2. ANALYSIS ON INTERVALS

In this chapter we focus on continuous functions on closed, bounded intervals. In many cases, the results are a consequence of the Bolzano-Weierstrass theorem holding for these intervals; in the Part A *Metric Spaces* course you will recognize these proofs afresh in the more general setting of *compact* spaces. Our first result though, the IVT, relies on the *connectedness* of intervals.

Theorem 2.1 *(**Intermediate Value Theorem**, Bolzano, 1817) Let $f\colon [a,b] \to \mathbb{R}$ be continuous and such that $f(a) < K < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = K$.*

Thoughts: The setting of this theorem is very general. We only know that the function f is continuous, nothing more specific, so we are certainly not looking to determine c constructively.

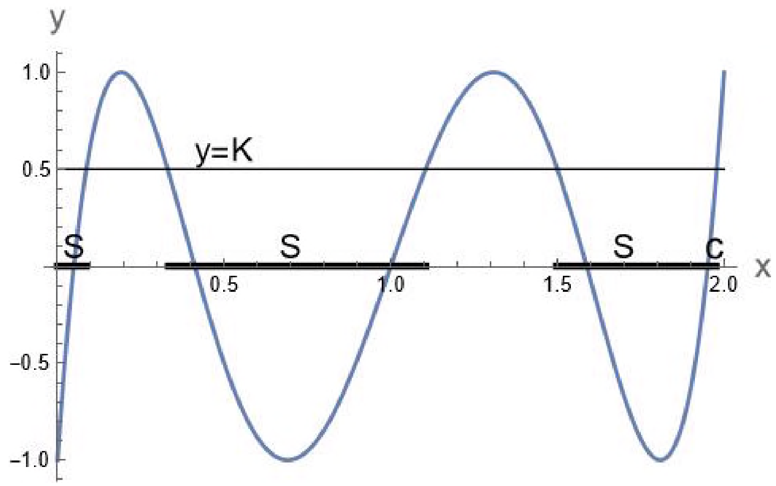

Fig. 2.1. – the IVT

As values above K are achieved, it seems reasonable to focus on the 'first' or 'last' time that this happens. As f is continuous, these notions of first and last turn out to be well-defined, but this wouldn't be the case for a general function. In the proof below, we set

$$c = \sup \{x \in [a,b] \mid f(x) \leqslant K\},$$

which is the last/largest root of $f(x) = K$. For the first/smallest root, we could have instead set

$$c = \inf \{x \in [a,b] \mid f(x) \geqslant K\}.$$

Proof. We define $S = \{x \in [a,b] \mid f(x) \leqslant K\}$. Note that $a \in S$ and that S is bounded above by b, so we may set $c = \sup S$.

By the approximation property, there exists a sequence (α_n) in S such that $\alpha_n \to c$. As $\alpha_n \in S$, then $f(\alpha_n) \leqslant K$, so we have

$$\begin{aligned} f(c) &= f(\lim \alpha_n) \\ &= \lim f(\alpha_n) \qquad \text{[by continuity]} \\ &\leqslant K \qquad \text{[as lim respects } \leqslant\text{]}. \end{aligned}$$

In particular, this means $c < b$; so there is a sequence (β_n) in $[a,b]$ with $c < \beta_n$ and $\beta_n \to c$. Note $\beta_n \notin S$ and so $f(\beta_n) > K$. Finally

$$\begin{aligned} f(c) &= f(\lim \beta_n) \\ &= \lim f(\beta_n) \qquad \text{[by continuity]} \\ &\geqslant K \qquad \text{[as lim respects } \geqslant \text{].} \end{aligned}$$

By antisymmetry, $f(c) = K$. ■

Below we give an alternative proof based on repeatedly bisecting the interval $[a,b]$.

