C5.9 - Mathematical Mechanical Biology

Sheet 2 — HT22

Section A - For practice

1. **From 3D to 2D.** Starting with the exact equation of a rod in the plane given by

$$F' + f = \rho A \ddot{x}, \tag{1}$$

$$G' + g = \rho A \ddot{y}, \tag{2}$$

$$EI\theta'' + G\cos\theta - F\sin\theta = \rho I \ddot{\theta}, \tag{3}$$

Derive a general equation for the beam. (You may assume that x is held fixed at the $s = 0$ end of the rod.) Specify what boundary conditions should be chosen in the cases of (a) a clamped-clamped beam (where the tangent are held horizontal), (b) a pinned beam (where the position of rod at the end is fixed but there is no moment applied at the boundary), and (c) a clamped-free beam, where one end is completely free.

Solution: Along with the equations above we have the geometric equations

$$\frac{\partial x}{\partial s} = \cos\theta \tag{4}$$

$$\frac{\partial y}{\partial s} = \sin\theta. \tag{5}$$

We use the hint from the problem sheet; namely, to consider $|\theta| \ll 1$. In this case, we can consider the corresponding asymptotic behavior in the limit of $\theta \to 0$:

$$\frac{\partial x}{\partial s} \sim 1 \Rightarrow x = s \tag{6}$$

$$\frac{\partial y}{\partial s} \sim \theta. \tag{7}$$

Since by assumption in the problem, the x-position of one of the rod is held fixed (which can be scaled to correspond to $x = 0$), here we have used that $x(0) = 0$ for all time to eliminate a time dependent constant. Thus, the moment balance equation becomes:

$$EI\frac{\partial^3 y}{\partial x^3} + G - F\theta = \rho I \frac{\partial^3 y}{\partial x \partial t^2}. \tag{8}$$

Now, differentiating this with respect to x, we find:

Derek Moulton: moulton@maths.ox.ac.uk

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\partial G}{\partial x}-\theta\frac{\partial F}{\partial x}-F\frac{\partial \theta}{\partial x}=EI\frac{\partial^4 y}{\partial x^4}+\rho A\frac{\partial^2 y}{\partial t^2}-g+f\frac{\partial y}{\partial x}-F\frac{\partial^2 y}{\partial x^2}=\rho I\frac{\partial^4 y}{\partial x^2\partial t^2}. \quad (9)$$

In the scenario of a static beam with no applied loads ($f = g = 0$), this reduces to:

$$EI\frac{\partial^4 y}{\partial x^4}-F\frac{\partial^2 y}{\partial x^2}=0, \quad (10)$$

which is the classical Euler-Bernoulli equation for a beam.

Considering the possible boundary conditions we can impose on our system, clamped boundary conditions correspond to not only fixing the position of the beam at its ends, but also its direction. In other words, y and $y'(x)$ are specified at both $x = 0$ and $x = L$. For a pinned beam, the positions are fixed, so y is specified, but are done so in a way that the beam is still free to rotate. This means the beam is moment free. In the nonlinear version, we would impose $m = EI\theta'(s) = 0$, this translates to the boundary condition $y'' = 0$.

Finally, if the beam is completely free at one end, then both moment and forces vanish. Zero moment means $y'' = 0$. Equation (8) gives the other boundary condition by setting $F = G = 0$. For a static rod, zero forces at one end implies $y''' = 0$.

2. **Monkey-saddle.** Draw and compute all the curvatures (principal, mean, Gaussian) for the monkey-saddle defined by $z = x^3 - 3xy^2$. Show that every point has negative Gaussian curvature, except the origin.

Solution: This could be done by hand, but would be quite tedious. This can be done much more easily with the assistance of the Mathematica script "Monkey saddle.nb" (provided on the course webpage).

