{8} + \frac{1}{81} + \cdots$$

converges absolutely whilst $|a_{n+1}/a_n|$ does not have a limit.

Example 6.26 *(**Exponential Series**) For all $x \in \mathbb{C}$, the exponential series*

$$\sum_0^\infty \frac{x^n}{n!}$$

converges absolutely.

Solution. The case $x = 0$ is trivial. If $x \neq 0$ then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0 < 1 \text{ as } n \to \infty$$

and apply the ratio test. ■

Example 6.27 *The series*

$$\sum_1^\infty (\sinh n)\, x^n$$

converges absolutely for $|x| < e^{-1}$ and diverges for $|x| \geqslant e^{-1}$.

Solution. By definition $\sinh n = (e^n - e^{-n})/2$ and so

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\sinh(n+1)}{\sinh n}|x| \\ &= \frac{e^{n+1} - e^{-n-1}}{e^n - e^{-n}}|x| \\ &= \frac{e - e^{-2n-1}}{1 - e^{-2n}}|x| \\ &\to e\,|x|\end{aligned}$$

as $n \to \infty$. If $x = e^{-1}$ then the ratio test is inconclusive but

$$a_n = \sinh n \times e^{-n} \to \frac{1}{2} \neq 0$$

and so the series does not converge. ■

Theorem 6.28 *(**Leibniz Alternating Series Test, 1676**) Let (a_n) be a non-negative decreasing series which tends to 0. Then*

$$\sum_{n=0}^\infty (-1)^n a_n$$

converges.

Proof. If we consider the partial sums $s_n = \sum_{k=0}^{n} (-1)^k a_k$ we see that

$$\begin{aligned} s_{2k} &= \underbrace{(a_0 - a_1)}_{\geqslant 0} + \underbrace{(a_2 - a_3)}_{\geqslant 0} + \cdots + \underbrace{(a_{2k-2} - a_{2k-1})}_{\geqslant 0} + a_{2k} \\ &= a_0 + \underbrace{(-a_1 + a_2)}_{\leqslant 0} + \underbrace{(-a_3 + a_4)}_{\leqslant 0} + \cdots + \underbrace{(-a_3 + a_4)}_{\leqslant 0} \\ &\leqslant a_0. \end{aligned}$$

Hence s_{2k} is an increasing sequence bounded above by a_0 and so s_{2k} converges to a limit L. We also have

$$s_{2k+1} = s_{2k} - a_{2k+1} \to L - 0 = L$$

by AOL. Hence s_k converges to L by Sheet 5, Exercise 4(i). ■

Remark 6.29 *Nothing we have done so far lets us tackle series like* $\sum_2^\infty \frac{1}{n(\log n)^2}$, *to evaluate* $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ *or define general exponents. In the remainder of this section we deal with these: but in order to do so we need to make use of the properties of integration and logarithms. We will define logarithms and general powers in the next chapter but we will not meet integration rigorously until Analysis III in Trinity. At the end of the year you will be able to persuade yourself that these properties which we now use do not depend on any of the results of this section, and that no circular arguments have been made. Basically, it is just impatience that forces us to deal with this test now and not wait until Trinity Term.*

Theorem 6.30 *Let* $K \in \mathbb{N}$ *and let* $f\colon [K, \infty) \to [0, \infty)$ *be continuous and decreasing. For* $n > K$ *we define*

$$\delta_n = \sum_{K}^{n-1} f(k) - \int_K^n f(x)\, \mathrm{d}x.$$

Then for $n > K$

$$0 \leqslant \delta_n \leqslant \delta_{n+1} \leqslant f(K)$$

and hence δ_n *converges.*

Corollary 6.31 ***(The Integral Test)*** *With* f *as above, the series* $\sum_K^\infty f(k)$ *is convergent if and only if* $\int_K^n f(x)\, \mathrm{d}x$ *is convergent.*

We postpone the proof for now and instead apply the integral test to a few series.

Example 6.32 $a_n = 1/n^\alpha$ *where* $\alpha \in \mathbb{R}$. *(We will not properly define general exponents until the next chapter.) If* $\alpha \leqslant 0$ *then* a_n *does not tend to* 0 *and so* $\sum a_n$ *diverges. Let* $\alpha > 0$. *Consider the function* $f(x) = x^{-\alpha} \geqslant 0$ *which is continuous and decreasing on* $(0, \infty)$. *We take* $K = 1$ *and note if* $\alpha \neq 1$ *that*

$$\int_1^n f(t)\, \mathrm{d}t = \left[\frac{t^{1-\alpha}}{1-\alpha}\right]_1^n = \frac{n^{1-\alpha} - 1}{1-\alpha}$$

which converges as $n \to \infty$ *if* $\alpha > 1$ *and diverges if* $\alpha < 1$. *If* $\alpha = 1$ *then*

$$\int_1^n f(t)\,\mathrm{d}t = [\log t]_1^n = \log n$$

which diverges. Hence

$$\sum_1^\infty \frac{1}{n^\alpha}$$

converges when $\alpha > 1$ *and diverges for* $\alpha \leqslant 1$.

Example 6.33 $a_n = (n \log n)^{-1}$ *for* $n \geqslant 2$. *Hence we define* $f(x) = \frac{1}{x \log x}$ *on* $(2, \infty)$, *and note* $f(x)$ *is decreasing as* x *and* $\log x$ *are increasing. Then*

$$\int_2^n \frac{1}{x \log x}\,\mathrm{d}x = \log\log n - \log\log 2 \to \infty \quad \textit{as} \quad n \to \infty.$$

Therefore $\sum \frac{1}{n \log n}$ *is divergent.*

Proof. (Of Theorem 6.30) We set

$$\delta_n = \sum_K^{n-1} f(k) - \int_K^n f(x)\,\mathrm{d}x.$$

In the diagram below, which includes a graph of $y = 1/x$ for $x \geqslant K = 1$ we can see δ_4 as the "excess area" above the graph between $1 \leqslant x \leqslant 4$.

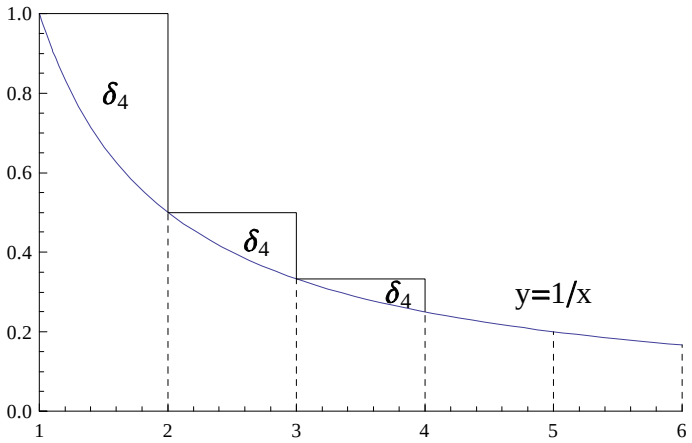

Fig. 6.1 – Proving the Integral Test

As f is decreasing,

$$f(k+1) \leqslant f(x) \leqslant f(k), \text{ if } \; k \leqslant x \leqslant k+1$$

We use the following properties of integration:

- $\int$ preserves weak inequalities;
- $\int_n^{n+1} 1\,\mathrm{d}x = 1$;

- $\int$ is additive: $\int_a^b = \int_a^c + \int_c^b$;
- $\int$ is a linear map on the space of integrable functions.

