C5.9 - Mathematical Mechanical Biology

Sheet 4 — MT

Section A - For practice

1. **Gravitropism plus autotropism.** Consider the model of a plant stem evolving its reference curvature to align with gravity, Section 3.3 in the typed Biogrowth notes. Here we modify this model to include the effect of *prioprioception*, or *autotropism*, a phenomenon by which the plant stem resists curvature. We suppose that the intrinsic curvature evolves according to

$$\frac{\partial \hat{\mathsf{u}}_2}{\partial t} = -\beta \sin\phi - \nu \mathsf{u}_2 \tag{1}$$

where the second term models the autotropic effect. As before ϕ is the angle between the tangent and the vertical. If the stem is clamped at the origin with angle $\phi = \phi_0$ such that $0 < \phi_0 \ll 1$, derive a leading order asymptotic problem for the shape and time evolution of $\hat{\mathsf{u}}_2$. This can be solved as a series solution, but it is very tedious. Instead, solve the full nonlinear system numerically and compare with the trajectory of a purely gravitropic stem to determine the effect of the proprioception.

Solution: Similarly to the purely gravitropism case in the notes, the kinematics of the stem is described by

$$\frac{\partial x}{\partial s} = \sin\phi \tag{2}$$
$$\frac{\partial y}{\partial s} = \cos\phi \tag{3}$$
$$\frac{\partial \phi}{\partial s} = \mathsf{u}_2 \tag{4}$$
$$\frac{\partial \mathsf{u}_2}{\partial t} = -\beta \sin\phi - \nu \mathsf{u}_2 \tag{5}$$

In the limit of small ϕ (nearly vertical), we expand $\sin\phi \approx \phi$, $\cos\phi \approx 1$, which leads to the single equation for $x(s,t)$:

$$x_{sst} = -\beta x_s - \nu x_{ss}. \tag{6}$$

This can be integrated once with s:

$$x_{st} + \beta x + \nu x_s = c(t), \tag{7}$$

and the function $c(t)$ is determined from the boundary conditions

$$x(0) = 0, \phi(0) = \phi_0 \Rightarrow x_s(0,t) = \phi_0 \Rightarrow x_{st}(0,t) = 0$$

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from which we conclude $c = \nu\phi_0$ is a constant.

In completing the system for x, we also have initial condition $x(s, 0) = \phi_0 s$, which comes from an initially straight configuration, $\mathsf{u}_2(s, 0) = 0$.

To solve the full nonlinear system, the simplest method is to integrate the system in space for a given curvature u_2, and then update the curvature by discretising (5), i.e.

$$\mathsf{u}_2(s, t + \Delta t) = \mathsf{u}_2(s, t) - \Delta t(\beta \sin\phi(s, t) + \nu \mathsf{u}_2(s, t)).$$

A sample Mathematica code, Tropism.nb is available on the course website.

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Section B - Regular problems

2. **Axon pulling.** In this problem we model an axon being pulled as a one-dimensional extensible growing rod. Following the notation used in lectures, let S be the grown reference arclength variable, L the total grown length, s the current arclength variable , and l the total current length. The axon is anchored at the point $S = s = 0$ and is only allowed to deform along its length. We assume that the tension n_3 in the axon is due to both an applied tension σ at the end $s = l$ and to a linear adhesion force between the substrate and the axon that can be modelled by a simple Hookean law

$$\mathsf{f}_3 = -k(s(S) - S). \tag{8}$$

Supposing that the $S = 0$ end remains fixed, i.e. $s(0) = 0$, solve for the tension $\mathsf{n}_3(S)$.

Next, consider the following growth law for the evolution of $S = S(t)$:

$$\frac{\partial \gamma}{\partial t} = \gamma \hat{k}(\mathsf{n}_3 - \sigma^*)\mathrm{H}(\mathsf{n}_3 - \sigma^*), \tag{9}$$

where $H(\cdot)$ is the Heaviside function, and $\gamma = \frac{\partial S}{\partial S_0}$. Give a physical interpretation of the growth law.

Obtain an expression for the material velocity $V(S, t) = \partial_t S$.

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3. **Growing rod with diffusion.** Consider a growing inextensible rod of initial length $2L_0$ at time $t = 0$. The rod is parameterised by $S_0 \in [-L_0, L_0]$ and constrained so that it can only grow along its length. Its growth rate is proportional to the growth stretch and the concentration of a nutrient, $u = u(s,t)$, that is kept constant at the two ends of the rod. The nutrient diffuses through the rod according to the law

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial s^2} - Q, \tag{10}$$

where $Q > 0$ is a constant uptake, $D > 0$ is the diffusion coefficient, and $s = s(S_0, t)$ is the position of a material point initially located at S_0. The boundary conditions are $u(\pm l, t) = U$ where $l = s(L_0, t)$ and $U > 0$ is a constant.

