Analysis II

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0. INTRODUCTORY MATERIAL

Syllabus

Definition of the function limit. Definition of continuity of functions on subsets of $\mathbb{R}$ and $\mathbb{C}$ in terms of ε and δ. The algebra of continuous functions; examples, including polynomials. [3]

Intermediate Value Theorem for continuous functions on intervals. Boundedness, maxima, minima for continuous functions on closed, bounded intervals. Uniform continuity for continuous functions on closed, bounded intervals. Monotone functions on intervals and the Continuous Inverse Function Theorem. [4]

Sequences and series of functions, uniform convergence. Weierstrass's M-test for uniformly convergent series of functions. Uniform limit of a sequence of continuous functions is continuous. Continuity of functions defined by power series. [2]

Definition of the derivative of a function of a real variable. Algebra of derivatives, examples to include polynomials and inverse functions. Vanishing of the derivative at a local maximum or minimum. Rolle's Theorem, Mean Value Theorem, and Cauchy's (Generalized) Mean Value Theorem with applications: Constancy Theorem. L'Hôpital's Formula. [3.5]

The derivative of a function defined by a power series is given by the derived series (proof not examinable). Exponential function and trigonometric functions. Taylor's Theorem with remainder in Lagrange's form; examples. The binomial expansion with arbitrary index. [3.5]

Reading list

(1) *Introduction to real analysis,* Robert Bartle, Donald Sherbert, Wiley 4th ed 2011
(2) *Real analysis and infinity,* H. Sedghat, OUP 2022
(3) *Guide to analysis*, Mary Hart, Macmillan 2nd ed 2001
(4) *A radical approach to real analysis,* David Bressoud, MAA 2007
(5) *Mathematical analysis: a straightforward approach,* K. G. Binmore, CUP 2nd ed 1982
(6) *Mathematical analysis,* Tom Apostol, Pearson, 2nd ed 1974

Further Reading

(1) *Understanding analysis*, Stephen Abbott, Springer 2nd ed 2015
(2) *A very short introduction to mathematical analysis,* Richard Earl, OUP 2023
(3) *A companion to analysis,* T. W. Körner, AMS 2003

0.1 Introduction and Historical Background

The story of analysis, as a separate subject within mathematics, begins in the nineteenth century. Three mathematicians, who might reasonably share the title of 'fathers of analysis' are Bolzano, Cauchy and Weierstrass, who were working around 1817, 1821 and 1861 respectively. Their names each arise throughout this year's analysis courses but, however seminal their work, it was around 150 years later than ideal.

During your degree you will see that analysis has ideas and themes of its own – this will be especially apparent when you meet *metric spaces* and *complex analysis* in the second year. But the biggest need driving a focus on analysis in the nineteenth century was the lack of rigor in mathematics, particularly with regard to calculus. The origins of calculus are typically associated with the work of Leibniz and Newton in the 1680s, though their work built on the work of many others, especially Fermat. Their work was a great leap forward but was definitely not the final word in the development of the calculus: neither Leibniz nor Newton could frame their work without reference to infinitesimals, fluxions or other ill-defined terms. At this time there were still no formal definitions of *convergence* or *limit*. Given the widespread impact of calculus within mathematics and science, there was a desperate need for rigorous foundations; Newton's and Leibniz's methods had been widely applied with success for 100 years or so, but by the nineteenth century the need for more clarity and careful definitions was becoming paramount.

Euler, in his seminal *Introductio in Analysin Infinitorum* of 1748, moved the discussion forward significantly. Euler placed functions – as defined locally by power series – at the centre of his work, and during his life calculated an amazing array of infinite sums. But still there was no formal defintion of a limit. And whilst it was known that the general solution to the wave equation was

$$y(x,t) = f(x+ct) + g(x-ct),$$

where f and g are *arbitrary functions*, what did this phrase mean? Certainly, in the case of a plucked string, something different from Euler's definition of what a function is. The physical and mathematical descriptions just didn't quite match.

Around 1816-17 in Prague, the mathematician and philosopher, Bernard Bolzano, first gave the modern definition of a limit, introduced so-called ε-δ analysis, defined the notion of 'greatest lower bound' and proved the *intermediate value theorem* and *Bolzano-Weierstrass theorem*. Unfortunately Bolzano's work would go unrecognized during his lifetime and only surface some fifty years later. Instead the next (widely recognized) step forward would be Augustin-Louis Cauchy's *Cours d'Analyse* of 1821. The text implicitly involved ε-δ arguments, and defined continuous functions, but still Cauchy referred to infinitely small quantities and his proof of the Fundamental Theorem of Calculus is incomplete through a lack of appreciation of uniform continuity.

Finally we come to the work of Karl Weierstrass, lecturing in Berlin around 1861. Weierstrass explicitly employs ε-δ arguments and defines convergence without any reference to infinitesimals. He appreciated the difference between pointwise and uniform convergence of functions, and between continuity and uniform continuity. He proved that a continuous function is bounded on a closed bounded interval and achieves its bounds – which is often referred to as

Weierstrass' Theorem (HT) – and he gave an example of a function which is continuous but nowhere differentiable.

But still there would be much work needed, even within real analysis, especially in the development of theories of integration (Riemann 1854, Darboux 1875, Lebesgue 1902).

0.2 Notation

IMPORTANT SETS

$\mathbb{N}$ – the set of *natural* numbers $\{0, 1, 2, \ldots\}$.
$\mathbb{Z}$ – the set of *integers* $\{0, \pm 1, \pm 2, \ldots\}$.
$\mathbb{Q}$ – the set of *rational* numbers.
$\mathbb{R}$ – the set of *real* numbers.
$\mathbb{C}$ – the set of *complex* numbers.
$\mathbb{Z}_n$ – the *integers, modulo* $n \geqslant 2$.
$\mathbb{R}^n$ – *n-dimensional real space* – the set of all real n-tuples $(x_1, x_2, \ldots, x_n)$.
$\mathbb{R}[x]$ – the set of *polynomials* in x with real coefficients.
$\varnothing$ – the empty set

For $a, b \in \mathbb{R}$ with $a < b$ we define
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.
$(a, b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$.
$[a, b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$.
$[a, b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$.

SET THEORETIC NOTATION

$X \cup Y$ – the *union* of X and Y – $\{s \mid s \in X \text{ or } s \in Y\}$.
$X \cap Y$ – the *intersection* of X and Y – $\{s \mid s \in X \text{ and } s \in Y\}$.
$X \times Y$ – the *Cartesian product* of X and Y – $\{(x, y) \mid x \in X \text{ and } y \in Y\}$.
Y^c – the *complement* of a subset Y.
$X \backslash Y$ – the *complement* of Y in X – $\{s \mid s \in X \text{ and } s \notin Y\}$.
$\mathcal{P}(X)$ – the *power set* of a set X, that is the set of subsets of X.
$\in$ – is an *element* of, e.g. $\sqrt{2} \in \mathbb{R}$ and $\pi \notin \mathbb{Q}$.
$\subset, \subseteq$ – is a *subset* of, e.g. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
$f \colon X \to Y$ – f is a *function*, map, mapping from a set X (the domain) to a set Y (the codomain).
$f \colon X \hookrightarrow Y$ – f is a *injective function* from a set X to a set Y.
$f \colon X \twoheadrightarrow Y$ – f is a *surjective function* from a set X to a set Y.
$f(X)$ – the *image* or *range* of the function f – i.e. the set $\{f(x) \mid x \in X\}$.
$g \circ f$ – the *composition* of the maps g and f – apply f first and then g.
$|X|$ – the *cardinality* (size) of the set X.