So we get:

$$f(k+1) \leqslant \int_k^{k+1} f(x)\,\mathrm{d}x \leqslant f(k)$$

and we can add such equations to get

$$f(K+1) + f(K+2) + \cdots + f(n) \leqslant \int_K^n f(t)\,\mathrm{d}t \leqslant f(K) + f(K+1) + \cdots + f(n-1).$$

So using the second inequality above we have

$$0 \leqslant \sum_{r=K}^{n-1} f(r) - \int_K^n f(t)\,\mathrm{d}t$$

which shows $0 \leqslant \delta_n$. Using the first inequality we have

$$\delta_n = \sum_{r=K}^{n-1} f(r) - \int_K^n f(t)\,\mathrm{d}t \leqslant \sum_{r=K}^{n-1} f(r) - \sum_{r=K+1}^{n} f(r) = f(K) - f(n) \leqslant f(K).$$

We also have

$$\delta_{n+1} - \delta_n = f(n) - \int_n^{n+1} f(t)\,\mathrm{d}t \geqslant 0.$$

Hence (δ_n) is bounded above, increasing and so convergent.

Finally

$$\sum_K^{n-1} f(k) \quad \text{and} \quad \int_K^n f(x)\,\mathrm{d}x$$

differ by a convergent sequence. Therefore they both converge or both diverge by AOL. ■

Example 6.34 *(**Euler's Constant** γ, **1734**) If we apply Theorem 6.30 to $f(x) = 1/x$ we get*

$$\begin{aligned}\gamma_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \int_1^n \frac{\mathrm{d}x}{x} \\ &= \delta_n + \frac{1}{n}\end{aligned}$$

is convergent. This limit is called Euler's constant, and often denoted as γ:

$$\gamma = \lim_{n\to\infty} \left(\sum_1^n \frac{1}{k} - \log n \right).$$

The approximate numerical value of γ is

$$0.57721566490153286060\ldots$$

Relatively little is known about γ – for example, it is an open problem still as to whether γ is irrational.

Example 6.35 *We make use of γ in Sheet 6, Exercise 6 to show that*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\log 2.$$

Example 6.36 ***(Euler's Number** e) In Sheet 4, Exercise 5, we showed that*

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots=\sum_{r=0}^{\infty}\frac{1}{r!}$$

converges to an irrational number e. Its approximate numerical value is

$$2.7182818284590452353\ldots$$

In fact, the constant e had been studied well before Euler, with some interest in the constant shown by Napier, Harriot and Huygens. The constant was explicitly defined by Jacob Bernoulli in 1683 as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ while investigating 'continuous compounding' but it was Euler who recognized the importance of the constant and its connection with the 'antilogarithm' function.

Proposition 6.37 *(Euler, 1748)*

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Proof. Let

$$\alpha_n=\left(1+\frac{1}{n}\right)^n \quad \text{and} \quad \beta_n=\sum_{k=0}^{n}\frac{1}{k!}.$$

It was shown in Sheet 4, Exercise 5 that $\lim\beta_n$ exists and we defined e as this limit. It was also shown in Sheet 1, Exercise 6, that α_n is an increasing sequence bounded above and so also converges. By the binomial theorem

$$\begin{aligned}
\alpha_n &= 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots+\frac{1}{n^n}\\
&= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\frac{1}{n}\\
&\leqslant 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=\beta_n.
\end{aligned}$$

From this we have $\lim\alpha_n\leqslant e$. On the other hand for $1\leqslant m<n$ and focusing on the first $m+1$ terms in the binomial expansion of α_n we see

$$1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\frac{1}{m!}\leqslant\alpha_n.$$

Fixing m and letting $n\to\infty$ we have, using AOL and recalling that limits respect weak inequalities,

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{m!}\leqslant\lim\alpha_n.$$

Finally letting $m\to\infty$ we have $e\leqslant\lim\alpha_n$ and the result follows. ■

Remark 6.38 *It's important to note why we took the first $m+1$ terms in the binomial expansion of α_n earlier. In that expansion there are $n+1$ terms and so, as n variees, the number of terms varies. AOL applies to a fixed finite number of terms – fixed in the sense of not depending on the variable that's tending. For exampl it's clear*

$$1 = \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}}.$$

If AOL could be applied to a varying number of terms, letting $n \to \infty$ we would find

$$1 = 0 + 0 + 0 + \cdots = 0,$$

which is false.

Whilst the tests are useful series are not usually met in such a straightforward way that a single convergence test can be employed. If they can be employed at all, some combination of the tests may be needed.

Example 6.39 *Discuss the convergence or divergence of the following series.*

- $$\sum \frac{\cos\left(n^2+1\right)}{n^2+\log n}.$$

  We note that

  $$0 \leqslant \left|\frac{\cos\left(n^2+1\right)}{n^2+\log n}\right| \leqslant \frac{1}{n^2+\log n} \leqslant \frac{1}{n^2}$$

  and so the series is AC by comparison with $\sum n^{-2}$.

- $$\sum (-1)^n \frac{\log\left(n^2+1\right)}{\sqrt{n+2}}$$

  If $y(x) = \log\left(x^2+1\right)(x+2)^{-1/2}$ then

  $$\begin{aligned} y'(x) &= \frac{1}{\sqrt{x+2}}\frac{2x}{\left(x^2+1\right)} - \frac{1}{2(x+2)^{3/2}}\log\left(x^2+1\right) \\ &= \frac{1}{(x+2)^{3/2}}\left[\frac{2x(x+2)}{x^2+1} - \frac{1}{2}\log\left(x^2+1\right)\right] \\ &< \frac{1}{(x+2)^{3/2}}\left[4 - \frac{1}{2}\log\left(x^2+1\right)\right] \\ &< 0 \text{ for } x > e^4. \end{aligned}$$

  So $y(n)$ is eventually decreasing – using a result from Analysis II – and by Leibniz's Test a tail of the series converges. Hence the whole series converges.

- $$\sum \frac{1}{\sqrt{n^2+n}}$$

  We see
  $$\frac{1}{\sqrt{n^2+n}} > \frac{1}{\sqrt{2n^2}} = \frac{1}{n\sqrt{2}}$$
  and so the series diverges by comparison with the harmonic series.