Proof. We recursively define sequences (a_n) and (b_n) as follows: $a_0 = a$ and $b_0 = b$ with

$$\begin{aligned} &\text{if } f\left(\frac{a_n+b_n}{2}\right) > K \text{ we set } a_{n+1} = a_n, \qquad b_{n+1} = \frac{a_n+b_n}{2}; \\ &\text{if } f\left(\frac{a_n+b_n}{2}\right) \leqslant K \text{ we set } a_{n+1} = \frac{a_n+b_n}{2}, \qquad b_{n+1} = b_n. \end{aligned}$$

By construction, (a_n) is bounded and increasing so converges; likewise (b_n) is bounded and decreasing, and so converges. Further, as

$$|b_n - a_n| = \frac{|b-a|}{2^n} \to 0 \text{ as } n \to \infty,$$

then (a_n) and (b_n) have the same limit, c say.

Also by construction, $f(a_n) \leqslant K$ and $f(b_n) \geqslant K$ for all n. As f is continuous, then

$$\begin{aligned} f(c) &= f(\lim a_n) = \lim f(a_n) \leqslant K; \\ f(c) &= f(\lim b_n) = \lim f(b_n) \geqslant K. \end{aligned}$$

Thus $f(c) = K$. ■

Corollary 2.2 *Let $f\colon I \to \mathbb{R}$ be continuous and injective. Then f is strictly monotone.*

Proof. SFAC that f is not monotone. Using De Morgan's laws, this means that there are $a < b < c$ in I such that

- $f(a) = f(b)$ or $f(b) = f(c)$, both of which would contradict injectivity or
- $f(a) < f(b)$ and $f(b) > f(c)$ or
- $f(a) > f(b)$ and $f(b) < f(c)$.

WLOG suppose that $f(a) < f(b)$ and $f(b) > f(c)$. By the IVT, for any

$$\max\{f(a), f(c\} < K < f(b),$$

there exist $x_1 \in (a,b)$ and $x_2 \in (b,c)$ such that

$$f(x_1) = K = f(x_2),$$

contradicting the injectivity of f. ■

Corollary 2.3 $[0, 1]$ *cannot be written as the disjoint union of two non-empty closed sets.*

Proof. Suppose that $[0, 1] = A \cup B$ is a partition of $[0, 1]$ as two non-empty closed sets. Consider the function $\mathbf{1}_A$. Its restriction to A is continuous, being the constant function 1. Its restriction to B is continuous, being the constant function 0. Sheet 2, Exercise 10 then implies $\mathbf{1}_A$ is continuous; this contradicts the IVT, as $\mathbf{1}_A$ takes values 0 and 1 but not the value 1/2. ■

Corollary 2.4 *The continuous image of an interval is an interval.*

Proof. Given $y_1, y_2 \in f(I)$, where f is continuous and I is an interval, then there exists $x_i \in I$ such that $f(x_i) = y_i$. By the IVT, for any $y \in [y_1, y_2]$ there exists $x \in [x_1, x_2] \subseteq I$ such that $f(x) = y$. Hence $f(I)$ has the interval property. ■

Example 2.5 *There exists unique* $\alpha > 0$ *such that* $\alpha^2 = 2$.

Solution. Recall how this was a first, significant application of the completeness axiom in Analysis I, and was a non-trivial result. Now we can note that $f(x) = x^2 - 2$ is strictly increasing and continuous of $[0, \infty)$, so there is at most one root α of $f(\alpha) = 0$. As $f(0) = -2 < 0$ and $f(2) = 2 > 0$, by IVT there is some root $\alpha \in [0, 2]$. ■

Example 2.6 *Let* $f\colon (0, 1) \to (0, 1)$ *be a continuous function such that* $f(x) \neq x$ *for all* $x \in (0, 1)$. *Show that*

$$\lim_{x\to 0} f(x) = 0 \qquad \textit{or} \qquad \lim_{x\to 1} f(x) = 1.$$

Solution. By applying the IVT to the function $f(x) - x$ we must either have $f(x) > x$ for all $x \in (0, 1)$ or $f(x) < x$ for all $x \in (0, 1)$. In the case when $f(x) < x$ for all $x \in (0, 1)$, we can apply sandwiching to

$$0 < f(x) < x$$

as $x \to 0$. If $f(x) > x$ for all $x \in (0, 1)$, we similarly note $x < f(x) < 1$ and let $x \to 1$. ■

Example 2.7 *Functions with intermediate value property are called* ***Darboux functions.*** *The IVT does not have a converse, so there are discontinuous Darboux functions. An example of a discontinuous Darboux function is*

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

At the discontinuiy of a Darboux function, the left or right limit (or both) must not exist. There exist Darboux functions which are nowhere continuous.