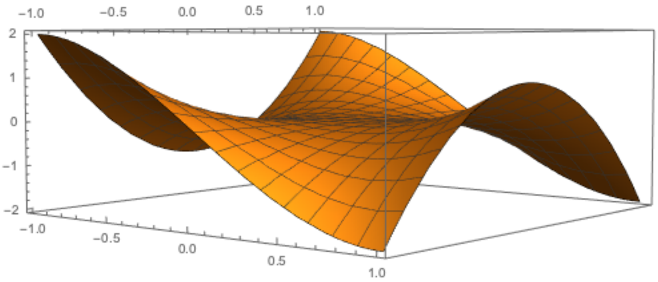
We can parametrize the surface as

$$\begin{aligned} x &= u && (11)\\ y &= v && (12)\\ z &= u^3 - 3uv^2, && (13) \end{aligned}$$

Derek Moulton: moulton@maths.ox.ac.uk

and it is a small change to the provided script to produce the surface and compute the curvatures.

The surface has the shape:



The Gaussian curvature, K_G, and mean curvature, H, are computed to be:

$$K_G = -\frac{36\left(u^2+v^2\right)}{\left(1+9u^4+18u^2v^2+9v^4\right)^2} \tag{14}$$

$$H = \frac{54\left(u^5-2u^3v^2-3uv^4\right)}{\left(1+9u^4+18u^2v^2+9v^4\right)^{\frac{3}{2}}}. \tag{15}$$

Note that the denominator of K_G is positive for all real values of u and v, while the numerator is greater than 0 for all values of u and v except when $u = v = 0$, where it is equal to 0. This then implies that the Gaussian curvature is negative everywhere except at the origin, as required.

Section B - To be marked

3. **Helices.** Consider an unshearable rod, such that we align $\mathbf{d}_3$ with the tangent $\boldsymbol{\tau}$.

   (a) Show that the curvature vector $\mathbf{u}$, satisfying $\mathbf{d}_i'(S) = \mathbf{u} \times \mathbf{d}_i$, $\quad i = 1, 2, 3$, can be written in the basis $\{\mathbf{d}_i\}$ as

   $$\mathbf{u} = (\mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3) = (\kappa \sin\varphi, \kappa\cos\varphi, \tau + \varphi'(s)).$$

   (b) Consider an elastic rod with helical shape in its reference configuration – this is characterised by intrinsic curvature $\hat{\mathbf{u}} = (0, \hat{\kappa}, \hat{\tau})$, where $\hat{\kappa}$ and $\hat{\tau}$ are constants. Suppose that the $s = 0$ end of the rod is located at the origin, and with material frame satisfying $\mathbf{d}_i(0) = \mathbf{d}_{i_0}$, $\ i = 1, 2, 3$. By directly integrating the kinematic equations, obtain explicitly the reference shape, i.e. in the absence of any external forces, moments, or boundary constraints, compute the centerline curve and material frame.

4. **Axon injury.** During injury, an axon can be extended and deformed so that it buckles. The axon lies inside a tissue. As a simple model of this process, we model the axon as a beam subject to an axial force and resting on a foundation that pulls the beam down; in other words we imagine the axon is on a substrate that resists its deformation from the flat state. The equation for deflection $W = W(x)$ of such a beam (Equation (85) in Biofilaments typed notes with $B = EI$, $F = -P$ and an extra kW term for the foundation) is given by

   $$BW'''' + PW'' + kW = 0. \tag{16}$$

   We will further assume that the beam is clamped at $x = \pm L$, that is $W(\pm L) = W'(\pm L) = 0$. First non-dimensionalise the equation so that it reads

   $$w'''' + \lambda w'' + \beta w = 0, \tag{17}$$

   and study its solution by looking at solutions of the form $w = ae^{i\omega x}$. It can be shown, and you may thus assume, that non-trivial solutions only exist in the range $\lambda^2 - 4\beta > 0$. Restricting to this case, compute the solution and show that λ must satisfy one of two relations.

Derek Moulton: moulton@maths.ox.ac.uk

5. **Area and length invariance.** Show that the definitions of area (Eq. 6 in Biomembrane typed notes) and length (Eq. 9 in Biomembrane typed notes) are invariant under a change of parameterisation.

   Getting started: Suppose that our original parameterisation is in the variables (x, y), and that we change variables to (u, v). For this transformation, we have the Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}. \tag{18}$$

   If we define the vectors

$$d\mathbf{u} = \begin{pmatrix} du \\ dv \end{pmatrix}, \quad d\mathbf{x} = \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

   It follows that

$$d\mathbf{u} = J d\mathbf{x}.$$

   Next, use the chain rule to show that if G is the original metric tensor in (x, y), and $\hat{G}$ is the metric tensor in (u, v), then

$$G = J^T \hat{G} J.$$

   Now, for arclength to be invariant, we need $d\hat{s}^2 = ds^2$, where ds is an arc length element in (x, y) and $d\hat{s}$ is that in (u, v)....