- $$\sum \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$$

  Note
  $$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$
  and hence
  $$\sum_1^N \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1 - \frac{1}{\sqrt{N+1}} \to 1 \quad as \quad N \to \infty.$$

Proposition 6.40 ***(Stirling's Approximation, 1730)*** *(Proof off-syllabus) As $n \to \infty$ then*

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \to 1.$$

The same result is often written as

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Proof. Firstly we note

$$\log n! = \log 2 + \log 3 + \cdots + \log n.$$

We can find a good approximation to the sum on the RHS by applying the trapezium rule to $\log x$ on the interval $[1, n]$. Let $f(x)$ denote the approximating function to $\log x$ whose integral the trapezium rule determines using $n-1$ steps – that is $f(x)$ satisfies $f(k) = \log k$ for each integer $k = 1, 2, \ldots, n$ and is piecewise linear between those values.

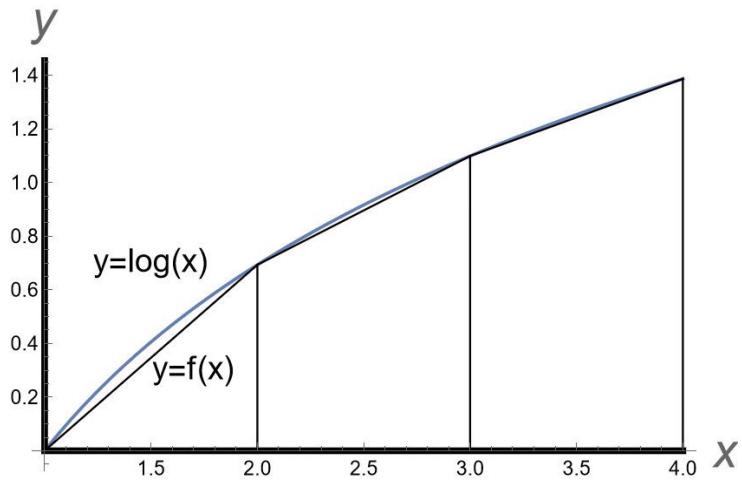

Fig. 6.2 – Trapezium Rule for $\log x$

Note for x in the range $k \leqslant x \leqslant k+1$ we have

$$\frac{1}{k+1} \leqslant \frac{1}{x} \leqslant \frac{1}{k}$$

and so integrating we have

$$\int_k^x \frac{\mathrm{d}t}{k+1} \leqslant \int_k^x \frac{\mathrm{d}t}{t} \leqslant \int_k^x \frac{\mathrm{d}t}{k}$$

or equivalently

$$\log k + \left(\frac{x-k}{k+1}\right) \leqslant \log x \leqslant \log k + \left(\frac{x-k}{k}\right).$$

Now $\log x$ is concave (that is, a chord connecting two points of the graph lies under the graph), and so $f(x) \leqslant \log x$ on the interval $[k, k+1]$. Further as $f\,'(x) \geqslant (k+1)^{-1}$ on the interval (that being the minimum gradient of $\log x$ whilst $f\,'(x)$ has the average gradient) we have

$$\log k + \left(\frac{x-k}{k+1}\right) \leqslant f(x) \leqslant \log x \qquad \text{for } k \leqslant x \leqslant k+1.$$

So we have the inequalities

$$0 \leqslant \log x - f(x) \leqslant \left(\frac{1}{k} - \frac{1}{k+1}\right)(x-k) \qquad \text{for } k \leqslant x \leqslant k+1,$$

and integrating on the interval $[k, k+1]$ we find

$$0 \leqslant \int_k^{k+1} (\log x - f(x))\ \mathrm{d}x \leqslant \left(\frac{1}{k} - \frac{1}{k+1}\right)\int_k^{k+1} (x-k)\ \mathrm{d}x = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Summing up the contributions from the intervals $[1,2], [2,3], \ldots, [n-1, n]$ we find

$$0 \leqslant \int_1^n (\log x - f(x))\ \mathrm{d}x \leqslant \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right),$$

as most of the terms in the above sum cancel consecutively.

Recalling an antiderivative of $\log x$ to be $x\log x - x$ and using the formula for the trapezium rule we then have

$$\begin{aligned} I_n &= \int_1^n (\log x - f(x))\ \mathrm{d}x \\ &= [x\log x - x]_1^n - 1\left(\frac{\log 1}{2} + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{\log n}{2}\right) \\ &= n\log n - n + 1 - \left(\log n! - \frac{1}{2}\log n\right) \\ &= \left(n + \frac{1}{2}\right)\log n - n + 1 - \log n!. \end{aligned}$$

So we have

$$0 \leqslant \int_1^n (\log x - f(x))\, \mathrm{d}x = \left(n + \frac{1}{2}\right) \log n - n + 1 - \log n! \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

(I_n) is an increasing sequence of numbers which we see are bounded above by $1/2$ and hence they converge to some L.

Applying the exponential function we find

$$e^{L-1} = \lim \frac{(n/e)^n \sqrt{n}}{n!}.$$

Whilst in Sheet 6, Exercise 11(ii) , we proved

$$\binom{2n}{n} \frac{\sqrt{n}}{2^{2n}} \to \frac{1}{\sqrt{\pi}}.$$

We can combine these facts to note

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} &= \lim \binom{2n}{n} \frac{\sqrt{n}}{2^{2n}} \\
&= \lim \frac{(2n)!}{n!n!} \frac{\sqrt{n}}{2^{2n}} \\
&= \sqrt{2} \times \lim \left(\frac{(2n!)}{\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}} \right) \times \lim \left(\frac{\sqrt{n}\left(\frac{n}{e}\right)^n}{n!} \right)^2 \\
&= \sqrt{2} \times \left(e^{L-1}\right)^{-1} \times \left(e^{L-1}\right)^2 \\
&= \sqrt{2} e^{L-1}
\end{aligned}$$

Hence $e^{1-L} = \sqrt{2\pi}$ and

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \to \frac{e^{1-L}}{\sqrt{2\pi}} = 1.$$

■

Remark 6.41 *In terms of relative error, Stirling's formula is a very accurate underestimate. For $n = 10$ the relative error is under 1%.*

An improvement on the above approximation is

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

and there are yet more accurate approximations

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left\{ \sum_{k=2}^{m} \frac{(-1)^k B_k}{k\,(k-1)\,n^{k-1}} + O\left(\frac{1}{n^m}\right) \right\},$$

where B_k is the kth Bernoulli number (see Sheet 7, Exercise 9).

7. POWER SERIES

7.1 The Disc and Radius of Convergence

Definition 7.1 *By a **power series** we will mean a series of the form*

$$\sum_{n=0}^{\infty} a_n z^n$$

where $(a_n)_0^\infty$ is a complex sequence and $z \in \mathbb{C}$. We consider (a_n) as fixed for this series, and z as a variable. Clearly the series might converge for some values of z and not for others.

Remark 7.2 *The above power series is a power series centred at the origin. Given $z_0 \in \mathbb{C}$ then we can also consider power series centred at z_0, which take the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Though we will not consider such power series in this chapter, the theory we will develop literally translates to an identical theory for power series centred at $z_0 \neq 0$.