LOGICAL NOTATION

$\forall$	for all	$\Longrightarrow$	implies, is sufficient for, only if
$\exists$	there exists	$\Longleftarrow$	is implied by, is necessary for, if
$\exists!$	there exists unique	$\Longleftrightarrow$	if and only if, is logically equivalent to
$\neg$	negation, not	: or | or s.t.	such that
$\vee$	or	$\square$ or QED	found at the end of a proof
$\wedge$	and		

ANALYTICAL NOTATION

$\operatorname{Re} z$ – the real part of a complex number.
$\operatorname{Im} z$ – the imaginary part of a complex number.
(x_n), $(x_n)_1^\infty$ – a sequence of elements from a set, usually real or complex numbers
x_k – the kth term of a sequence (x_n)
$x_n \to L$ – the sequence (x_n) converges to L.
$\lim x_n$ – the limit of the convergent sequence (x_n).
$(\exists\, N)\ (\forall n \geqslant N)$ – eventually, or for a tail of the natural numbers
$(\forall\, N)\ (\exists n \geqslant N)$ – for infinitely many, or for arbitrarily large, natural numbers
AC – absolutely convergent
$\sum x_n$ – the infinite series $x_1 + x_2 + x_3 + \cdots$, whether convergent or not.
$\sum_1^\infty x_n$ – when denoting a (real or complex) number, the infinite sum of the series.
$O(n)$ – large O notation.
$o(n)$ – small o notation.

The Greek Alphabet

A, α	alpha	H, η	eta	N, ν	nu	T, τ	tau
B, β	beta	Θ, θ	theta	Ξ, ξ	xi	Y, υ	upsilon
Γ, γ	gamma	I, ι	iota	O, o	omicron	Φ, ϕ, φ	phi
Δ, δ	delta	K, κ	kappa	Π, π	pi	X, χ	chi
E, ϵ	epsilon	Λ, λ	lambda	P, ρ	rho	Ψ, ψ	psi
Z, ζ	zeta	M, μ	mu	$\Sigma, \sigma, \varsigma$	sigma	$, \omega$	omega

0.3 Acknowledgements

The lecture notes are relatively original, to the extent notes on such well-trodden mathematics can be original. The exercise sheets have many sources. In particular, I would like to thank Paul Balister who has contributed a number of exercises and advised on others.

1. LIMITS AND CONTINUITY

1.1 Limits on Intervals

Definition 1.1 *By an* ***interval*** $I \subseteq \mathbb{R}$, *we mean a subset such that if* $x, y \in I$ *and* $x \leqslant z \leqslant y$ *then* $z \in I$. *It can be readily shown that the subsets with the 'interval property' are of the form*

$$\varnothing, \quad (a,b), \quad [a,b), \quad (a,b], \quad [a,b], \quad [a,\infty), \quad (a,\infty), \quad (-\infty,b), \quad (-\infty,b], \quad \mathbb{R}.$$

(See Sheet 1, Exercise 1.) In what follows, when we denote a subset I *we are assuming it to be an interval. We will write* $\overline{I}$ *for the interval* I *together with any missing endpoints. So*

$$\overline{(a,b)} = \overline{(a,b]} = [a,b], \qquad \overline{(-\infty,b)} = \overline{(-\infty,b]} = (-\infty,b].$$

Note $-\infty$ *is in no way a 'missing endpoint' as it is not a real number.*

In *Analysis I* we met the notion of a limit L of a sequence (a_n) as $n \to \infty$. The sequence was not defined for ∞, rather our definition required that the sequence became arbitrarily close to L in some tail. Recall that a (basic) neighbourhood of L has the form $(L-\varepsilon, L+\varepsilon)$ and a (basic) neighbourhood of ∞ has the form (X,∞) and does not include ∞. So, in terms of neighbourhoods, $a_n \to L$ means: given any neighbourhood of L, the sequence is in that neighbourhood for a neighbourhood of ∞. We look to make similar definitions now for real and complex functions, initially on intervals and later on general domains.

Definition 1.2 *Given a function* $f\colon I \to \mathbb{R}$ *and* $p \in \overline{I}$, *we say that* f *has* ***limit*** L *as* $x \to p$ *if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I \quad \textit{such that} \quad 0 < |x-p| < \delta \quad \textit{then} \quad |f(x) - L| < \varepsilon.$$

This is written as

$$f(x) \to L \quad \textit{as} \quad x \to p \qquad \textit{or} \qquad \lim_{x \to p} f(x) = L.$$

We are requiring: given any neighbourhood of L, that $f(x)$ is in that neighbourhood for x in some neighbourhood of p, not including p. Note that this does not imply that $L = f(p)$, nor even that $p \in I$, so $f(p)$ may not even be defined.

For $f\colon I \to \mathbb{R}$ to have no limit at $p \in \overline{I}$ means

$$\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \in I \quad \text{s.t.} \quad 0 < |x-p| < \delta \quad \text{and} \quad |f(x) - L| \geqslant \varepsilon.$$

Proposition 1.3 ***(Uniqueness of limits)*** *Let* $f\colon I \to \mathbb{R}$ *and* $p \in \overline{I}$. *Say* $f(x) \to L$ *as* $x \to p$ *and* $f(x) \to M$ *as* $x \to p$. *Then* $L = M$.

Proof. SFAC that $L \neq M$ and set $\varepsilon = \frac{1}{2}|L - M|$. There exists $\delta_L, \delta_M > 0$ such that

$$\begin{aligned} 0 &< |x-p| < \delta_l &\implies& |f(x) - L| < \varepsilon, \\ 0 &< |x-p| < \delta_M &\implies& |f(x) - M| < \varepsilon. \end{aligned}$$

Set $\delta = \min\{\delta_L, \delta_M\} > 0$. For $0 < |x-p| < \delta$ we have by the triangle inequality

$$2\varepsilon = |L - M| \leqslant |L - f(x)| + |f(x) - M| < 2\varepsilon,$$

a contradiction. ■

Example 1.4 *(a) The function* $f_1 \colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$f_1(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

has a limit $L = 1$ *at* $p = 0$. *Note that this does not equal* $f_1(0)$. *See Fig. 1.1.*
(b) The function $f_2 \colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$f_2(x) = \begin{cases} 2 & x > 0 \\ 1 & x \leqslant 0 \end{cases}$$

has no limit at 0. *It is the case though that* f_2 *has a* ***right-hand limit*** *of* 2 *and a* ***left-hand limit*** *of* 1 *at* $x = 0$. *The right-hand limit is denoted as*

$$f_2(x) \to 2 \quad \text{as } x \to 0^+ \quad \text{or} \quad f_2(0^+) = 2 \quad \text{or} \quad \lim_{x \searrow 0} f_2(x) = 2.$$

See Fig. 1.2.

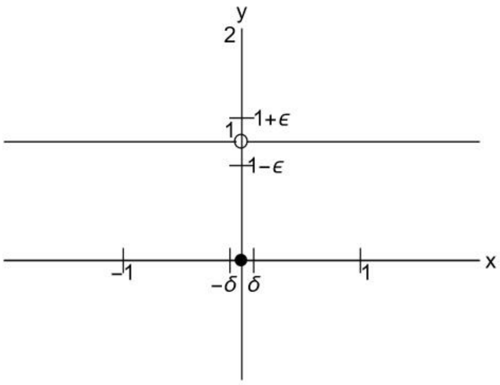

Fig. 1.1.