We make explicit use of the BWT in the next theorem. This *boundedness theorem* very much relies on the *compactness* of a closed bounded interval.

Theorem 2.8 *(**Boundedness Theorem**, Weierstrass, c. 1861) Let* $f\colon [a, b] \to \mathbb{R}$ *be continuous. Then* f *is (a) bounded and (b) attains its bounds.*

Thoughts: When proven by contradiction, the logic of the definitions clearly leads to an application of the BWT.

Proof. (a) SFAC that f is not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. By the BWT, (x_n) has a convergent subsequence (x_{n_k}). Finally, by the continuity of $|f|$, we have both

$$|f(x_{n_k})| \to |f(\lim x_{n_k})| \qquad \text{and} \qquad |f(x_{n_k})| > n_k \to \infty,$$

as $k \to \infty$, a contradiction. There is an alternative proof of (a) set in Sheet 3, Exercise 10.

(b) SFAC that f does not attain its supremum K. Then $K - f(x) > 0$ on $[a, b]$ and so

$$g(x) = \frac{1}{K - f(x)}$$

is continuous. By (a), $g(x)$ is bounded above and so there exists $M > 0$ such that $g(x) < M$ for all x. This rearranges to

$$f(x) < K - \frac{1}{M},$$

but this contradicts K being the supremum. Similarly, we can show that f attains its infimum.

(b') Here is an alternative proof of (b). As f is bounded above, we can set $K = \sup f$. By the approximation property, there is a sequence (x_n) such that $f(x_n) \to K$. By BWT, (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \to c \in [a, b]$ as $k \to \infty$. By continuity,

$$f(c) = f(\lim x_{n_k}) = \lim f(x_{n_k}) = K$$

and so f achieves its maximum. ■

Example 2.9 *Let $f : [0, \infty) \to \mathbb{R}$ be continuous and such that $f(x) \to 0$ as $x \to \infty$. Show that f is bounded and attains its infimum or its supremum. Give examples to show that f need not have both a maximum and a minimum, but may have.*

Thoughts: This is typical of many analysis problems. The nature of the exercise leads us to consider the bounded theorem, but the boundedness theorem does not apply on $[0, \infty)$. However, we have an extra hypothesis, namely that $\lim f(x) = 0$ at ∞. Our challenge is to use this extra hypothesis to reduce the problem to one where the boundedness theorem can be applied. Informally, as $\lim f(x) = 0$, then f is small in some neighbourhood (X, ∞) of ∞, so that we can focus on $[0, X]$ where the boundedness theorem can be applied.

Solution. If f is identically zero, then every x is a maximum and a minimum. WLOG assume that $f(x_0) > 0$ at some $x_0 \geqslant 0$. Taking $\varepsilon = f(x_0)$ we see that $|f(x)| < \varepsilon$ on some (X, ∞). By the boundedness theorem, f achieves it maximum on $[0, X]$ which is necessarily at least $f(x_0)$. We then see this maximum is in fact a global maximum for f on $[0, \infty)$.