Example 7.3

- *$a_n = 1$: $\sum_0^\infty z^n$: **Geometric series** : as we have already seen (Example 6.6), this series is convergent when $|z| < 1$ and divergent when $|z| \geqslant 1$.*
- *$a_n = 1/n!$: $\sum_0^\infty z^n/n!$: **Exponential series** : we have shown (Example 6.26) that this series is convergent for all $z \in \mathbb{C}$.*
- *$a_n = 1/n$: $\sum_{n=1}^\infty z^n/n$: **Logarithmic series** : convergent for $|z| < 1$. This follows from the ratio test as*

$$\left| \frac{z^{n+1}/(n+1)}{z^n/n} \right| = \frac{n\,|z|}{n+1} \to |z|\,.$$

The series converges at $z = -1$ (Leibniz test and Sheet 6, Exercise 6) and diverges at $z = 1$ (as it's the harmonic series). What about for other values where $|z| = 1$? Well at $z = i$ we have

$$\sum_1^{2N} \frac{i^n}{n} = \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{(-1)^N}{2N} \right) + i \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{N-1}}{2N-1} \right)$$

and we see that both real and imaginary parts converge by the Leibniz test. In fact we know the above partial sums to converge to $-\frac{1}{2}\log 2 + \frac{i\pi}{4}$ (Sheet 6, Exercise 6 and Sheet 5, Exercise 6). More generally it can be shown that the logarithmic series converges on the circle $|z| = 1$ except at $z = 1$.

- $\left.\begin{array}{rcl} a_{2n} & = & (-1)^n/(2n)! \\ a_{2n+1} & = & 0 \end{array}\right\} : \sum \frac{(-1)^n}{(2n)!} z^{2n}$: ***Cosine series*** *: convergent for all z by the ratio test.*
- $\left.\begin{array}{rcl} a_{2n} & = & 0 \\ a_{2n+1} & = & (-1)^n/(2n+1)! \end{array}\right\} : \sum \frac{(-1)^n}{(2n+1)!} z^{2n+1}$: ***Sine series*** *: convergent for all z by the ratio test again.*

Definition 7.4 *Given a power series $\sum a_n z^n$ the set*

$$S = \left\{ z \in \mathbb{C} \mid \sum a_n z^n \text{ converges} \right\} \subseteq \mathbb{C}$$

*is either bounded or unbounded. Note also, that S is non-empty as $0 \in S$. We define the power series' **radius of convergence** R as*

$$R = \begin{cases} \sup\{|z| \mid z \in S\} & \text{when } S \text{ is bounded,} \\ \infty & \text{when } S \text{ is unbounded.} \end{cases}$$

Lemma 7.5 *Suppose that the power series $\sum a_n (z_0)^n$ converges. Then $\sum a_n z^n$ converges absolutely when $|z| < |z_0|$.*

Proof. As $\sum a_n (z_0)^n$ converges then $a_n (z_0)^n \to 0$ and in particular the sequence $a_n (z_0)^n$ is bounded; say $|a_n (z_0)^n| < M$ for all n. Then, for $|z| < |z_0|$,

$$|a_n z^n| = |a_n (z_0)^n| \left|\frac{z}{z_0}\right|^n < M \left|\frac{z}{z_0}\right|^n$$

and so $\sum |a_n z^n|$ converges by comparison with the convergent geometric series $\sum M \, |z/z_0|^n$. ■

Theorem 7.6 *Given a power series $\sum a_n z^n$ with radius of convergence R,*

- *$\sum a_n z^n$ is AC when $|z| < R$,*
- *$\sum a_n z^n$ diverges when $|z| > R$.*

Note that when $R = \infty$ then $\sum a_n z^n$ converges absolutely for all $z \in \mathbb{C}$.

Proof. If $|z| < R$ then, by the approximation property, $|z| < |z_0| < R$ for some $z_0 \in S$ and hence $\sum a_n z^n$ is AC by the previous lemma. On the other hand if $|z| > R$ then $z \notin S$ and hence $\sum a_n z^n$ diverges. ■

Definition 7.7 *The set S is called the **disc of convergence**.*

Remark 7.8 *So a power series is AC strictly within its radus of convergence and diverges strictly beyond the disc of convergence. For z on the boundary $|z| = R$ the series may converge or diverge. It's quite easy to construct power series that converge at only finitely many points of the boundary, or power series that converge everywhere on the boundary except finitely many points. The general question – for which subsets of $|z| = R$ is there a power series which converges exactly on that subset? – remains an open problem.*

Remark 7.9 *Commonly we will use the ratio test to determine the radius of convergence, but it is not hard to produce examples where the ratio test can not be employed, or at least has to be used more subtly. See the third example below.*

Remark 7.10 *(Off-syllabus) As a consequence of Cauchy's root test (Sheet 6, Exercise 8(i)) an exact formula for the radius of convergence is*

$$R = \left(\limsup \sqrt[n]{|a_n|}\right)^{-1}. \tag{7.1}$$

Example 7.11 *Find the radius of convergence of the following examples, and consider the series' convergence on the disc's boundary.*

- $\sum_1^\infty z^n/n^2$. *If we set* $a_n = z^n/n^2$ *then*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|^{n+1}/(n+1)^2}{|z|^n/n^2} = \left(1+\frac{1}{n}\right)^2 |z| \to |z|.$$

  Hence, by the ratio test the series converges absolutely when $|z| < 1$ *but diverges when* $|z| > 1$. *In fact, by comparison with* $\sum n^{-2}$ *we see that the series is AC for all* $|z| = 1$.

- $\sum z^n/n$. *If we set* $a_n = z^n/n$ *we can argue as above to see* $R = 1$. *This is the logarithmic series and we have commented that it diverges at* $z = 1$ *and otherwise converges on* $|z| = 1$.

- $\sum z^p$ *where the sum is taken over all primes* p. *Then* $R = 1$. *To see this we can note* z^p *does not tend to* 0 *when* $|z| \geqslant 1$. *On the other hand* $\sum z^p$ *is AC when* $|z| < 1$ *by comparison with the geometric series* $\sum z^n$.

- *Cosine series:* $\sum_0^\infty (-1)^n z^{2n}/(2n)!$ *If we set* $a_n = (-1)^n z^{2n}/(2n)!$ *then, for all* $z \in \mathbb{C}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|^{2n+2}/(2n+2)!}{|z|^{2n}/(2n)!} = \frac{|z|^2}{(2n+2)(2n+1)} \to 0 \quad \text{as } n \to \infty.$$

  Hence by the ratio test the cosine series is AC for all z.

- *Sine series:* $\sum_0^\infty (-1)^n z^{2n+1}/(2n+1)!$ *If we set* $a_n = (-1)^n z^{2n+1}/(2n+1)!$ *then, for all* $z \in \mathbb{C}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|^{2n+3}/(2n+3)!}{|z|^{2n+1}/(2n+1)!} = \frac{|z|^2}{(2n+2)(2n+3)} \to 0 \quad \text{as } n \to \infty.$$

  Hence by the ratio test the sine series is AC for all z.