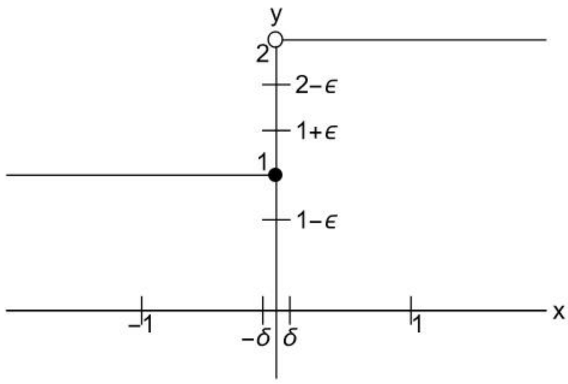

Fig 1.2

(c) The function $f_3(x) = \frac{\sin x}{x}$ *defined on* $(0, \infty)$ *has a limit* $L = 1$ *at* $p = 0$. *See Fig. 1.3.*

(d) The function $f_4(x) = \sin \frac{1}{x}$ *defined on* $(0, \infty)$ *has no limit at* 0. *See Fig. 1.4.*

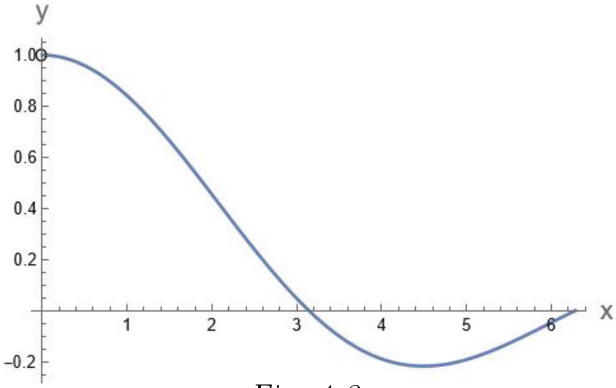

Fig. 1.3.

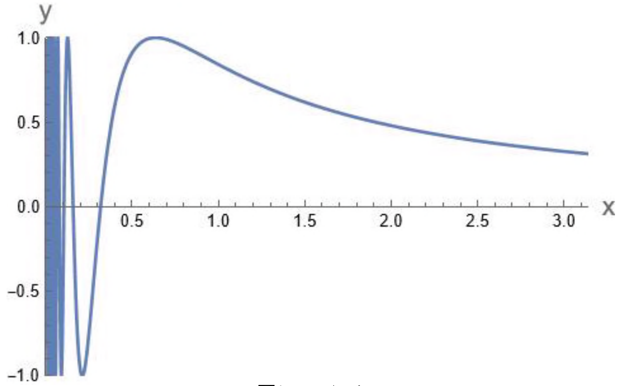

Fig. 1.4.

Solution. (a) Given any $\varepsilon > 0$, we may take any $\delta > 0$ and note $0 < |x| < \delta$ implies

$$|f_1(x) - 1| = |1 - 1| = 0 < \varepsilon$$

and hence $L = 1$. Clearly $L \neq f_1(0) = 0$.

(b) SFAC that L is a limit of f_2 at 0. Take $\varepsilon = \frac{1}{3}$ and $\delta > 0$. Then $\pm\delta/2$ are two points satisfying $0 < |x| < \delta$, so we would need both

$$\left| f_2\left(-\frac{\delta}{2}\right) - L \right| < \frac{1}{3} \qquad \text{and} \qquad \left| f_2\left(\frac{\delta}{2}\right) - L \right| < \frac{1}{3}.$$

As $f(-\delta/2) = 1$ and $f(\delta/2) = 2$ then the first inequality implies $L \in (2/3, 4/3)$ and the second inequality implies $L \in (5/3, 7/3)$. As these intervals are disjoint, no limit L exists.

However it is the case that $f_2(x) \to 2 \quad \text{as } x \to 0^+$. Formally this means

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in (0, \delta) \quad |f_2(x) - 2| < \varepsilon.$$

We can take $\delta = \varepsilon$ for example. And $f_2(x) \to 1 \quad \text{as } x \to 0^-$ as

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in (-\delta, 0) \quad |f_2(x) - 1| < \varepsilon.$$

(c)

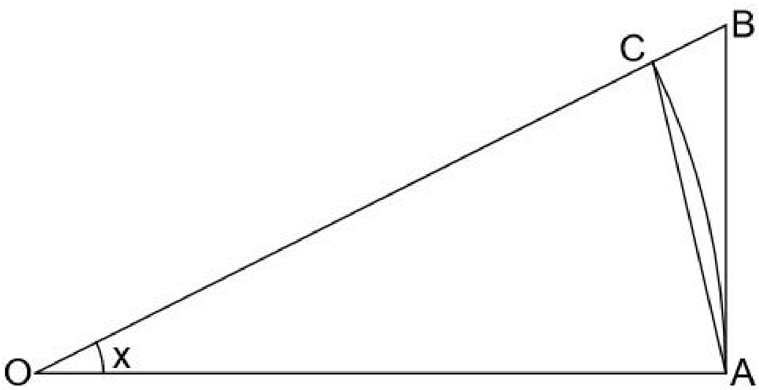

Fig. 1.5.

In the Fig. 1.5. OA has unit length. Considering the areas of the triangle OAC, the sector OAC and the triangle OAB, we see

$$\frac{1}{2}1^2 \sin x < \frac{x}{2} < \frac{1}{2} \times 1 \times \tan x.$$

Rearranging this we find

$$\cos x < \frac{\sin x}{x} < 1.$$

Further

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!}\left(1 - \frac{x^2}{5 \times 6}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{9 \times 10}\right) + \cdots > 1 - \frac{x^2}{2}$$

for small positive x. Hence by sandwiching (which we will formally cover in the next section) we find $f_3(x) \to 1$ as $x \to 0$.

(d) SFAC that L is a limit of f_4 at 0. Take $\varepsilon = 1$. There would then be $\delta > 0$ such that

$$|f_4(x) - L| < 1 \quad \text{for all } x \in (0, \delta).$$

However, by the Archimedean property, there is $N \in \mathbb{N}$ such that $x_N = 2/[(4N+1)\pi] < \delta$. We would then have

$$f_4(x_N) = \sin \frac{1}{x_N} = \sin\left((4N+1)\frac{\pi}{2}\right) = 1,$$

meaning that $L \in (0, 2)$. But we also have $2\pi/(4N+3) < \delta$ and

$$f_4\left(\frac{2}{(4N+3)\pi}\right) = -1,$$

meaning that $L \in (-2, 0)$. As these intervals are disjoint, no such limit L exists. ■

In a moment we will meet a familiar set of 'algebra of limits'. We won't prove them afresh as we can conclude these new AOL from last term's sequential AOL via the next result.

Proposition 1.5 *Let $f\colon I \to \mathbb{R}$, (x_n) be a sequence in I and $p \in \overline{I}$.*

(a) If $f(x) \to L$ as $x \to p$ and $x_n \to p$ as $n \to \infty$ with $x_n \neq p$, then $f(x_n) \to L$ as $n \to \infty$.

(b) If, whenever $x_n \to p$ with $x_n \neq p$ it's the case that $f(x_n) \to L$, then $f(x) \to L$ as $x \to p$.

Proof. (a) Let $\varepsilon > 0$. As $f(x) \to L$ as $x \to p$, then there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in I$ and $0 < |x - p| < \delta$. As $x_n \to p$ and $x_n \neq p$, there exists N such that $0 < |x_n - p| < \delta$ when $n \geqslant N$. Hence $|f(x_n) - L| < \varepsilon$ for $n \geqslant N$, showing $f(x_n) \to L$.