$f(x) = e^{-x}$ is an example of a continuous function with only a maximum and no minimum. In fact, the above proof shows f has a maximum (resp. minimum) if and only f is somewhere positive (resp. negative). The function $f(x) = e^{-x} \sin x$ therefore has a maximum and a minimum. ■

Example 2.10 *Let* $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. *Show that any continuous function* $f\colon X \to \mathbb{R}$ *is bounded and attains its bounds.*

Solution. Because X is closed and bounded, the BWT holds in X. Therefore the proof of the boundedness theorem holds for X. ■

Example 2.11 *Let* $S \subseteq \mathbb{R}$ *be such that every continuous function* $f\colon S \to \mathbb{R}$ *is bounded. Show that* S *is closed and bounded.*

Solution. This is left to Sheet 3, Exercise 11. ■

Another consequence of BWT holding in closed, bounded intervals is that continuous functions are in fact *uniformly continuous.*

Definition 2.12 *Let* $S \subseteq \mathbb{R}$. *A function* $f\colon S \to \mathbb{R}$ *is* ***uniformly continuous*** *if*

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \forall p \in S \quad |x - p| < \delta \quad \Longrightarrow \quad |f(x) - f(p)| < \varepsilon.$$

Remark 2.13 *Compare this with the definition of* $f\colon S \to \mathbb{R}$ *being continuous which is*

$$\forall p \in S \quad \forall \varepsilon > 0 \quad \exists \delta_{\varepsilon,p} > 0 \quad |x - p| < \delta \quad \Longrightarrow \quad |f(x) - f(p)| < \varepsilon.$$

Uniform continuity ensures that the δ *may depend only on* ε *which is a stronger requirement than mere continuity. Typically* δ *depends on* ε *and* p. *The smaller* ε *is, the smaller* δ *usually needs to be; the more* f *is 'varying', the smaller* δ *needs to be. Loosely put, if* f *exhibits maximal variation at some point then the* δ *that works, for that maximal level and a given* ε, *will work for the given* ε *at all points. If there is no upper bound to the variation which* f *shows then, for a given* ε, *the infimum of possible* $\delta_{\varepsilon,p}$ *as* p *varies will be zero. This is not a permissible value of* δ.

Example 2.14 *(a) The function* $f(x) = 1/x$ *on* $(0, \infty)$ *is continuous, but not uniformly so.*
(b) The function $g(x) = x^2$ *on* $(0, 1)$ *is uniformly continuous.*
(c) The function $h(x) = x^2$ *on* $(0, \infty)$ *is continuous, but not uniformly so.*

Solution. (a) Let $\varepsilon > 0$ and $p > 0$. With $\delta < p$ the greatest difference in $|f(x) - f(p)|$ on $(p - \delta, p + \delta)$ is towards the left end of the interval. We require of $\delta_{\varepsilon,p}$ that

$$\frac{1}{p - \delta} - \frac{1}{p} < \varepsilon.$$

This rearranges to

$$\delta < \frac{\varepsilon p^2}{\varepsilon p + 1}.$$

This shows continuity, but there is no δ meeting this inequality as p becomes arbitrarily small as the RHS $\to 0$ as $p \to 0$.

(b) Let $\varepsilon > 0$ and $0 < p < 1$. The greatest difference in $|g(x) - g(p)|$ on $(p - \delta, p + \delta) \subseteq (0, 1)$ is towards the right end of the interval. We require of $\delta_{\varepsilon,p}$ that

$$(p + \delta)^2 - p^2 < \varepsilon.$$

This rearranges to $2p\delta + \delta^2 < \varepsilon$. As $\delta < 1$ then we may choose

$$\delta_{\varepsilon,p} = \frac{\varepsilon}{1+2p}.$$

This shows continuity with $\delta_{\varepsilon,p}$ becoming smaller as p increases. But, as $p < 1$, we see $\delta = \varepsilon/3$ works for the given ε and all $p \in (0,1)$.