6. **Eigenvalues of L.** The matrices G and K associated with the first and second fundamental form are symmetric. However, the combination $L = G^{-1}K$ is not necessarily symmetric. Prove that, nevertheless, the eigenvalues of L are real and that eigenvectors with distinct eigenvalues are orthogonal with respect to the inner product $<\mathbf{a}, \mathbf{b}> = \mathbf{a}^T G \mathbf{b}$.

Derek Moulton: moulton@maths.ox.ac.uk

Section C - Tougher problems, optional

7. **Euler's theorem.** Prove Euler's theorem given in Eq. (17) in the Biomembrane typed notes.

   Solution: There are a number of ways to show Euler's theorem for the normal curvature, however we will derive the result as follows:

   The normal curvature is defined as:

$$k_n = -\boldsymbol{n} \cdot \frac{d\boldsymbol{t}}{ds}. \tag{19}$$

   Using the orthogonality of $\boldsymbol{t}$ and $\boldsymbol{n}$, (i.e. expanding $0 = \frac{d}{ds}(\mathbf{n} \cdot \mathbf{t})$) we can write this as:

$$k_n = -\boldsymbol{n} \cdot \frac{d\boldsymbol{t}}{ds} = \boldsymbol{t} \cdot \frac{d\boldsymbol{n}}{ds} = \frac{d\boldsymbol{x} \cdot d\boldsymbol{n}}{ds^2}. \tag{20}$$

   To proceed, recall that

$$d\mathbf{x} = \mathbf{r}_1 d\xi^1 + \mathbf{r}_2 d\xi^2. \tag{21}$$

   If we take take a differential of

$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{||\mathbf{r}_1 \times \mathbf{r}_2||}$$

   we will pick up a lot of terms, however any differential of the denominator will vanish when dotted with $d\mathbf{x}$, and differentials of the numerator, dotted with $d\mathbf{x}$, can then be shuffled using the vector triple product to obtain terms of the form

$$-\mathbf{n} \cdot \frac{\partial \mathbf{r}_i}{\partial \xi^j}$$

   which are the components of the second fundamental form K. Putting it together, we get

$$k_n = -\frac{L\left(d\xi^1\right)^2 + 2M d\xi^1 d\xi^2 + N\left(d\xi^2\right)^2}{E\left(d\xi^1\right)^2 + 2F d\xi^1 d\xi^2 + G\left(d\xi^2\right)^2}, \tag{22}$$

   where L, M, and N are the entries of the second fundamental form and E, F and G are the entries of the first fundamental form. In terms of the matrices K and G, we would write $L = K_{11}$, $M = K_{12}$, $N = K_{22}$, and $E = G_{11}$, $F = G_{12}$, $G = G_{22}$[1].

   For simplicity, let us consider a surface such that $M = F = 0$, so our normal curvature is now:

$$k_n = -\frac{L\left(d\xi^1\right)^2 + N\left(d\xi^2\right)^2}{E\left(d\xi^1\right)^2 + G\left(d\xi^2\right)^2}. \tag{23}$$

Derek Moulton: moulton@maths.ox.ac.uk

We can find the principal curvatures, k_1 and k_2, of this by considering two isoparametric curves, along one of which k_1 is constant and, along the other, k_2 is constant. In the present proof, we use a parametrization such that ξ^1 and ξ^2 are coordinates in the directions of these curves, so we can write our isoparametric curves as $\xi^1 = A$ and $\xi^2 = B$, where A and B are both constants.

As such, if we consider traveling along the isoparametric curve given by $\xi^1 = A$, then $d\xi^2 = 0$ and hence $k_1 = -\frac{L}{E}$, whilst traveling along the other curve, given by $\xi^2 = B$, would then imply $d\xi^1 = 0$ which then gives $k_2 = -\frac{N}{G}$.