(b) SFAC that $f(x) \nrightarrow L$. As a quantified statement '$f(x) \to L$ as $x \to p$' means

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I \quad 0 < |x - p| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Negating this gives

$$\exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x \in I \ \text{ s.t. } \ 0 < |x - p| < \delta \quad \text{and} \quad |f(x) - L| \geqslant \varepsilon.$$

For this ε, we take $\delta = 1/n$. So there exists x_n such that

$$0 < |x_n - p| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - L| \geqslant \varepsilon.$$

But then $x_n \to p$ yet $f(x_n) \nrightarrow L$ which is the required contradiction. ■

1.2 Algebra of Limits

Proposition 1.6 *(**Algebra of Limits (AOL)**) Let $f, g \colon I \to \mathbb{R}$ and $p \in \overline{I}$. Say $f(x) \to L$ and $g(x) \to M$ as $x \to p$. Then*

(a) $f(x) + g(x) \to L + M$ as $x \to p$.

(b) $f(x)\,g(x) \to LM$ as $x \to p$.

(c) if $M \neq 0$, then $1/g(x) \to 1/M$ as $x \to p$.

Proof. We will only prove (c), first using sequential AOL, and then proving it directly from the recent definitions.

First proof: Take a sequence such that $x_n \to p$ with $x_n \neq p$. As $M \neq 0$ it follows that $g(x_n) \neq 0$ in at least a tail of the sequence. By the corresponding sequential AOL, we have that

$$\frac{1}{g(x_n)} \to \frac{1}{M} \quad \text{as} \quad n \to \infty.$$

Finally, by Proposition 1.5 (b), we have that

$$\frac{1}{g(x)} \to \frac{1}{M} \quad \text{as} \quad x \to p.$$

Second proof: Let $\varepsilon > 0$. As $g(x) \to M$ as $x \to p$, there is $\delta_1 > 0$ such that

$$\forall x \in I \quad 0 < |x - p| < \delta_1 \quad \Longrightarrow \quad |g(x) - M| < \frac{|M|}{2}.$$

By the triangle inequality

$$\forall x \in I \quad 0 < |x - p| < \delta_1 \quad \Longrightarrow \quad |g(x)| > \frac{|M|}{2}.$$

For such x we have

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|g(x) - M|}{|g(x)|\,|M|} < \frac{2\,|g(x) - M|}{|M|^2}$$

and we would like RHS $< \varepsilon$. As $g(x) \to M$ as $x \to p$, there is $\delta_2 > 0$ such that

$$\forall x \in I \quad 0 < |x - p| < \delta_2 \quad \Longrightarrow \quad |g(x) - M| < \frac{|M|^2 \varepsilon}{2}.$$

By taking $\delta = \min\{\delta_1, \delta_2\} > 0$, we obtain

$$\forall x \in I \quad 0 < |x - p| < \delta \quad \Longrightarrow \quad \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \frac{2\,|g(x) - M|}{|M|^2} < \frac{2}{|M|^2} \times \frac{|M|^2 \varepsilon}{2} = \varepsilon,$$

demonstrating the required result. ■

Proposition 1.7 *(**Sandwich rule**) Say that $f, g, h \colon I \to \mathbb{R}$ and $p \in \overline{I}$ are such that*

$$f(x) \leqslant g(x) \leqslant h(x) \qquad \text{for all } x \in I.$$

If $f(x) \to L$ and $h(x) \to L$ as $x \to p$ then $g(x) \to L$ as $x \to p$.

Proof. Let (x_n) be a sequence in I such that $x_n \to p$ and $x_n \neq p$ for all n. Then

$$f(x_n) \leqslant g(x_n) \leqslant h(x_n)$$

and

$$\lim f(x_n) = L = \lim h(x_n)$$

by Proposition 1.5(a). By the sequential sandwich rule, $g(x_n) \to L$ as $n \to \infty$. By Proposition 1.5(b), $g(x) \to L$ as $x \to p$. ■

Proposition 1.8 *(**Limits respect weak inequalities**) Say that $f, g \colon I \to \mathbb{R}$ and $p \in \overline{I}$ are such that*

$$f(x) \leqslant g(x) \qquad \textit{for all } x \in I.$$

If $f(x) \to L$ and $g(x) \to M$ as $x \to p$, then $L \leqslant M$.

Proof. Let (x_n) be a sequence in I such that $x_n \to p$ and $x_n \neq p$ for all n. Then $f(x_n) \leqslant g(x_n)$, so by Proposition 1.5(a) and the sequential weak inequality result,

$$L = \lim f(x_n) \leqslant \lim g(x_n) = M.$$

■

Proposition 1.9 *Let $f, g \colon I \to \mathbb{R}$ be continuous. Then $|f|$, $\max\{f, g\}$ and $\min\{f, g\}$ are continuous.*

Proof. See Sheet 1, Exercise 2. ■

1.3 Continuity

In many important cases, the limit of a function at a point p will equal $f(p)$. This is precisely what it is for f to be *continuous* at p.

Definition 1.10 *Let $f \colon I \to \mathbb{R}$ and $p \in I$.*

*(a) We say that f is **continuous** at p if $f(x) \to f(p)$ as $x \to p$. Alternatively, this is equivalent to $f(p+h) \to f(p)$ as $h \to 0$. Quantified, this means*

$$\forall \varepsilon > 0 \quad \exists \dot{\delta} > 0 \quad \forall x \in I \quad s.t. \ |x - p| < \delta \quad \Longrightarrow \quad |f(x) - f(p)| < \varepsilon.$$

*(b) We say that f is **continuous on** I if f is continuous at each $p \in I$.*

*(c) We say that f is **discontinuous** at p if it is not continuous at p. Quantified this means*

$$\exists \varepsilon > 0 \quad \forall \dot{\delta} > 0 \quad \exists x \in I \quad s.t. \ |x - p| < \delta \quad \textit{and} \quad |f(x) - f(p)| \geqslant \varepsilon.$$

As a consequence of Proposition 1.5 we also have:

Corollary 1.11 *Let $f\colon I \to \mathbb{R}$. Then f is continuous at $p \in I$ if and only if whenever (p_n) is a sequence in I such that $p_n \to p$, then $f(p_n) \to f(p)$.*