(c) The same argument applies as in (b) to show that h is continuous, but no $\delta_{\varepsilon,p}$ works for all $p > 0$. Solving the quadratic inequality in (b), we require

$$\delta < \sqrt{p^2+\varepsilon} - p = \frac{\varepsilon}{\sqrt{p^2+\varepsilon}+p}.$$

However the RHS $\to 0$ as $p \to \infty$, so there is no positive δ meeting this inequality for all $p > 0$. ■

Theorem 2.15 ***(Heine 1872)*** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

Remark 2.16 *This theorem is often referred to as Heine's theorem, after Eduard Heine, a student of Weierstrass. He was the first to publish it but claimed no priority on it. Heine himself credited Cantor with the notion of uniform continuity, and so the theorem is also commonly referred to as the Heine-Cantor theorem. The proof is essentially the same as a proof Dirichlet had given in 1854, which Weierstrass was aware of. And Bolzano had proved the same result in 1830, but his proof wasn't published until 1930.*

Proof. Recall that the quantified definition of uniform continuity is

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in [a,b] \quad |x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

The negation of this is therefore

$$\exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists x, y \in [a,b] \quad |x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| \geqslant \varepsilon.$$

SFAC that f is not uniformly continuous. For the ε above, and for each $n \geqslant 1$, we can choose $\delta = 1/n$ to find x_n, y_n such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geqslant \varepsilon.$$

By the BWT, there is a convergent subsequence (x_{n_k}) of (x_n) with limit $L \in [a,b]$. As $|x_n - y_n| < n^{-1}$, then (y_{n_k}) also converges to L. Finally, by the triangle inequality and continuity, we have

$$\varepsilon \leqslant |f(x_{n_k}) - f(y_{n_k})| \leqslant |f(x_{n_k}) - f(L)| + \left|f(L) - f(y_{n_k)}\right| \to 0 + 0 = 0 \quad \text{as} \quad k \to \infty.$$

This is the required contradiction. ■

Remark 2.17 *Uniform continuity will prove particularly useful in Analysis III next term when studying (Riemann) integration. As a continuous function f on $[a,b]$ is uniformly continuous, for any ε there is a single δ and $n > (b-a)/\delta$ such that*

$$|x-y| < \frac{b-a}{n} \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

From this we can conclude that a continuous function can be uniformly approximated with a 'step function' (Sheet 5, Exercise 8).

Uniform continuity is an important, if nuanced, aspect of integration theory. In fact, the logical gap in Cauchy's 1821 proof of the Fundamental Theorem of Calculus was because he didn't appreciate the difference between continuity and uniform continuity.

Heine's theorem gives an alternative proof of the following.

Corollary 2.18 *A continuous function $f\colon [a,b] \to \mathbb{R}$ is bounded.*

Remark 2.19 *Note that the above argument only relies on the Bolzano-Weierstrass theorem being true for the domain of f. Therefore the above result applies to any closed and bounded subset of $\mathbb{R}^n$.*

Proof. Let $\varepsilon = 1$. By uniform continuity there exists $\delta > 0$ such that

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

By the Archimedean property, there is $n \in \mathbb{N}$ such that $|b-a|/n < \delta$. For $0 \leqslant k \leqslant n$, set

$$x_k = a + \frac{k(b-a)}{n} \quad \text{and} \quad M = \max\{|f(x_k)| : 0 \leqslant k \leqslant n\}.$$

For any $x \in [a,b]$ there exists k such that $|x - x_k| < \delta$ and hence $|f(x) - f(x_k)| < 1$. Hence $|f(x)| < M + 1$ for all $x \in [a,b]$. ■

Definition 2.20 *A function $f\colon S \to \mathbb{R}$ is* ***Lipschitz continuous****, or simply* ***Lipschitz****, if there exists $K > 0$ such that*

$$|f(x) - f(y)| \leqslant K|x-y|,$$

for all $x, y \in S$. This definition is named after the German analyst Rudolf Lipschitz (1832-1903).