(a) Write an equation for the evolution of growth $\gamma = \partial s/\partial S_0$.

(b) Assuming that growth is slow compared to diffusion, find the equilibrium concentration of u. Show that there are two regimes depending on the critical length l^* defined as the smallest length for which the concentration at the origin vanishes. Sketch typical solutions for $l < l^*$, $l = l^*$, and $l > l^*$.

(c) Derive an equation for the position $s(S_0, t)$ of a point S_0 as a function of time, and the corresponding equation for the evolution of the length l. Solve the equation for the length in the asymptotic regime $l \ll l^*$.

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4. **Buckling of an infinite planar rod on foundation.** Consider the problem of the growth of a rod on a foundation, such that the foundation is on the x-axis. If f and g are the x and y components of the foundation force

$$\mathbf{f} = \frac{h(\Delta)}{\gamma\Delta}\left[(x - S/\gamma)\mathbf{e}_x + (y - y_0)\mathbf{e}_y\right],$$

where $\Delta^2 = (x - S/\gamma)^2 + (y - y_0)^2$. We take $y_0 = 0$ and expand $h(\Delta) \approx h(0) + h'(0)\Delta = -Ek\Delta$, so that

$$f = -\frac{Ek}{\gamma}(x - S/\gamma), \quad g = -\frac{Ek}{\gamma}y.$$

To solve this problem, first show that there is a grown ($\gamma > 1$) yet trivial solution that is a straight compressed rod:

$$x^{(0)} = S/\gamma, \; y^{(0)} = \theta^{(0)} = G^{(0)} = 0, \; F^{(0)} = EA\frac{1-\gamma}{\gamma}. \tag{11}$$

To find the critical value of γ where a bifurcation first occurs, expand the variables in power series, e.g. $x = x^{(0)} + \epsilon x^{(1)} + \mathrm{O}(\epsilon^2)$ and linearise the system around the trivial compressed state. Show that to first order, $x^{(1)} = F^{(1)} = 0$, and that the remaining equations reduce to a single fourth order differential equation for $\theta^{(1)}$

$$\frac{d^4\theta^{(1)}}{dS^4} + 2a\frac{d^2\theta^{(1)}}{dS^2} + b^2\theta^{(1)} = 0, \tag{12}$$

where a and b should be defined. Now look for modes of the form $\theta^{(1)} \sim e^{i\omega S}$. Obtain the first bifurcation condition (i.e. smallest $\gamma > 1$) for which there exist oscillatory modes on an infinite domain, and an explicit expression for γ in the case of a circular cross section with radius r.

5. **The growing cuboid.** Consider an isotropic incompressible cuboid constrained between two plates and growing along its axis as shown in Figure 1. The growth deformation transforms the reference state of a cuboid initially of length L_0 into a longer one of length L but the plates prevent this expansion so that its current length remains $l = L_0$, thus leading to elastic stresses. The strain energy function for the cuboid is $W(\alpha_1, \alpha_2, \alpha_3)$. Using the Cartesian basis as shown in the figure, the deformation tensors are of the form

$$\mathbf{F} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \; \mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3), \; \mathbf{G} = \mathrm{diag}(1, 1, \gamma^2). \tag{13}$$

Show that the axial stress generated during growth is given by

$$t_3 = -\frac{\gamma}{2}\hat{W}'(\gamma), \tag{14}$$

where $\hat{W}$ is defined by $\hat{W}(\gamma) = W(\gamma, \gamma, 1/\gamma^2)$.

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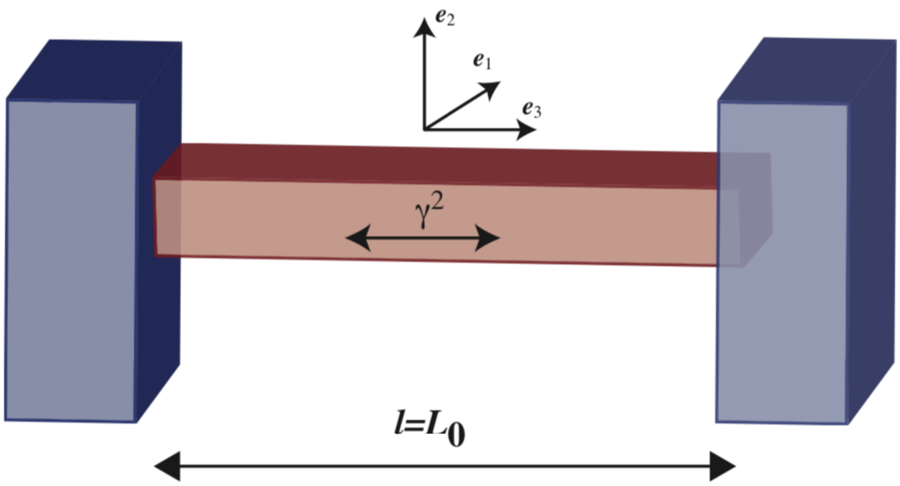

Figure 1: Growth of a block constrained between two plates.