Example 7.12 *Use (7.1) to determine the radii of convergence of the series*

$$\sum_1^\infty \frac{z^n}{n}, \qquad \sum_{prime\ p} z^p, \qquad \sum_0^\infty \frac{z^n}{n!}.$$

Solution.

- $a_n = \frac{1}{n}$. Now $n^{1/n} \to 1$ (Sheet 3, Exercise 3) and so

$$\limsup \sqrt[n]{|a_n|} = \lim \left(\sqrt[n]{n}\right)^{-1} = 1$$

so that $R = 1$.

- $a_p = 1$. As there are infinitely many primes then $\limsup \sqrt[n]{|a_n|} = \limsup 1 = 1$ and so $R = 1$.

- $a_n = 1/n!$. By Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

and hence

$$\limsup \sqrt[n]{|a_n|} = \limsup \frac{1}{\sqrt[n]{n!}} = \lim \frac{1}{\sqrt[2n]{2\pi n}}\left(\frac{e}{n}\right) = 0,$$

giving $R = \infty$.

■

The following theorem is beyond the scope of this course, but will be proved in Hilary Term. This theorem will prove very useful when proving various properties of the elementary functions in the next section.

Theorem 7.13 *(**Term-by-term differentiation**) Suppose the (real or complex) power series $\sum_0^\infty a_n z^n$ has radius of convergence R. Then the power series defines a differentiable function on $|z| < R$.*

Term-by-term differentiation is valid within $|z| < R$ so that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_0^\infty a_n z^n\right) = \sum_1^\infty n a_n z^{n-1} = \sum_0^\infty (n+1)\, a_{n+1} z^n.$$

*The power series $\sum_0^\infty (n+1)\, a_{n+1} z^n$ is called the **derived series** and also has radius of convergence R.*

Remark 7.14 *(**Uniqueness of Coefficients**) Say that a function $f(x) = \sum_0^\infty a_n x^n$ is defined on the interval $|x| < R$. By repeated differentiation we see that*

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

*So, if a function is locally defined by a power series, that is $f(x)$ is **analytic**, then the coefficients a_n are unique.*

As a corollary to this, if an analytic function satisfies $f'(x) = 0$ for all x then $a_n = 0$ for all $n \geqslant 1$, and $f(x) = a_0$ is constant.

Remark 7.15 *(**Existence of Coefficients**) A real function is said to be analytic (at* 0*) if it can be locally defined by a power series on some* $(-R, R)$*. As we may differentiate term-by-term, then* $f(x)$ *is necessarily* ***smooth*** *– that is,* $f(x)$ *has derivatives of all orders. Unfortunately smoothness is not a sufficient condition though. For example, the function*

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

can be shown to have derivatives of all orders at $x = 0$ *with* $f^{(n)}(0) = 0$ *for all* $n \geqslant 0$*. So if* $f(x)$ *could be defined by a power series on some* $(-R, R)$ *then we'd have*

$$f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n = 0,$$

but $f(x) \neq 0$ *except at* $x = 0$*. So* $f(x)$ *is smooth, but isn't analytic.*

In the Part A Complex Analysis course, you will see that the situation is very different for complex functions. A complex function which is differentiable (just once!) on an open disc about the origin will be analytic.

Proposition 7.16 *A product of analytic functions is analytic.*

Proof. Say that

$$f(z) = \sum_{n=0}^\infty a_n z^n, \qquad g(z) = \sum_{n=0}^\infty b_n z^n,$$

both converge for $|z| < R$. By Cauchy's formula for the product of series

$$f(z)\, g(z) = \sum_{n=0}^\infty c_n$$

converges where

$$c_n = \sum_{k=0}^n \left(a_k z^k\right)\left(b_{n-k} z^{n-k}\right) = \left(\sum_{k=0}^n a_k b_{n-k}\right) z^n.$$

Setting $d_n = c_n / z^n$ we see that

$$f(z)\, g(z) = \sum_{n=0}^\infty d_n z^n$$

is analytic for $|z| < R$. ■

Proposition 7.17 *Let* $f(x) = \sum_0^\infty a_n x^n$ *converge on* $(-R, R)$*.*

(a) $f(x)$ *is an even function if and only if* $a_{2n+1} = 0$ *for each* $n \geqslant 0$*.*

(b) $f(x)$ *is an odd function if and only if* $a_{2n} = 0$ *for each* $n \geqslant 0$*.*

Proof. (a) If $a_{2n+1} = 0$ for each $n \geqslant 0$, then

$$f(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} a_{2n} (-x)^{2n} = f(-x)$$

is even. Conversely say that $f(x)$ is even. Then $f^{(n)}(x)$ is even when n is even and odd when n is odd – these facts follow from the chain rule. So $f^{(2n+1)}(x)$ is odd and in particular $f^{(2n+1)}(0) = 0$. Hence

$$a_{2n+1} = \frac{f^{(2n+1)}(0)}{(2n+1)!} = 0$$

as required. The proof of (b) is almost identical. ■

7.2 The Elementary Functions

The *elementary functions* include polynomials, rational functions, exponentials, logarithms and trigonometric functions. In contrast there are *special functions* such as Bessel functions (Sheet 7, Exercise 4), Gauss's *error function,* the *gamma function,* etc. and there are deep theorems showing the special functions cannot be expressed in terms of the elementary functions.

In this section we give rigorous definitions for exponentials, logarithms, general exponents and the trigonometric and hyperbolic functions.

Definition 7.18 ***(Exponential Function)*** *The exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ *is defined by the infinite series*

$$\exp(z) = \sum_{0}^{\infty} \frac{z^n}{n!}.$$

1. For all $z \in \mathbb{C}$, $\sum z^n/n!$ is convergent by the ratio test (Example 6.26): so $R = \infty$.
2. $\exp(0) = 1$
3. $\exp(1) = e$
4. $\exp'(z) = \exp(z)$.

   Proof. We use Theorem 7.13 for this. Note

   $$\frac{\mathrm{d}}{\mathrm{d}z} \exp z = \frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{0}^{\infty} \frac{z^n}{n!} \right) = \sum_{0}^{\infty} \frac{(n+1)\, z^n}{(n+1)!} = \sum_{0}^{\infty} \frac{z^n}{n!} = \exp z.$$

   ■
5. $\exp(x+y) = \exp(x)\exp(y)$.