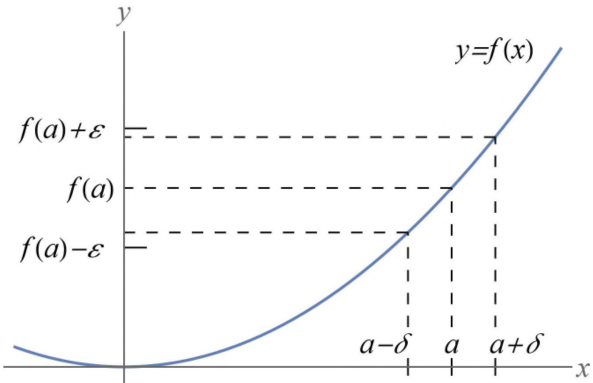

Fig. 1.6. f is continuous at a

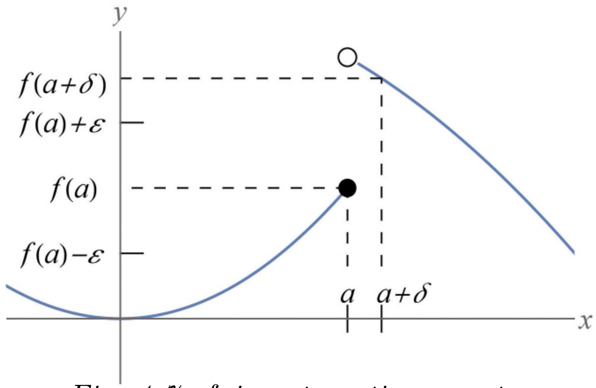

Fig. 1.7. f is not continuous at a

Example 1.12 *In Fig. 1.6 the function f is continuous at a. A particular choice of ε is shown with a δ such that f maps $(a-\delta, a+\delta)$ into $(f(a)-\varepsilon, f(a)+\varepsilon)$. To show continuity at a, we would need to show that this can be done for every $\varepsilon > 0$. In Fig. 1.7 the function f is discontinuous at a. It is the case that $f(x) \to f(a)$ as $x \to a_-$ so that $f(a^-) = f(a)$. However, if we take ε to be less than the jump discontinuity $f(a^+) - f(a)$, we see that there are values of x in any $(a, a+\delta)$ such that $f(x) > f(a) + \varepsilon$.*

Example 1.13 *From first principles, show that*
(a) $f(x) = 3x+2$ is continuous on $\mathbb{R}$.
(b) $f(x) = x^2$ is continuous on $\mathbb{R}$.
(c) $f(x) = 1/x$ is continuous on $(0,\infty)$.
(d)

$$f(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leqslant 0, \end{cases}$$

is discontinuous at 0, but otherwise continuous.

Solution. (a) Take $p \in \mathbb{R}$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$|x-p| < \delta \quad \Longrightarrow \quad |((3x+2)-(3p+2))| = 3|x-p| < \varepsilon.$$

Thus we may choose $\delta = \varepsilon/3$ to see f is continuous at p.

(b) Take $p \in \mathbb{R}$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$|x-p| < \delta \quad \Longrightarrow \quad \left|x^2 - p^2\right| < \varepsilon. \tag{1.1}$$

Note that in the interval $(p-\delta, p+\delta)$ the greatest difference in $|f(x) - f(p)|$ is achieved towards an end-point of the interval. If $\delta < 1$, then

$$\begin{aligned}(p+\delta)^2 - p^2 &= 2\delta p + \delta^2 < 2p\delta + \delta = (2p+1)\,\delta; \\ (p-\delta)^2 - p^2 &= -2\delta p + \delta^2 < -2p\delta + \delta = (1-2p)\,\delta.\end{aligned}$$

Thus if

$$\delta = \min\left\{\frac{\varepsilon}{1+2\,|p|}, 1\right\},$$

then (1.1) follows.

(c) Take $p > 0$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$|x-p| < \delta \quad \Longrightarrow \quad \left|\frac{1}{x} - \frac{1}{p}\right| < \varepsilon. \tag{1.2}$$

Note that in the interval $(p-\delta, p+\delta)$ the greatest difference in $|f(x) - f(p)|$ is achieved towards the end-point $p - \delta$. If $\delta < \frac{p}{2}$ then the greatest difference is

$$\left|\frac{1}{x} - \frac{1}{p}\right| \leqslant \frac{1}{p-\delta} - \frac{1}{p} = \frac{\delta}{(p-\delta)\,p} < \frac{\delta}{(p-p/2)\,p} = \frac{2\delta}{p^2}.$$

So taking

$$\delta = \min\left\{\frac{p^2\varepsilon}{2}, \frac{p}{2}\right\}$$

then (1.2) follows.

(d) f is locally constant at every $p \neq 0$, so continuity is an easy check at such p. At $p = 0$, take $\varepsilon = 1/2$. Given any $\delta > 0$, we may take $x = \delta/2$ and note

$$|f(x) - f(p)| = |1 - 0| = 1 \geqslant \varepsilon.$$

Hence f is discontinuous at 0. ■

Remark 1.14 *Typically δ depends on ε and p. The smaller ε is, the smaller δ usually needs to be; the more f is 'varying', the smaller δ needs to be. So (b) is more typical; we see for the same given ε that δ will need to decrease as $|p|$ increases. This is because x^2 varies more quickly at larger positive and negative values. This is not the case with (a). Here, given ε, the same $\delta = \varepsilon/3$ works independently of p. This is because $x \mapsto 3x+2$ is 'uniformly continuous', a stronger form of continuity we will meet later in the course.*

Example 1.15 *Show that $\sqrt{x}$ is continuous on $[0,\infty)$.*

Solution. In Analysis I, Sheet 4, Exercise 3, we showed that if $a_n \to L \geqslant 0$ then $\sqrt{a_n} \to \sqrt{L}$. This would be sufficient to show $\sqrt{x}$ is continuous at L.

Alternatively, from the definition of continuity, we can note for $0 < h < p$ that

$$\left|\sqrt{p+h} - \sqrt{p}\right| = \frac{h}{\left|\sqrt{p+h} + \sqrt{p}\right|} < \frac{h}{2\sqrt{p}} \to 0 \text{ as } h \to 0.$$

Finally

$$\left|\sqrt{0+h} - \sqrt{0}\right| = \sqrt{h} \to 0 \text{ as } h \to 0.$$

■

Example 1.16 *Show that* $\cos x$ *is continuous on* $\mathbb{R}$*.*

Solution. Let $p \in \mathbb{R}$. From standard trigonometric identities we have

$$|\cos(p+h) - \cos p| = \left|2\sin\left(p + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right| \leqslant 2\left|\sin\left(\frac{h}{2}\right)\right| \leqslant 2\frac{|h|}{2} = |h|,$$

which tends to 0 as $h \to 0$. ■

Example 1.17 *Let* $f, g\colon \mathbb{R} \to \mathbb{R}$ *be continuous functions and define* $h\colon \mathbb{R} \to \mathbb{R}$ *by*

$$h(x) = \begin{cases} f(x) & x \in \mathbb{Q}, \\ g(x) & x \notin \mathbb{Q}. \end{cases}$$

(a) Show that h *is continuous at* α *if and only if* $f(\alpha) = g(\alpha)$*.*
(b) Hence show the ***Dirichlet function*** $D\colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$D(x) = \begin{cases} f(x) & x \in \mathbb{Q}, \\ g(x) & x \notin \mathbb{Q}, \end{cases}$$

is discontinuous everywhere.

Solution. (a) Suppose that h is continuous at α, so that the limit $\lim_{x\to\alpha} h(x)$ exists and equals $h(\alpha)$. By the continuity of f and g, we also have

$$\begin{aligned} h(\alpha) &= \lim_{x\to\alpha} h(x) = \lim_{x\to\alpha, x\in\mathbb{Q}} h(x) = \lim_{x\to\alpha, x\in\mathbb{Q}} f(x) = f(\alpha); \\ h(\alpha) &= \lim_{x\to\alpha} h(x) = \lim_{x\to\alpha, x\notin\mathbb{Q}} h(x) = \lim_{x\to\alpha, x\notin\mathbb{Q}} g(x) = g(\alpha). \end{aligned}$$

Hence $f(\alpha) = g(a)$.

Conversely suppose $f(\alpha) = g(\alpha)$ and let $\varepsilon > 0$. As f and g are continuous, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |x-\alpha| &< \delta_1 & \Longrightarrow && |f(x) - f(\alpha)| < \varepsilon; \\ |x-\alpha| &< \delta_2 & \Longrightarrow && |g(x) - g(\alpha)| < \varepsilon. \end{aligned}$$

So, if $|x - \alpha| < \min\{\delta_1, \delta_2\}$, whether x is rational or irrational, we have $|h(x) - h(\alpha)| < \varepsilon$. Hence h is continuous at α.