Let us now consider the angle between our tangent vector $\boldsymbol{\xi} = (\xi^1, \xi^2)$ and the principal axis where k_1 is constant, which we will define as $\boldsymbol{u} = (u_1, 0)$ with a component only in the ξ^1 direction. Taking the inner product of these vectors and using the generalized definitions of the dot product and arc-length that uses the first fundamental form, we write:

$$\cos\theta = \frac{\boldsymbol{\xi}\cdot\boldsymbol{u}}{|\boldsymbol{\xi}||\boldsymbol{u}|} = \frac{Ed\xi^1 du_1 + Fd\xi^2 du_1}{\sqrt{E\left(d\xi^1\right)^2 + 2Fd\xi^1 d\xi^2 + G\left(d\xi^2\right)^2}\sqrt{Edu_1^2}} = E\frac{d\xi^1}{ds}\frac{du_1}{ds_1}, \tag{24}$$

where we have used the fact that $F = 0$ and the ξ^2 component of $\boldsymbol{u}$ is 0, and where we have defined $ds = \sqrt{E\left(d\xi^1\right)^2 + G\left(d\xi^2\right)^2}$ for $\boldsymbol{\xi}$ and $ds_1 = \sqrt{Edu_1^2}$ for our other vector, $\boldsymbol{u}$.

Using this definition for ds_1, we can further simplify as:

$$\cos\theta = \sqrt{E}\frac{d\xi^1}{ds}, \tag{25}$$

and, substituting this into $ds = \sqrt{E\left(d\xi^1\right)^2 + G\left(d\xi^2\right)^2}$, we find that:

$$ds^2 = E\left(\frac{1}{\sqrt{E}}\cos\theta ds\right)^2 + G\left(d\xi^2\right)^2 \Rightarrow \sin\theta = \sqrt{G}\frac{d\xi^2}{ds}. \tag{26}$$

Lastly, we can calculate $k_1\cos^2\theta + k_2\sin^2\theta$ using the above equations and our previous results for k_1 and k_2:

$$\begin{aligned} k_1\cos^2\theta + k_2\sin^2\theta &= -\frac{L}{E}E\left(\frac{d\xi^1}{ds}\right)^2 - \frac{N}{G}G\left(\frac{d\xi^2}{ds}\right)^2 = -\frac{L\left(d\xi^1\right)^2 + N\left(d\xi^2\right)^2}{ds.ds} \\ &= -\frac{L\left(d\xi^1\right)^2 + N\left(d\xi^2\right)^2}{E\left(d\xi^1\right)^2 + G\left(d\xi^2\right)^2} = k_n, \end{aligned} \tag{27}$$

as required.

8. **Slightly deformed sphere.** Compute the mean and Gaussian curvatures of a slightly deformed sphere. That is, find to order $\mathcal{O}(\epsilon)$, the curvatures of

$$\mathbf{x}(\theta,\phi) = R(1+\epsilon h(\theta,\phi))\,[\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)] \tag{28}$$

and show that they can be expressed as

$$H = \frac{\mathrm{tr}(L)}{2} = \frac{1}{R}\left(1 - \frac{\epsilon}{2}\Delta(\theta,\phi)\right) + \mathcal{O}(\epsilon^2) \tag{29}$$

$$K = \det(L) = \frac{1}{R^2}\left(1 - \epsilon\Delta(\theta,\phi)\right) + \mathcal{O}(\epsilon^2). \tag{30}$$

where

$$\Delta = 2h + \frac{1}{\sin^2\theta}\partial_{\phi\phi}h + \cot\theta\ \partial_\theta h + \partial_{\theta\theta}h. \tag{31}$$

Solution: It is highly recommended that you use a symbolic algebra package like Mathematica or Maple to help you complete this problem; the algebra becomes messy very quickly. There are some similarities in computing the curvatures as in question 2, however, the main difference will be consistently working to first order in ϵ.