   Proof. We proved this in Example 6.19. We can also use Theorem 7.13 to show this: for fixed $c \in \mathbb{C}$ we define

   $$F(z) = \exp(z+c)\exp(-z).$$

By the product rule

$$F'(z) = \exp(z+c)\exp(-z) - \exp(z+c)\exp(-z) = 0.$$

So $F(z)$ is constant by Remark 7.14 and, as $F(0) = \exp(c)$, then

$$\exp(z+c)\exp(-z) = \exp(c) \qquad \text{for all } z \in \mathbb{C}.$$

Set $c = x + y$ and $z = -y$ for the required result. ■

6. $\exp(z) \neq 0$. (In fact we will see below that the image of exp is $\mathbb{C}\setminus\{0\}$.)

   Proof. For any $z \in \mathbb{C}$ we have

$$\exp(z)\exp(-z) = \exp(0) = 1.$$

   ■

7. $\exp(q) = e^q$ for rational q.

   Proof. Say $q = m/n$ then

$$\left(\exp\left(\frac{m}{n}\right)\right)^n = \exp\left(n\frac{m}{n}\right) = \exp(m \times 1) = (\exp 1)^m = e^m.$$

   By the uniqueness of positive nth roots, we have $\exp(q) = \exp(m/n) = \sqrt[n]{e^m} = e^q$. ■

It seems appropriate to make the following definitions here, though some of what follows requires theory from Hilary Term. We now restrict our attention to the real exponential $\exp\colon \mathbb{R} \to \mathbb{R}$. It is clear from the power series definition of exp that $\exp x > 0$ if $x \geqslant 0$. Further if $x < 0$ then

$$\exp(x) = \frac{1}{\exp(-x)} > 0$$

also. So $\exp(\mathbb{R}) \subseteq (0, \infty)$.

Now $\exp' x = \exp x > 0$ and so exp is an increasing function; in particular this means that $\exp\colon \mathbb{R} \to (0,\infty)$ is injective. Also for $x > 0$, $\exp x > x$ and so exp takes arbitrarily large values of x and similarly $\exp(-x) = 1/\exp(x)$ takes arbitrarily small positive values. So, by the Intermediate Value Theorem (proved in HT), we have

- $\exp\colon \mathbb{R} \to (0,\infty)$ is a bijection and hence invertible.
- The inverse is denoted as $\log\colon (0,\infty) \to \mathbb{R}$, and by a HT result log is differentiable.

Definition 7.19 *The* ***natural logarithm*** $\log x$, *or* $\ln x$, *is the inverse of the real exponential function* $\exp\colon \mathbb{R} \to (0,\infty)$.

Proposition 7.20 *For* $x > 0$,

$$\log' x = \frac{1}{x}.$$

Proof. As $\exp(\log x) = x$ on $(0, \infty)$ then, by the chain rule,

$$\log'(x) \times \exp(\log x) = 1$$

and the result follows. ■

Example 7.21 *The image of* $\exp\colon \mathbb{C} \to \mathbb{C}$ *is* $\mathbb{C}\setminus\{0\}$.

Solution. We previously showed 0 is not in the image. Take $z = re^{i\theta} \neq 0$. We need to find $w = x + iy \in \mathbb{C}$ such that $\exp(w) = z$. This means $e^x e^{iy} = re^{i\theta}$. Setting

$$x = \log r \qquad \text{and} \qquad y = \theta,$$

gives one solution to $\exp(w) = z$. ■

Definition 7.22 ***(General Exponents)*** *Given* $a > 0$ *and* $x \in \mathbb{R}$*, we define*

$$a^x = \exp(x \log a).$$

Note, with this definition,

$$e^x = \exp x \text{ for } x \in \mathbb{R}.$$

Proposition 7.23 *Let* $a, b > 0$ *and* $x \in \mathbb{R}$*. Then*

$$\log(ab) = \log a + \log b, \qquad \log(a^x) = x \log a.$$

Proof. Note

$$\exp(\log a + \log b) = \exp(\log a)\exp(\log b) = ab = \exp(\log(ab))$$

and then take the log of both sides. Also

$$\log(a^x) = \log(\exp(x \log a)) = x \log a.$$

■

Proposition 7.24 *Let* $a > 0$ *and* $x, y \in \mathbb{R}$*. Then*

$$a^{x+y} = a^x a^y, \qquad (a^x)^y = a^{(xy)}.$$

Proof. Note

$$\begin{aligned} a^{x+y} &= \exp((x+y)\log a) \\ &= \exp((x\log a) + (y \log a)) \\ &= \exp(x \log a)\exp(y \log a) \\ &= a^x a^y. \end{aligned}$$

Also

$$\log(a^x)^y = y \log(a^x) = y(x \log a) = (xy)\log a = \log\left(a^{(xy)}\right)$$

and apply the exponential to both sides. ■

Proposition 7.25 *For $x > 0$ and real a,*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a\right) = ax^{a-1}.$$

Proof. By the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(a\log x\right)\right) = \frac{a}{x}\exp\left(a\log x\right) = ax^{-1}x^a = ax^{a-1}.$$

■

Definition 7.26 ***(The Trigonometric and Hyperbolic Functions)***

1. For all $z \in \mathbb{C}$ we define *cosine* and *sine* by

$$\cos z = \frac{\exp\left(iz\right) + \exp\left(-iz\right)}{2}, \qquad \sin z = \frac{\exp\left(iz\right) - \exp\left(-iz\right)}{2i}$$

2. Then

$$\cos z = \sum_0^\infty \frac{(-1)^n z^{2n}}{(2n)!} \qquad \text{and} \qquad \sin z = \sum_0^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

with these series converging for all $z \in \mathbb{C}$.

Proof.

$$\begin{aligned}
\frac{\exp\left(iz\right) + \exp\left(-iz\right)}{2} &= \frac{1}{2}\sum_0^\infty \frac{\left(i^n + (-i)^n\right)}{n!} z^n \\
&= \frac{1}{2}\sum_{k=0}^\infty \frac{(-1)^k + (-1)^k}{(2k)!} z^{2k} = \sum_0^\infty \frac{(-1)^k z^{2k}}{(2k)!}; \\
\frac{\exp\left(iz\right) - \exp\left(-iz\right)}{2i} &= \frac{1}{2i}\sum_0^\infty \frac{\left(i^n - (-i)^n\right)}{n!} z^n \\
&= \frac{1}{2i}\sum_{k=0}^\infty \frac{(-1)^k i + (-1)^k i}{(2k+1)!} z^{2k+1} = \sum_0^\infty \frac{(-1)^k z^{2k+1}}{(2k+1)!}.
\end{aligned}$$

■

3. $\cos 0 = 1$, $\sin 0 = 0$.

4.

$$\exp\left(iz\right) = \cos z + i\sin z.$$

Proof.

$$\cos z + i\sin z = \left(\frac{\exp\left(iz\right) + \exp\left(-iz\right)}{2}\right) + i\left(\frac{\exp\left(iz\right) - \exp\left(-iz\right)}{2i}\right) = \exp\left(iz\right).$$

■

5. $\cos z = \cos(-z)$ and $\sin z = -\sin(-z)$.