(b) For the Dirichlet function $f(x) = 1$ and $g(x) = 0$. As these functions are never equal then D is nowhere continuous. ■

Some of the functions in Example 1.13 could have been dealt with as we now have an algebra of continuous functions because of the previous AOL. We now know:

Corollary 1.18 *(**Algebra of continuous functions**) Say that* $f, g\colon I \to \mathbb{R}$ *are continuous at* $p \in I$*. Then the functions* $f + g$ *and* fg *are continuous at* p *and, provided* $g(p) \neq 0$*, then* f/g *is continuous at* p*.*

Proof. These results follow from the AOL and the definition of continuity. For example, we know

$$\lim_{x \to p} f(x) = f(p) \quad \text{and} \quad \lim_{x \to p} g(x) = g(p),$$

so by AOL we have

$$\lim_{x \to p} f(x) g(x) = f(p) g(p),$$

which is to say that fg is continuous at p. ■

Remark 1.19 *Using these algebraic properties it's immediately clear that $3x + 2$ on $\mathbb{R}$, x^2 on $\mathbb{R}$ and $1/x$ on $(0, \infty)$ are continuous because the identity function and constant functions are readily seen to be continuous.*

Corollary 1.20 *Any polynomial on $\mathbb{R}$ is continuous.*

Proposition 1.21 *Let $f \colon I_1 \to \mathbb{R}$ and $g \colon I_2 \to \mathbb{R}$ be continuous functions such that $f(I_1) \subseteq I_2$. Then $g \circ f$ is continuous.*

Proof. Let $\varepsilon > 0$ and $p \in I_1$. As g is continuous at $f(p) \in I_2$, there exists $\Delta > 0$ such that

$$\forall y \in I_2 \ \text{ s.t. } \ |y - f(p)| < \Delta \quad \Longrightarrow \quad |g(y) - g(f(p))| < \varepsilon.$$

As f is continuous at $p \in I_1$ there exists $\delta > 0$ such that

$$\forall x \in I_1 \ \text{ s.t. } \ |x - p| < \delta \quad \Longrightarrow \quad |f(x) - f(p)| < \Delta.$$

Combining these we find

$$\forall x \in I_1 \ \text{ s.t. } \ |x - p| < \delta \quad \Longrightarrow \quad |g(f(x)) - g(f(p))| < \varepsilon.$$

Hence $g \circ f$ is continuous at p and the result follows. ■

Proposition 1.22 *Let $f \colon \mathbb{R} \to \mathbb{R}$ be continuous and $f(a) > 0$ (resp. $f(a) < 0$). Then there exists $\delta > 0$ such that $f(x) > 0$ (resp. $f(x) < 0$) on $(a - \delta, a + \delta)$.*

Proof. Let $\varepsilon = f(a)/2 > 0$. By continuity there exists $\delta > 0$ such that on $(a - \delta, a + \delta)$

$$|f(x) - f(a)| < \frac{f(a)}{2} \quad \Longrightarrow \quad f(x) > \frac{f(a)}{2} > 0.$$

■

Remark 1.23 *A useful consequence of this previous result is in integration. It helps demonstrate the following: if $f \colon [a, b] \to \mathbb{R}$ is continuous, non-negative and*

$$\int_a^b f(x) \, \mathrm{d}x = 0,$$

then f is identically zero. Should $f(x) > 0$ at any point then the above shows a positive integral is accrued on $(a - \delta, a + \delta)$. The result is clearly untrue of discontinuous functions as a single $f(x)$ might be positive.

1.4 Infinite Limits

Definition 1.24 *(**Infinite limits**) Let $f\colon I \to \mathbb{R}$ and $p \in \overline{I}$. We write $f(x) \to \infty$ as $x \to p$ if*

$$\forall M \in \mathbb{R} \quad \exists \delta > 0 \quad \forall x \in I \quad 0 < |x - p| < \delta \quad \Longrightarrow \quad f(x) > M.$$

We write $f(x) \to -\infty$ as $x \to p$ if

$$\forall M \in \mathbb{R} \quad \exists \delta > 0 \quad \forall x \in I \quad 0 < |x - p| < \delta \quad \Longrightarrow \quad f(x) < M.$$

Definition 1.25 *(**Limits at Infinity**) Let I be an interval, which is not bounded above. We write $f(x) \to L$ as $x \to \infty$ if*

$$\forall \varepsilon > 0 \quad \exists X \in \mathbb{R} \quad \forall x \in I \quad x > X \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

We write $f(x) \to \infty$ as $x \to \infty$ if

$$\forall M \in \mathbb{R} \quad \exists X \in \mathbb{R} \quad \forall x \in I \quad x > X \quad \Longrightarrow \quad f(x) > M.$$

We write $f(x) \to -\infty$ as $x \to \infty$ if

$$\forall M \in \mathbb{R} \quad \exists X \in \mathbb{R} \quad \forall x \in I \quad x > X \quad \Longrightarrow \quad f(x) < M.$$

Proposition 1.26 *Let $f, g\colon I \to \mathbb{R}$ and $p \in \overline{I}$.*
(a) If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to p$, then $f(x) + g(x) \to \infty$ as $x \to p$.
(b) If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to p$, then $f(x)\,g(x) \to \infty$ as $x \to p$.
(c) If $f(x) \to \infty$ as $x \to p$, then $1/f(x) \to 0$ as $x \to p$.
(d) If $f(x) \to \infty$ and $g(x) \to L$ as $x \to p$, then $f(x) + g(x) \to \infty$.

Proof. These are left as exercises. ■

Proposition 1.27 *(**Algebra of limits at infinity**) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ and say $f(x) \to L$ and $g(x) \to M$ as $x \to \infty$. Then*

- *$f(x) + g(x) \to L + M$ as $x \to \infty$.*
- *$f(x)\,g(x) \to LM$ as $x \to \infty$.*
- *if $M \neq 0$ then $f(x)/g(x) \to L/M$ as $x \to \infty$.*

Proof. These are left as exercises. ■

Example 1.28 *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a real, non-constant polynomial with positive leading coefficient a_n. Then*
(a) $p(x) \to \infty$ as $x \to \infty$.
(b) $p(x) \to \pm\infty$ as $x \to -\infty$ depending on whether n is even/odd.

Solution. (a) For $x \neq 0$ we can write

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \right).$$

By AOL,

$$a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \to a_n \qquad \text{as } x \to \infty.$$

Take $M > 0$. There exists $X > 0$ such that, for $x > X$,

$$\left| \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \right) - a_n \right| < \frac{a_n}{2} \quad \Longrightarrow \quad \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \right) > \frac{a_n}{2}.$$

Hence, for $x > \max\{X, 2M/a_n, 1\}$ we have

$$p(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \right) > x\frac{a_n}{2} > \frac{2M}{a_n}\frac{a_n}{2} = M.$$

This shows $p(x) \to \infty$ as $x \to \infty$.