Proposition 2.21 *Lipschitz continuity implies uniform continuity. The converse is false.*

Proof. Let $\varepsilon > 0$ and set $\delta = \varepsilon/K$. If $|x-y| < \delta$ then

$$|f(x) - f(y)| \leqslant K(\varepsilon/K) = \varepsilon,$$

showing f is uniformly continuous. We see with the next example that the converse is false. ■

Example 2.22 *$f\colon [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz continuous.*

Solution. As f is continuous on the closed bounded interval $[0,1]$, then f is uniformly continuous. However, for $0 < x < 2x < 1$, we have

$$\left|\frac{f(2x)-f(x)}{2x-x}\right| = \frac{\left(\sqrt{2}-1\right)\sqrt{x}}{x} = \frac{\sqrt{2}-1}{\sqrt{x}}.$$

Were f Lipschitz with constant K, we would need

$$K \geqslant \frac{\sqrt{2}-1}{\sqrt{x}} \qquad \text{for all } 0 < x < \frac{1}{2}.$$

But the RHS tends to infinity as $x \to 0$, a contradiction. ■

Remark 2.23 ***(Underlying topology)*** *As will be seen in Part A, the results of this chapter are consequences of a closed bounded interval being 'compact' and 'connected'.*

A space is ***compact*** *if the Bolzano-Weierstrass theorem holds in that space – that is, a sequence in the space has a subsequence which converges in that space. In general, for a compact space:*

- *Continuous functions are bounded and attain their bounds.*
- *Continuous functions are uniformly continuous.*

A subset $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded – we proved this in Remark 1.44.

The intermediate value theorem is a consequence of intervals being ***connected****. A space is connected if every continuous integer-valued map is constant. On a connected space:*

- *The image of a continuous real function is an interval.*

The connected subsets of $\mathbb{R}$ are precisely the intervals.

Our focus now moves to considering what continuous functions are invertible, which naturally also leads to a discussion of monotone functions.

Theorem 2.24 *Let $f\colon [a,b] \to \mathbb{R}$ be an increasing function and let $c \in (a,b)$. Then the one-sided limits*

$$f\left(c^-\right) = \lim_{x \nearrow c} f(x) \qquad \textit{and} \qquad f\left(c^+\right) = \lim_{x \searrow c} f(x)$$

both exist and

$$f\left(c^-\right) \leqslant f(c) \leqslant f\left(c^+\right).$$

Further, f is continuous at c if and only if

$$f\left(c^-\right) = f(c) = f\left(c^+\right)$$

Thoughts: Figure 2.2 below suggests how to proceed. We note that $f(c^-)$ is the supremum of all values achieved prior to c. We need to verify that this supremum exists and then show it meets the definition of being the left-hand limit.

Proof.

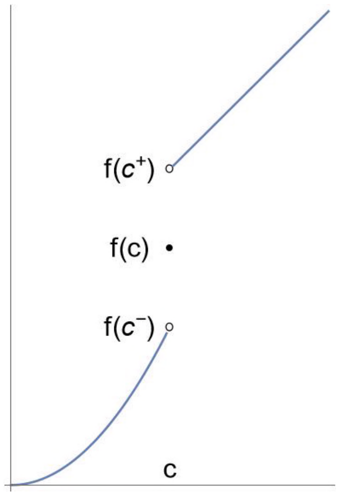

Fig. 2.2. – Monotone discontinuity

We will show that

$$f\left(c^-\right) = \sup\{f(x) \mid x < c\}.$$

Note that the set $\{f(x) \mid x < c\}$ is bounded above by $f(c)$, as f is increasing, and contains $f(a)$. Hence the supremum exists; denote it L and note that $L \leqslant f(c)$. By the approximation property, for any $\varepsilon > 0$ there exists $x_0 < c$ such that

$$L - \varepsilon < f(x_0) \leqslant L.$$

Set $\delta = c - x_0 > 0$. As f is increasing, we see that $f(x) \in (L-\varepsilon, L]$ when $x \in (c-\delta, c)$. Hence $L = f(c^-)$.

A similar proof shows that

$$f\left(c^+\right) = \inf\{f(x) \mid c < x\}$$

and that $f(c^+) \geqslant f(x)$.