   Proof. The powers series of cos (resp. sin) involves only even (resp. odd) powers. ■

6. $$\cos'(z) = -\sin z \qquad \text{and} \qquad \sin'(z) = \cos z.$$

   Proof. Using $\exp' z = \exp z$ then

   $$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(\cos z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\exp(iz)+\exp(-iz)}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{i\exp(iz)-i\exp(-iz)}{2}\right) = -\sin z, \\ \frac{\mathrm{d}}{\mathrm{d}z}(\sin z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\exp(iz)-\exp(-iz)}{2i}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\exp(iz)+\exp(-iz)}{2}\right) = \cos z. \end{aligned}$$

   or we can just calculate the derived series of $\cos z$ and $\sin z$. ■

7. $$\begin{aligned} \sin(z+w) &= \sin z\cos w + \cos z \sin w, \\ \cos(z+w) &= \cos z\cos w - \sin z \sin w. \end{aligned}$$

   Proof. By definition $\sin z\cos w + \cos z \sin w$ equals

   $$\begin{aligned} &\left(\frac{\exp(iz)-\exp(-iz)}{2i}\right)\left(\frac{\exp(iw)+\exp(-iw)}{2}\right) \\ &+\left(\frac{\exp(iz)+\exp(-iz)}{2}\right)\left(\frac{\exp(iw)-\exp(-iw)}{2i}\right) \end{aligned}$$

   which rearranges to

   $$\begin{aligned} &\frac{1}{4i}\left[2\exp(iz)\exp(iw) - 2\exp(-iz)\exp(-iw)\right] \\ =\ &\frac{\exp(iz+iw)-\exp(-iz-iw)}{2i} \\ =\ &\sin(z+w). \end{aligned}$$

   The second identity can be proved in a similar manner or by differentiating the first identity with respect to z. ■

8. $$\cos^2 z + \sin^2 z = 1$$

   Proof. Set $z = -w$ in the previous identity for $\cos(z+w)$. Alternatively, differentiating gives

   $$\frac{\mathrm{d}}{\mathrm{d}z}\left(\cos^2 z + \sin^2 z\right) = -2\cos z\sin z + 2\sin z\cos z = 0.$$

   So $\cos^2 z + \sin^2 z$ is constant by Remark 7.14 and takes value $1^2 + 0^2 = 1$ at $z = 0$. Or we can argue

   $$\cos^2 z + \sin^2 z = (\cos z + i\sin z)(\cos z - i\sin z) = \exp(iz)\exp(-iz) = \exp(iz - iz) = 1.$$

   ■

9. It is easy to note that $\cos 0 = 1$ and that

$$\begin{aligned}
\cos 2 &= \sum_0^\infty \frac{(-1)^n\, 2^{2n}}{(2n)!} \\
&= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \cdots \\
&= \underbrace{1 - 2 + \frac{2}{3}}_{<0} - \underbrace{\frac{2^6}{6!}\left(1 - \frac{2^2}{7\times 8}\right)}_{>0} - \underbrace{\frac{2^{10}}{10!}\left(1 - \frac{2^2}{11\times 12}\right)}_{>0} - \cdots \\
&< 0.
\end{aligned}$$

It follows from theorems we will meet in Hilary Term that there exists a smallest positive root to the equation $\cos x = 0$. We will define $\pi/2$ as the smallest root of cosine.

10. As $\cos^2 z + \sin^2 z = 1$ then $\sin(\pi/2) = \pm 1$ (in fact it equals 1 as we know) and

$$\exp(\pi i/2) = \cos(\pi/2) + i\sin(\pi/2) = \pm i.$$

Then

$$\exp(z + 2\pi i) = \exp z \left(\exp \frac{\pi i}{2}\right)^4 = (\exp z)(\pm i)^4 = \exp z.$$

Hence exp has period $2\pi i$ and cosine and sine have period 2π – i.e.

$$\cos(z + 2\pi) = \cos(z) \quad \text{and} \quad \sin(z + 2\pi) = \sin(z).$$

11. For the other trigonometric functions we define

$$\begin{aligned}
\sec x &= \frac{1}{\cos x}, \qquad \tan x = \frac{\sin x}{\cos x}, \\
\csc x &= \frac{1}{\sin x}, \qquad \cot x = \frac{\cos x}{\sin x}.
\end{aligned}$$

12. We define *hyperbolic cosine* and and *hyperbolic sine* by

$$\begin{aligned}
\cosh z &= \frac{\exp(z) + \exp(-z)}{2} = \sum_0^\infty \frac{z^{2n}}{(2n)!}, \\
\sinh z &= \frac{\exp(z) - \exp(-z)}{2} = \sum_0^\infty \frac{z^{2n+1}}{(2n+1)!},
\end{aligned}$$

where these series converge for all $z \in \mathbb{C}$. Note that

$$\begin{aligned}
\cos iz &= \cosh z, \qquad \cosh iz = \cos z, \\
\sin iz &= i\sinh z, \qquad \sinh iz = i\sin z, \\
\cosh(-z) &= \cosh z, \qquad \sinh(-z) = -\sinh z, \\
\cosh' z &= \sinh z, \qquad \sinh' z = \cosh z, \\
\sin(x + iy) &= \sin x \cosh y + i\cos x \sinh y, \\
\cos(x + iy) &= \cos x \cosh y - i\sin x \sinh y, \\
\cosh^2 z - \sinh^2 z &= 1.
\end{aligned}$$

Example 7.27 *(See also Sheet 7, Exercise 9.) Find the power series for* $\tan z$ *up to the* z^5 *term.*

Solution. As $\tan z$ is odd then we only have to calculate coefficients for z, z^3 and z^5 (Proposition 7.17). One approach would involve differentiating repeatedly, but we would then need to calculate the fifth derivative of $\tan z$. Instead we will use the binomial theorem.

Recall that

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7), \qquad \cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6),$$

so that

$$\tan z = \frac{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)}{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}.$$

By the binomial theorem, $(1-y)^{-1} = \sum_0^\infty y^n$ for $|y| < 1$, for suitably small z,

$$\begin{aligned}
&\left(1 - \left(\frac{z^2}{2} - \frac{z^4}{24} + O(z^6)\right)\right)^{-1} \\
= \;& 1 + \left(\frac{z^2}{2} - \frac{z^4}{24} + O(z^6)\right) + \left(\frac{z^2}{2} - \frac{z^4}{24} + O(z^6)\right)^2 + O(z^6) \\
= \;& 1 + \left(\frac{z^2}{2} - \frac{z^4}{24}\right) + \left(\frac{z^4}{4}\right) + O(z^6) \\
= \;& 1 + \frac{z^2}{2} + \frac{5z^4}{24} + O(z^6).
\end{aligned}$$

So

$$\begin{aligned}
\tan z &= \left(z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)\right)\left(1 + \frac{z^2}{2} + \frac{5z^4}{24} + O(z^6)\right) \\
&= z + \left(\frac{1}{2} - \frac{1}{6}\right)z^3 + \left(\frac{1}{120} - \frac{1}{12} + \frac{5}{24}\right)z^5 + O(z^7) \\
&= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + O(z^7).
\end{aligned}$$