(b) can be argued similarly and is left as Sheet 3, Exercise 2. ■

Example 1.29 *Determine if the following limits exist, evaluating those that do.*

$$\lim_{x\to\infty} \frac{\sin x}{x}, \qquad \lim_{x\to\infty} \frac{x^2}{2^x}, \qquad \lim_{x\to\infty} \frac{\log(2x)}{\log x},$$
$$\lim_{x\to 0} \frac{x^2+1}{x}, \qquad \lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

Solution. (a) We have that $-1 \leqslant \sin x \leqslant 1$ for all x, so

$$-\frac{1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x}.$$

Now $\pm 1/x \to 0$ as $x \to \infty$ so, by sandwiching, $\sin x/x \to 0$ as $x \to \infty$.

(b) Recall the exponential series and the definition of powers; for $x > 0$ we have that

$$0 < \frac{x^2}{2^x} = \frac{x^2}{e^{x \log 2}} < \frac{x^2}{(x \log 2)^3/3!} = \frac{6}{(\log 2)^3}\frac{1}{x} \to 0 \quad \text{as} \quad x \to \infty.$$

By sandwiching once more, $x^2/2^x \to 0$ as $x \to \infty$.

(c) Note, for $x > 1$, that

$$1 < \frac{\log(2x)}{\log x} = \frac{\log 2 + \log x}{\log x} = 1 + \frac{\log 2}{\log x} \to 1 \quad \text{as} \quad x \to \infty.$$

By sandwiching once more, $\log(2x)/\log x \to 1$ as $x \to \infty$.

(d) Note that

$$\lim_{x\to 0^+} \frac{x^2+1}{x} = \lim_{x\to 0^+} \left(x + \frac{1}{x} \right) = \infty$$
$$\lim_{x\to 0^-} \frac{x^2+1}{x} = \lim_{x\to 0^-} \left(x + \frac{1}{x} \right) = -\infty$$

and hence this limit does not exist.

(e) We saw, in Example 1.4(c), that $\sin x/x \to 1$ as $x \to 0$. Subsiituting $u = x - 1$ we find

$$\frac{\sin(x^2-1)}{x-1} = \frac{\sin(u^2+2u)}{u} = \left(\frac{\sin u^2}{u^2}\right) u \cos 2u + 2\cos u^2 \left(\frac{\sin 2u}{2u}\right).$$

By the AOL and continuity, as $x \to 1$ and so $u \to 0$, the above tends to

$$1 \times 0 \times \cos 0 + 2 \times \cos 0 \times 1 = 2.$$

■

1.5 Arbitrary Domains

Extending the notion of continuity from functions on intervals to functions more generally defined on subsets of $\mathbb{R}$ and $\mathbb{C}$ is not difficult.

Definition 1.30 *Let $p \in X \subseteq \mathbb{R}$ and $f \colon X \to \mathbb{R}$. We say that f is **continuous** at p if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t. whenever } |x-p| < \delta \text{ and } x \in X \quad \text{then} \quad |f(x) - f(p)| < \varepsilon.$$

We say that f is continuous (on X) if it is continuous for every $p \in X$.

Note that the following is immediate from the definition:

- Let $f \colon X \to \mathbb{R}$ be continuous (on X) and $Y \subseteq X$. Then $f|_Y$ is continuous (on Y).

Remark 1.31 *The previous algebra of limits for continuous functions, etc. all apply to continuous functions on X.*

Example 1.32 *Show that the function $f \colon \mathbb{Q} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 0 & x < \sqrt{2} \\ 1 & x > \sqrt{2} \end{cases}$$

is continuous.

Solution. Take $q \in \mathbb{Q}$ and $\varepsilon > 0$. If $q < \sqrt{2}$ take $\delta > 0$ such that $q + \delta < \sqrt{2}$. Then whenever $x \in \mathbb{Q}$ and $|x - q| < \delta$, we have $|f(x) - f(q)| = |0 - 0| = 0 < \varepsilon$. So f is continuous at q. A similar argument applies to $q > \sqrt{2}$. ■

Remark 1.33 *This may seem surprising initially given the apparent jump discontinuity. But a function $f \colon S \to \mathbb{R}$ is discontinuous iff f is discontinuous at some point of S. Clearly $\sqrt{2}$ is not in $\mathbb{Q}$ and the above proof shows that f is in fact locally constant at every point of $\mathbb{Q}$.*

Example 1.34 *Every function $f \colon \mathbb{Z} \to \mathbb{R}$ is continuous.*

Solution. Let $n \in \mathbb{Z}$, $\varepsilon > 0$ and set $\delta = \frac{1}{2}$. So if $m \in \mathbb{Z}$ and $|m - n| < \delta$, then $m = n$ and so $|f(m) - f(n)| = 0 < \varepsilon$ as required. ■

Definition 1.35 *Let $X \subseteq \mathbb{C}$ and $p \in X$.*
(a) We say that $f\colon X \to \mathbb{R}$ is continuous at p if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t. whenever } |x - p| < \delta \text{ and } x \in X \quad \text{then} \quad |f(x) - f(p)| < \varepsilon.$$

(b) We say that $g\colon X \to \mathbb{C}$ is continuous at p if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t. whenever } |x - p| < \delta \text{ and } x \in X \quad \text{then} \quad |g(x) - g(p)| < \varepsilon.$$

(c) By identifying $\mathbb{C}$ with $\mathbb{R}^2$, the above definitions extends continuity to functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^2$ to functions on subsets of $\mathbb{R}^2$. Explicitly

- *$f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at $\mathbf{p}$ if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |\mathbf{x} - \mathbf{p}| < \delta \quad \Longrightarrow \quad |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon.$$

- *$\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is continuous at $\mathbf{p}$ if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |\mathbf{x} - \mathbf{p}| < \delta \quad \Longrightarrow \quad |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{p})| < \varepsilon.$$

Remark 1.36 *The definitions of continuity for functions $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbb{C} \to \mathbb{C}$ appear almost identical. The only real difference is in the nuance of whether $||$ denotes the magnitude of a vector in $\mathbb{R}^2$, the modulus of a complex number, or the absolute value of a real number. The concept of continity is crucially the same with each definition with $|a - b|$ denoting the distance between two elements a and b, however distance is measured and understood.*

*More generally, our definition of continuity extends to any domains with a notion of distance, that is **metric spaces** (see Analysis I, Sheet 3, Exercise 7 and Analysis II, Sheet 2, Exercise 12). Continuity between metric spaces is defined as follows:*

*Let d_X and d_Y be metrics on two sets X and Y respectively. We say $f\colon X \to Y$ is **continuous** if*

$$\forall p \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad d_X(x, p) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(p)) < \varepsilon.$$

Proposition 1.37 *Let $X \subseteq \mathbb{C}$, $p \in X$ and $g\colon X \to \mathbb{C}$. Then g is continuous at p iff $\operatorname{Re} g\colon X \to \mathbb{R}$ and $\operatorname{Im} g\colon X \to \mathbb{R}$ are continuous at p.*

Proof. Assume that g is continuous. Let $p \in \mathbb{C}$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|z - p| < \delta \quad \Longrightarrow \quad |g(z) - g(p)| < \varepsilon.$$

As $|\operatorname{Re} w| \leqslant |w|$ and $|\operatorname{Im} w| \leqslant |w|$ for all $w \in \mathbb{C}$, then $\operatorname{Re} g$ and $\operatorname{Im} g$ are continuous at p.