Section C - Tougher problems, optional

6. **Growing cylindrical tube.** Consider the problem of a growing tube, that is a cylindrical shell, under loads. We assume that the tube is hyperelastic isotropic, and is subject to both an inflation pressure P and axial load N. The radius of the undeformed tube is defined by $A_0 \leq R_0 \leq B_0$ and its initial height is H_0. The tube after deformation is given by $a \leq r \leq b$ and height h. The deformation is fully described by the function $r(R_0)$, which gives the location of circles of radius R_0 after deformation, and the axial stretch $\zeta = h/H_0$.

   Obtain a set of two equations for the two unknown constants a and ζ in terms of the loads P and N, and the elements of the given growth tensor $\mathbf{G} = \text{diag}(\gamma_r, \gamma_\theta, \gamma_z)$.

   Give particular consideration to the form of your equations – think about how you would solve them, and try to obtain a form that could be solved in practice for given parameters and growth functions.

   Solution: The deformation gradient tensor, expressed in cylindrical coordinates, is $\mathbf{F} = \text{diag}(r'(R_0), r/R_0, \zeta)$. The elastic strain tensor is $\mathbf{A} = \text{diag}(\alpha_r, \alpha_\theta, \alpha_z)$, and the growth tensor for the symmetric growth is $\mathbf{G} = \text{diag}(\gamma_r, \gamma_\theta, \gamma_z)$. Here γ_r corresponds to radial growth, γ_θ to circumferential growth and γ_z to axial growth – each of these may be constant or functions of R_0, but we assume that they are input to the problem. This, along with the assumption that the tube remains cylindrical through the deformation, implies that the elastic stretches, i.e. components of $\mathbf{A}$, can also be functions of R_0, but not θ. Since $\mathbf{F} = \mathbf{AG}$, we have

$$\alpha_r = \frac{r'}{\gamma_r}, \quad \alpha_\theta = \frac{r}{R_0\gamma_\theta}, \quad \alpha_z = \frac{\zeta}{\gamma_z}. \tag{15}$$

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Material incompressibility implies det$\mathbf{A}$=1, that is $\alpha_r\alpha_\theta\alpha_z = 1$ which implies

$$rdr = \frac{R_0 g(R_0)}{\zeta} dR_0, \tag{16}$$

that is

$$r^2 = a^2 + \frac{2}{\zeta}\int_{A_0}^{R_0} g(\rho)\rho d\rho \tag{17}$$

where $g(R_0) = \det \mathbf{G} = \gamma_r\gamma_\theta\gamma_z$.

The equilibrium conditions are given by the non-vanishing equation for the divergence of the Cauchy stress:

$$\frac{dt_r}{dr} + \frac{1}{r}(t_r - t_\theta) = 0, \tag{18}$$

and the application of the boundary conditions. The radial inflation pressure P gives the boundary condition

$$t_r = \begin{cases} 0, & r = b, \\ -P & r = a, \end{cases}$$

from which we obtain

$$\int_a^b \frac{t_\theta - t_r}{r} dr = P,$$

while the axial load condition is

$$2\pi \int_a^b t_z r \, dr \, d\theta = N.$$

The difficulty is that the bounds of the integrals a and b are not known – these must be determined through the deformation! It is therefore simpler to reformulate these integrals in the initial configuration. To do this we use (16) to convert to an integral over R_0. If we define

$$\tau(R_0; \zeta, a) := \int_{A_0}^{R_0} \frac{t_\theta - t_r}{\zeta\, r^2(\tilde{R}_0)} g(\tilde{R}_0)\tilde{R}_0 d\tilde{R}_0, \tag{19}$$

$$\phi(R_0; \zeta, a) := 2\pi \int_{A_0}^{R_0} \frac{t_z}{\zeta} g(\tilde{R}_0)\tilde{R}_0 d\tilde{R}_0, \tag{20}$$

then the two equations to find ζ and a as a function of P and N are

$$\tau(B_0; \zeta, a) = P, \tag{21}$$

$$\phi(B_0; \zeta, a) = N, \tag{22}$$

where $r(R_0)$ is given by (17).

