Topics in Fluid Mechanics, CS.7

1.1 Navier Stokes Equations, recap

let $\underline{u}(\underline{x}, t)$ be the velocity at location $\underline{x}$, time t.

Then

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0 \quad , \quad \text{mass conservation}$$

(ρ: density)

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} + \underline{F} \quad , \quad \text{momentum conservation}$$

($-\nabla p$: pressure; $\mu \nabla^2 \underline{u}$: viscosity; $\underline{F}$: body force, eg. $\underline{F} = \rho \underline{g}$)

where $$\frac{D}{Dt}(\cdot) = \left[\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right](\cdot).$$

Non-dimensionalise

$$\underline{u}' = U \underline{u} \quad , \quad t' = \frac{L}{U} t \quad , \quad \underline{x}' = L \underline{x}$$

$$p' = p^* p \quad , \quad \underline{F}' = p^*/L \, \underline{F}$$

$$(Re) \frac{D\underline{u}'}{Dt'} = \left(\frac{p^* L}{\mu U}\right)\left(-\nabla' p' + \underline{F}'\right) + \nabla'^2 \underline{u}'$$

(Re) — Reynolds number $= \dfrac{\rho U^2/L}{\mu U/L^2} = \dfrac{\rho U L}{\mu}$

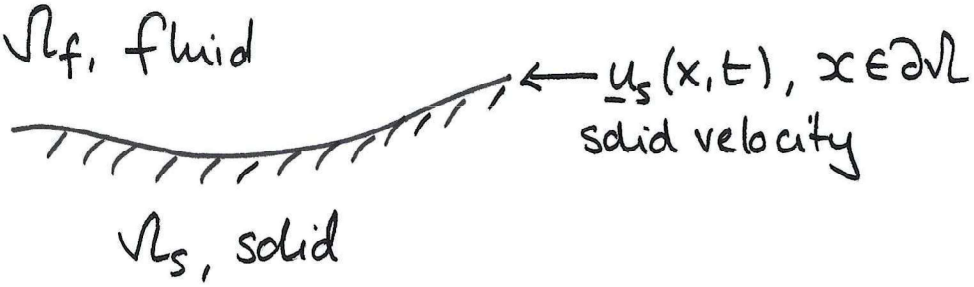
$Re \ll 1$, viscous dominated. Let $p^* = \mu U/L$ for balance

$Re \gg 1$, inertia dominated. Let $p^* = (Re)\mu \dfrac{U}{L}$ for balance.

Drop Primes henceforth

Boundary conditions

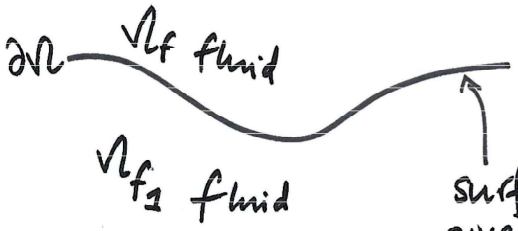
Boundary with a solid, with known motion



Then $\underline{u}(x,t) = \underline{u}_s(x,t)$, $x \in \partial\Omega$.

- Tangential components, no slip BC
- Normal Component, kinematic BC

Boundary with a fluid



surface often given via $z = f(x,y,t)$.

Continuity of velocity at $\partial\Omega$

$$\underline{u}(x,t) = \underline{u}_{f_1}(x,t), \quad x \in \partial\Omega$$

Then kinematic condition often enforced via

$$\underline{u} \cdot \underline{e}_z \Big|_{\partial\Omega} \equiv w|_{\partial\Omega} = \frac{Df}{Dt} = \partial f/\partial t + u\,\partial f/\partial x$$

Stress boundary conditions also required... will return to this later.

When $Re \ll 1$, Stokes Equations

$$0 = -\nabla p + \underline{F} + \nabla^2 \underline{u}$$

$Re \gg 1$, Eulers Equations $\dfrac{D\underline{u}}{Dt} = -\nabla p + \underline{F}$

Boussinesq Approximation: changes in density negligible except when coupled to $\underline{g}$ in a buoyancy body force

$$\therefore \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = \rho \nabla \cdot \underline{u} = 0 \quad \text{and we have}$$

$$\nabla \cdot \underline{u} = 0, \text{ fluid incompressibility}$$

Two dimensional flows: the streamfunction (ψ).

Planar flows

$$\underline{u}(x, y) = (u, v, 0) \quad \text{and we let } u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}$$

This ensures $\nabla \cdot \underline{u} = 0$.