Conversely, assume that $\operatorname{Re} g$ and $\operatorname{Im} g$ are continuous at p and let $\varepsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |z - p| &< \delta_1 \quad \Longrightarrow \quad |\operatorname{Re} g(z) - \operatorname{Re} g(p)| < \varepsilon/2, \\ |z - p| &< \delta_2 \quad \Longrightarrow \quad |\operatorname{Im} g(z) - \operatorname{Im} g(p)| < \varepsilon/2. \end{aligned}$$

As $|w| \leqslant |\operatorname{Re} w| + |\operatorname{Im} w|$, then $|z - p| < \min\{\delta_1, \delta_2\}$ implies $|g(z) - g(p)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. ■

Example 1.38 *Let $X = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and define $f\colon X \to \mathbb{C}$ by $f(z) = (z-1)^{-1}$. Show that f is continuous.*

Solution. It is a simple check to show $z \mapsto z$ is continuous and so $z \mapsto z-1$ and $z \mapsto (z-1)^{-1}$ are continuous by AOL as $1 \notin X$. Alternatively, though it's less efficient, we can note

$$\begin{aligned}
\operatorname{Re}\left(\frac{1}{z-1}\right) &= \operatorname{Re}\left(\frac{1}{x-1+iy}\right) = \frac{x-1}{(x-1)^2+y^2};\\
\operatorname{Im}\left(\frac{1}{z-1}\right) &= \operatorname{Im}\left(\frac{1}{x-1+iy}\right) = \frac{-y}{(x-1)^2+y^2},
\end{aligned}$$

are continuous by AOL when $y > 0$ and then apply the previous proposition. ■

Example 1.39 *Show that the function* $\max\colon \mathbb{R}^2 \to \mathbb{R}$ *is continuous.*

Solution. Take $(a,b) \in \mathbb{R}^2$ and let $\varepsilon > 0$. Note

$$\begin{aligned}
|x-a| &\leqslant \sqrt{(x-a)^2+(y-b)^2} = |(x,y)-(a,b)|,\\
|y-b| &\leqslant \sqrt{(x-a)^2+(y-b)^2} = |(x,y)-(a,b)|,
\end{aligned}$$

and recall that

$$\max\{x,y\} = \frac{x+y+|x-y|}{2}.$$

If $|(x,y)-(a,b)| < \varepsilon/2$, so that $|x-a| < \varepsilon/2$ and $|y-b| < \varepsilon/2$, then

$$\begin{aligned}
|\max\{x,y\} - \max\{a,b\}| &= \frac{1}{2}\left|(x+y+|x-y|) - (a+b+|a-b|)\right|\\
&= \frac{|(x-a)+(y-b)+(|x-y|-|a-b|)|}{2}.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
|x-y| &\leqslant |x-a| + |a-b| + |b-y|,\\
|a-b| &\leqslant |a-x| + |x-y| + |y-b|,
\end{aligned}$$

so that

$$||x-y| - |a-b|| \leqslant |x-a| + |y-b|.$$

Hence

$$|\max\{x,y\} - \max\{a,b\}| \leqslant \frac{2|x-a|+2|y-b|}{2} < \frac{2(\varepsilon/2)+2(\varepsilon/2)}{2} = \varepsilon.$$

Hence $\max\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (a,b). ■

1.6 Limit Points

When we were discussing functions on intervals, it was natural to discuss limits at end-points, even if those end-points were not in the function's domain. For the open, unit disc $|z| < 1$ it would seem reasonable to discuss any limits achieved on the boundary $|z| = 1$. Or for a function defined on $\mathbb{Q}$, we might seek to discuss potential limits at irrational values. For a more general domain, the notion of a 'missing end-point' is generalized to a *limit point.*

Definition 1.40 *Given a subset S of $\mathbb{R}$ or $\mathbb{C}$, we say p is a **limit point** (or **accumulation point**) of S if*

$$\forall \varepsilon > 0 \quad \exists x \in S \quad s.t. \quad 0 < |x - p| < \varepsilon.$$

That is, there are points of S, other than p, which are arbitrarily close to p.

*Given a limit point p of S, we say that $f\colon S \to \mathbb{R}$ has **limit** L at p if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad whenever \quad x \in S \quad 0 < |x - p| < \delta \quad then \quad |f(x) - L| < \varepsilon$$

- The point p need not be in S, and points of S need not be limit points of S. Any point of S, which is not a limit point of S, is called an **isolated point**.
- Note that p is a limit point of S if and only if there is a sequence in $S\backslash\{p\}$ which converges to p.

Definition 1.41 *A set is called **closed** if it contains all its limit points. We write $\overline{S}$ for the union of S and its set of limit points. This notation concurs with the earlier notation of $\overline{I}$ denoting an interval together with any missing endpoints.*

- Note that S is closed if and only if $S = \overline{S}$.
- Note that $p \in \overline{S}$ if and only if

$$\forall \varepsilon > 0 \quad \exists x \in S \quad \text{s.t.} \quad |x - p| < \varepsilon.$$

- So $p \in \overline{S}$ if and only there is a sequence in S which converges to p.

Example 1.42 *(a) Note that every real number x is a limit point of $\mathbb{Q}$. That is $\overline{\mathbb{Q}} = \mathbb{R}$. Pick, for example, a rational q_n in each interval $(x, x + n^{-1})$ so that $q_n \to x$.*

(b) $\mathbb{Z}$ is closed and all the points of $\mathbb{Z}$ are isolated. Any sequence in $\mathbb{Z}$ which converges to n is eventually constant as n. But that sequence is not in $\mathbb{Z}\backslash\{n\}$.

(c) The only limit point of $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ is 0. The points of S are isolated points, whilst the points of X as listed converge to 0.

Example 1.43 *(a) For $S = \{z \in \mathbb{C}\colon\ |z| < 1\}$, we have $\overline{S} = \{z \in \mathbb{C}\colon\ |z| \leqslant 1\}$. Clearly points on the boundary $|z| = 1$ are limit points, and as limits preserve weak inequalities, no z with $|z| > 1$ is the limit of a sequence in S.*

(b) For $S = \{e^{in} \mid n \in \mathbb{Z}\}$, we have $\overline{S} = S^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$. This is a little harder to show and relies on the irrationality of π.

(c) The hyperbola $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$ is closed, and so $\overline{S} = S$. Compare with Sheet 2, Exercise 7.

Remark 1.44 ***In what subsets of $\mathbb{R}$ and $\mathbb{C}$ does the Bolzano-Weierstrass theorem hold?***

For many of the theorems in the next chapter, the results will rely on the fact that the Bolzano-Weierstrass theorem holds in the given domain: that is, a sequence in the domain has a convergent subsequence which converges in that domain. For example, the BWT holds in a closed bounded interval. This leads us to consider, for what subsets of $\mathbb{R}$ or $\mathbb{C}$ does BWT hold? The answer is: for the closed and bounded subsets.

- *If a domain is not bounded, BWT does not hold. Given an unbounded subset of $\mathbb{R}$ or $\mathbb{C}$ we can construct a sequence (z_n) such that $|z_n| \to \infty$. Then (z_n) has no convergent subsequence.*
- *If a domain is not closed, BWT does not hold. Given a not-closed subset X of $\mathbb{R}$ or $\mathbb{C}$, there is a limit point p of X which is not in X. We can construct a sequence (z_n) in X such that $|z_n - p| \to 0$. All subsequences of (z_n) converge to p and so no subsequence converges in X.*
- *In a closed and bounded domain $X \subseteq \mathbb{R}$ or $\mathbb{C}$, BWT holds. A sequence in X is bounded and so, by BWT, has a convergent subsequence with a limit p. Then $p \in X = \overline{X}$ as X is closed. Hence BWT holds in X.*

This more generally applies to subsets of $\mathbb{R}^n$ where BWT holds precisely in the closed and bounded subsets of $\mathbb{R}^n$.