We now turn to the constitutive law

$$t_r = \alpha_r \frac{\partial W}{\partial \alpha_r} - p, \quad t_\theta = \alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - p, \quad t_z = \alpha_z \frac{\partial W}{\partial \alpha_z} - p. \tag{23}$$

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These enable us to write the stress components in terms of the elastic components, since W is a given function of the α_i, and the α_i in turn can be expressed in terms of the unknowns a and ζ. The problem is they also contain the unknown hydrostatic pressure p, so the trick is to eliminate p from the integrals in τ and ϕ.

Considering τ first: clearly, we can write

$$t_\theta - t_r = \alpha_\theta \frac{\partial W}{\partial \alpha_\theta} - \alpha_r \frac{\partial W}{\partial \alpha_r};$$

then, combining this with the relationships

$$\alpha_r = \frac{R_0 \gamma_\theta \gamma_z}{r(R_0)\zeta}, \quad \alpha_\theta = \frac{r(R_0)}{R_0 \gamma_\theta}, \quad \alpha_z = \frac{\zeta}{\gamma_z}, \tag{24}$$

the integrand of (19) is well defined with the only unknowns being the constants a and ζ. (Keep in mind that we are imagining that the growth functions γ_i are given, therefore $r(R_0)$ is known except for the constants a and ζ.)

In terms of expressing t_z only in terms of a and ζ in the integrand for ϕ, one approach is to note that the radial stress is obtained as the indefinite integral

$$t_r(R_0) = -P + \tau(R_0; \zeta, a), \tag{25}$$

Since this expression does not involve p, we can then eliminate p between t_r and t_z, to give

$$t_z = t_r - \alpha_r \frac{\partial W}{\partial \alpha_r} + \alpha_z \frac{\partial W}{\partial \alpha_z}.$$

In principle, one could complete the calculation this way, but in practice it is very messy, as the integral for ϕ involves the indefinite integral $\tau(R_0; \zeta, a)$ in the integrand!

A much better approach is as follows: Due to the incompressibility, $\det\mathbf{A} = 1$, we can write

$$\alpha_r = \alpha_z^{-1} \alpha_\theta^{-1}.$$

Then, defining

$$\hat{W}(\alpha_\theta, \alpha_z) = W(\frac{1}{\alpha_z \alpha_\theta}, \alpha_\theta, \alpha_z),$$

we can compute

$$\frac{\partial \hat{W}}{\partial \alpha_\theta} = -\frac{1}{\alpha_\theta^2 \alpha_z} \frac{\partial W}{\partial \alpha_r} + \frac{\partial W}{\partial \alpha_\theta},$$

and thus

$$\alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta} = -\alpha_r \frac{\partial W}{\partial \alpha_r} + \alpha_\theta \frac{\partial W}{\partial \alpha_\theta}.$$

Therefore, we can write

$$t_\theta - t_r = \alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta}.$$

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Similarly,

$$t_z - t_r = \alpha_z \frac{\partial \hat{W}}{\partial \alpha_z}.$$

Now, the big trick is to write

$$\int_a^b t_z r \, dr = \int_a^b r(t_z - t_r) + r t_r \, dr.$$

The first term can be replaced by $r\alpha_z \frac{\partial \hat{W}}{\partial \alpha_z}$, while we can integrate by parts on the second term and use (18):

$$\int_a^b r t_r \, dr = \frac{r^2}{2} t_r \bigg|_a^b - \int_a^b \frac{r^2}{2} \frac{t_\theta - t_r}{r} \, dr = \frac{Pa^2}{2} - \frac{1}{2} \int_a^b r(t_\theta - t_r) \, dr.$$

Bringing it together, the axial load condition becomes

$$\pi \int_a^b (2\alpha_z \frac{\partial \hat{W}}{\partial \alpha_z} - \alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta}) r \, dr = N - Pa^2.$$

Converting to reference variables, the function ϕ is updated to

$$\phi(R_0, A_0; \zeta, a) = \frac{\pi}{\zeta} \int_{A_0}^{R_0} (2\alpha_z \frac{\partial \hat{W}}{\partial \alpha_z} - \alpha_\theta \frac{\partial \hat{W}}{\partial \alpha_\theta}) g(R_0) R_0 \, dR_0,$$

and the two conditions to solve for a and ζ are updated to

$$\tau(B_0; \zeta, a) = P, \tag{26}$$

$$\phi(B_0; \zeta, a) = N - Pa^2. \tag{27}$$

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