■

Remark 7.28 *Note that many of the properties of complex sine and cosine differ significantly from their real counterparts. Whilst*

$$\cos^2 z + \sin^2 z = 1$$

is true for all z*, this does not mean that* $|\cos z| \leqslant 1$ *or* $|\sin z| \leqslant 1$*. In fact,* $\cos\colon \mathbb{C} \to \mathbb{C}$ *and* $\sin\colon \mathbb{C} \to \mathbb{C}$ *are onto, and so in, particular, are unbounded.*

Given $w \in \mathbb{C}$ *then*

$$\begin{aligned}
\sin z = w \quad &\iff \quad \frac{\exp(iz) - \exp(-iz)}{2i} = w \\
&\iff \quad \exp(iz)^2 - 2iw\exp(iz) - 1 = 0 \\
&\iff \quad \exp(iz) = i \pm \sqrt{1 - w^2}.
\end{aligned}$$

Recall exp *takes all values except* 0. *So unless* $w^2 = 1$, *the RHS represents two distinct complex numbers, so there is a solution* z *to at least one of the two equations. And if* $w^2 = 1$ *as* $i \neq 0$ *then* $\exp(iz) = i$ *has a solution. (See also Example 7.29.)*

Also, whilst $\exp(iz) = \cos z + i \sin z$ *is true for all complex* z, *it's not generally true that* $\cos z = \mathrm{Re}\,(\exp(iz))$.

Example 7.29 *Find all the solutions to* $\sin z = 2$.

Solution. Set $z = x + iy$ so that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y = 2.$$

Comparing real and imaginary parts, we have

$$\sin x \cosh y = 2, \qquad \cos x \sinh y = 0.$$

If $y = 0$ then $\sin x = 2$ which has no solutions. Hence $\cos x = 0$ and $x = (2n+1)\frac{\pi}{2}$. Then

$$2 = \sin\left((2n+1)\frac{\pi}{2}\right)\cosh y = (-1)^n \cosh y.$$

So n must be even and we have $y = \pm \cosh^{-1} 2$. So the solutions to $\sin z = 2$ are

$$z = (2n+1)\frac{\pi}{2} \pm i \cosh^{-1} 2.$$

■

Example 7.30 *Show that* $\cos z = \mathrm{Re}\,(\exp(iz))$ *holds if and only if* z *is real.*

Solution. We know this is true for real z. To prove the converse, say $z = x + iy$. Then

$$\begin{aligned} \cos(x+iy) &= \cos x \cosh y - i \sin x \sinh y, \\ \mathrm{Re}\,(\exp(iz)) &= \mathrm{Re}\,(\exp(-y+ix)) = e^{-y}\cos x. \end{aligned}$$

Comparing real and imaginary parts we have

$$e^{-y}\cos x = \cos x \cosh y, \qquad \sin x \sinh y = 0.$$

If $y = 0$ then both equations are satisfied and z is real. If $y \neq 0$ then $\sin x = 0$ and so $x = n\pi$ for some integer n, so that $\cos x = (-1)^n \neq 0$. Finally

$$\cosh y = e^{-y} \iff \frac{e^y + e^{-y}}{2} = e^{-y} \iff e^{2y} = 1 \iff y = 0,$$

and hence z is real. ■

Example 7.31 *Show that* $|\exp(iz)| = 1$ *if and only if* z *is real.*

Solution. Let $z = x + iy$. Note

$$1 = |\exp(iz)| = |\exp(-y + ix)| = e^{-y}$$

if and only if $y = 0$ and so z is real. ■

Definition 7.32 ***(Logarithmic Series)*** *Consider the power series*

$$\lambda(z) = \sum_1^\infty \frac{z^n}{n}.$$

The radius of convergence is 1 *(by the ratio test) and so converges for* $|z| < 1$.

For $-1 < x < 1$, by Theorem 7.13,

$$\lambda'(x) = \sum_1^\infty \frac{nx^{n-1}}{n} = \sum_1^\infty x^{n-1} = \frac{1}{1-x}.$$

Set

$$\mu(x) = (1-x)\exp(\lambda(x)).$$

By the chain and product rules,

$$\mu'(x) = -1 \times \exp\lambda(x) + (1-x)\lambda'(x)\exp\lambda(x) = 0.$$

It follows that $\mu(x)$ is constant and equals $\mu(0) = \exp\lambda(0) = \exp 0 = 1$. Hence

$$\exp(\lambda(x)) = \frac{1}{1-x} \qquad \text{for } -1 < x < 1$$

and, with using the definition of real logarithm (Definition 7.19)

$$\lambda(x) = \log\left(\frac{1}{1-x}\right) = -\log(1-x).$$

In terms of the branch L for complex logarithm defined earlier we have

$$\lambda(z) = -L(1-z) \qquad \text{for } |z| < 1.$$

Example 7.33 *Let* $\alpha, z \in \mathbb{C}$ *with* $|z| < 1$. *Find the power series of*

$$B(z,\alpha) = (1+z)^\alpha = \exp(\alpha L(1+z)).$$

Solution. The composition of two analytic functions is itself analytic (which I do not prove here), so we may set

$$B(z,\alpha) = \sum_0^\infty a_n z^n.$$

By the chain rule

$$B'(z,\alpha) = \alpha L'(1+z)\exp(\alpha L(1+z)) = \frac{\alpha B(z,\alpha)}{1+z}$$

and so

$$(1+z)B'(z,\alpha) = \alpha B(z,\alpha).$$

We note $a_0 = 1$ and, focusing on the z^n term on each side, we obtain the recurrence relation

$$(n+1)a_{n+1} + na_n = \alpha a_n$$

so that

$$a_0 = 1, \qquad a_{n+1} = \left(\frac{\alpha - n}{n+1}\right) a_n \quad \text{for } n \geqslant 0.$$

Hence

$$\begin{aligned} a_n &= \left(\frac{\alpha - n + 1}{n}\right) a_{n-1} \\ &= \left(\frac{\alpha - n + 1}{n}\right)\left(\frac{\alpha - n + 2}{n-1}\right) a_{n-2} \\ &= \cdots \\ &= \frac{(\alpha - n + 1)(\alpha - n + 2)\cdots(\alpha - 1)\alpha}{n(n-1)\times\cdots\times 2\times 1} a_0 \\ &= \frac{(\alpha - n + 1)(\alpha - n + 2)\cdots(\alpha - 1)\alpha}{n(n-1)\times\cdots\times 2\times 1}. \end{aligned}$$

If we denote this last expression as $\binom{\alpha}{n}$ then we have determined the **binomial series for a general exponent:**

$$B(z,\alpha) = (1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n.$$

Note that if α is a natural number then this is a finite sum and otherwise the above series is an infinite sum which converges for $|z| < 1$. ■