Suppose now that f is continuous at c. Then $\lim_{x\to c} f(x)$ exists and equals $f(c)$. This implies $f(c^-)$ and $f(c^+)$ both exist and equal $f(c)$. Conversely, assume that

$$f\left(c^-\right) = f(c) = f\left(c^+\right)$$

and let $\varepsilon > 0$. There exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} c - \delta_1 < x < c \quad &\Longrightarrow \quad f(c) - \varepsilon < f(x) \leqslant f(c); \\ c < x < c + \delta_2 \quad &\Longrightarrow \quad f(c) \leqslant f(x) < f(c) + \varepsilon. \end{aligned}$$

Set $\delta = \min\{\delta_1, \delta_2\} > 0$ so that

$$|x - c| < \delta \quad \Longrightarrow \quad |f(x) - f(c)| < \varepsilon.$$

Hence f is continuous at c. ■

Corollary 2.25 *Let $f\colon [a,b] \to [c,d]$ be strictly increasing and bijective. Then f is continuous.*

Proof. Take $x \in [a,b]$. If $f(x^-) < f(x)$ were the case, then f would achieve no values in the interval $(f(x^-), f(x))$ and so would not be bijective. Hence $f(x^-) = f(x)$ and by a similar argument $f(x) = f(x^+)$. By the previous theorem, f is continuous. ■

Corollary 2.26 *A monotonic function $f\colon \mathbb{R} \to \mathbb{R}$ has at most countably many discontinuities.*

Proof. WLOG assume that f is increasing. By the previous theorem, at any discontinuity one of the intervals $(f(c^-), f(c))$ and $(f(c), f(c^+))$ is non-empty. To each such 'jump interval' we can assign a rational number in that interval. By monotonicity, the jump intervals are disjoint. This defines an injective map from the set of discontinuities to $\mathbb{Q}$. Recalling that $\mathbb{Q}$ is countable, this implies the set of discontinuities is countable. ■

Example 2.27 *(a) The floor function on $\mathbb{R}$ is an increasing function with countably infinite discontinuities.*

(b) The (necessarily non-monotone) Dirichlet function has uncountably many discontinuities.

Theorem 2.28 ***(CIFT, Continuous Inverse Function Theorem)***

Let $f\colon [a,b] \to [f(a), f(b)]$ be a continuous, strictly increasing function. Then f is invertible and f^{-1} is strictly increasing and continuous.

Thoughts: Another proof where a diagram helps. That f is a bijection follows quickly. That f^{-1} is continuous can be seen in the diagram. Note that the y-values form the domain for f^{-1} and the x-values form the codomain. We need to find an open interval around y that maps into $(x-\varepsilon, x+\varepsilon)$ where $x = f^{-1}(y)$. What needs to be argued is clear in the diagram.

Proof. As f is strictly increasing then f is 1-1. Further, given $K \in [f(a), f(b)]$, by the IVT there exists $c \in [a,b]$ such that $f(c) = K$. Hence f is bijective and has inverse f^{-1}. It follows immediately that f^{-1} is strictly increasing.

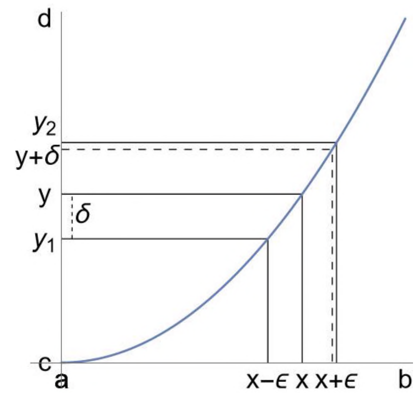

Fig. 3.3 – CIFT

Let $y \in [f(a), f(b)]$ and set $x = f^{-1}(y)$. Let $\varepsilon > 0$ and set

$$y_1 = f(x-\varepsilon), \qquad y_2 = f(x+\varepsilon).$$

As f is increasing, then f maps $(x-\varepsilon, x+\varepsilon)$ bijectively onto (y_1, y_2). Set $\delta = \min\{y - y_1, y_2 - y\}$, so that $(y-\delta, y+\delta) \subseteq (y_1, y_2)$. Hence f^{-1} maps $(y-\delta, y+\delta)$ into $(x-\varepsilon, x+\varepsilon)$ and f^{-1} is continuous at y. ■