To begin with, we start with our position vector, defined as:

$$\boldsymbol{x} = R(1+\epsilon h(\theta,\phi)\left\{\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)\right\}, \tag{32}$$

find the corresponding tangent vectors, $\boldsymbol{r}_\theta$ and $\boldsymbol{r}_\phi$:

$$\begin{aligned}\boldsymbol{r}_\theta = [&R\cos(\phi)\left(\epsilon\sin(\theta)h^{(1,0)}(\theta,\phi) + \epsilon\cos(\theta)h(\theta,\phi) + \cos(\theta)\right), \\ &R\sin(\phi)\left(\epsilon\sin(\theta)h^{(1,0)}(\theta,\phi) + \epsilon\cos(\theta)h(\theta,\phi) + \cos(\theta)\right), \\ &R\epsilon\cos(\theta)h^{(1,0)}(\theta,\phi) - R\sin(\theta)(\epsilon h(\theta,\phi)+1)],\end{aligned} \tag{33}$$

$$\begin{aligned}\boldsymbol{r}_\phi = [&R\sin(\theta)\left(\epsilon\cos(\phi)h^{(0,1)}(\theta,\phi) - \sin(\phi)(\epsilon h(\theta,\phi)+1)\right), \\ &R\sin(\theta)\left(\epsilon\sin(\phi)h^{(0,1)}(\theta,\phi) + \epsilon\cos(\phi)h(\theta,\phi) + \cos(\phi)\right), \\ &R\epsilon\cos(\theta)h^{(0,1)}(\theta,\phi)],\end{aligned} \tag{34}$$

and compute the unit normal as we did previously:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi\|}. \tag{35}$$

We can now calculate the first fundamental form, G, however, we only work up to $O(\epsilon)$ and neglect higher ordered terms. This can be done in Mathematica by using the "Series" command to expand in powers of ϵ. To first order, we find:

$$G = \begin{pmatrix} R^2(2\epsilon h(\theta,\phi)+1) & 0 \\ 0 & R^2(2\epsilon h(\theta,\phi)+1)\sin^2(\theta) \end{pmatrix}. \tag{36}$$

Calculating the second fundamental form, K, again up to $O(\epsilon)$ only, we find that the entries of K are given by:

$$\begin{aligned}
K_{11} &= R\left(\epsilon h(\theta,\phi) - \epsilon h^{(2,0)}(\theta,\phi) + 1\right) && (37)\\
K_{12} &= K_{21} = \frac{R\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{\sin(\theta)} && (38)\\
K_{22} &= R\left(\epsilon h(\theta,\phi)\sin^2(\theta) + \left(\sin(\theta) - \epsilon\cos(\theta)h^{(1,0)}(\theta,\phi)\right)\sin(\theta) - \epsilon h^{(0,2)}(\theta,\phi)\right) && (39)
\end{aligned}$$

where $h^{(1,0)}(\theta,\phi) \equiv \frac{\partial h}{\partial \theta}$, $h^{(0,1)}(\theta,\phi) \equiv \frac{\partial h}{\partial \phi}$ and so forth.

Lastly, we obtain the entries of the principal curvature matrix, $L = G^{-1}K$ by combining the first order results. We again only take terms to $O(\epsilon)$:

$$\begin{aligned}
L_{11} &= \frac{1}{R} - \frac{\epsilon\left(h(\theta,\phi) + h^{(2,0)}(\theta,\phi)\right)}{R} && (40)\\
L_{12} &= \frac{\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{R\sin(\theta)} && (41)\\
L_{21} &= \frac{\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{R\sin^3(\theta)} && (42)\\
L_{22} &= \frac{1}{R} - \frac{\epsilon\left(h^{(0,2)}(\theta,\phi)\csc^2(\theta) + h(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi)\right)}{R}. && (43)
\end{aligned}$$

Computing the Gaussian curvature, K_G, and mean curvature, H, to first order, we find:

$$\begin{aligned}
H &= \frac{\mathrm{tr}(L)}{2} = \frac{1}{R} - \frac{\epsilon\left(\left(h^{(2,0)}(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi) + \csc^2(\theta)h^{(0,2)}(\theta,\phi) + 2h(\theta,\phi)\right)\right)}{2R} && (44)\\
K_G &= \det(L) = \frac{1}{R^2} - \frac{\epsilon\left(h^{(2,0)}(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi) + \csc^2(\theta)h^{(0,2)}(\theta,\phi) + 2h(\theta,\phi)\right)}{R^2} && (45)
\end{aligned}$$

which are the required results.

A small Mathematica script which does this computation is provided on the course webpage.

Derek Moulton: moulton@maths.ox.ac.uk