Analogous streamfunctions for axisymmetric flows are in the on-line notes

1.2 Other conservation laws

- Concentration of a dissolved solute

$$\frac{DC}{Dt} = \nabla \cdot (\kappa \nabla C)$$

κ: diffusion coefficient (often constant)

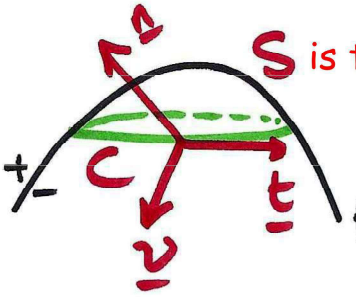
- Thermal energy

$$\frac{DT}{Dt} = \nabla \cdot (\kappa_T \nabla T)$$

neglecting viscous dissipation and/other heat sources (and taking $\nabla \cdot \underline{u} = 0$).

2.1 Surface Tension

Stress conditions at a fluid-fluid interface



S is the surface cap

Fluid interface e.g. water/oil

Surface tension: force at interface in direction $\underline{\nu}$, of magnitude γ per unit length.

Extend S to the interval $(-\epsilon, \epsilon)$ in the normal direction $\underline{n}$ to form the volume V.

$\underline{t}^+$ the force per unit area the upper fluid exerts on S

Force balance

$$\int_V \rho \frac{D\underline{u}}{Dt}\, dV = \underbrace{\int_V \underline{f}\, dV}_{\text{body force}} + \int_S (\underline{t}^+ + \underline{t}^-)\, dS + \int_C \gamma \underline{\nu}\, ds$$

$\underline{t}^-$ the force per unit area the lower fluid exerts on S

arclength

$\to 0$ as $\epsilon \to 0$

$\therefore$ let $\epsilon \to 0$,

$$0 = \int_S (\underline{t}^+ + \underline{t}^-)\, dS + \int_C \gamma \underline{\nu}\, ds$$

The stress tensor is

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \mu\left[\nabla \underline{u} + (\nabla \underline{u})^T\right]$$

Hence $\underline{t}^+ = \underline{n} \cdot \underline{\underline{T}}^+$, $\underline{t}^- = -\underline{n} \cdot \underline{\underline{T}}^-$

Define γ, $\underline{n}$ off-surface by an invariant extension in direction $\underline{n}$

Then (Q1, Problem Sheet 1)

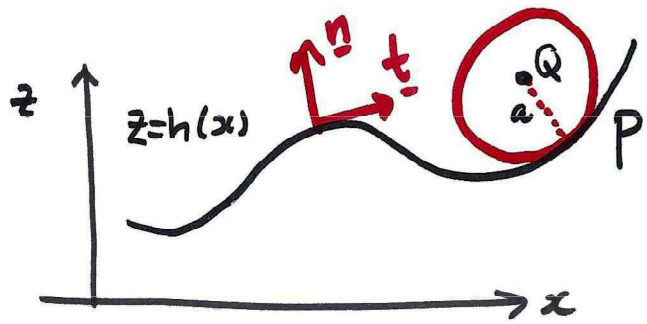
$$\int_C \gamma \underline{\nu} \, ds = \int_S \left[\nabla\gamma - (\nabla \cdot \underline{n}) \underline{n} \gamma \right] dS$$

With $\nabla_s := \nabla - \underline{n} \frac{\partial}{\partial n}$, $\nabla\gamma = \nabla_s \gamma$

$$\therefore \quad 0 = \int_S \left(\underline{n} \cdot \underline{\underline{T}}^+ - \underline{n} \cdot \underline{\underline{T}}^- + \nabla_s \gamma - \gamma \nabla \cdot \underline{n} \, \underline{n} \right) dS, \quad \text{all } S.$$

$$\boxed{\underline{n} \cdot \underline{\underline{T}}^+ - \underline{n} \cdot \underline{\underline{T}}^- = \gamma (\nabla \cdot \underline{n}) \underline{n} - \nabla_s \gamma} \qquad x \in S.$$

Curvature (2D)



In 2D the magnitude of the curvature at P is the reciprocal of the best fit circle radius, a.

With the surface $z = h(x)$, the unit normal and tangent are

$$\underline{n} = \frac{(-h', 1)}{(1+h'^2)^{1/2}} \qquad \underline{t} = \frac{(1, h')}{(1+h'^2)^{1/2}}$$

Recall $\underline{n}$ defined off-surface by invariant normal extension

Then

$$\frac{1}{a} = \frac{d}{dx}\left(\frac{h'}{(1+h'^2)^{1/2}}\right) = -\nabla\cdot\underline{n} = \frac{h''}{(1+h'^2)^{3/2}}$$

[as shown in Appendix of on-line notes]

More generally (see Appendix), in 2D or 3D, the curvature, κ, is given by

$$\kappa = -\nabla\cdot\underline{n}$$

Hence in 3D, for a surface $z = h(x,y)$,

$$\kappa = -\nabla\cdot\underline{n} \quad \text{with} \quad \underline{n} = \frac{\nabla(z - h(x,y))}{|\nabla(z-h(x,y))|}$$

so that $$\kappa = \frac{h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_x h_y h_{xy}}{(1+h_x^2+h_y^2)^{3/2}}.$$

Capillary Statics

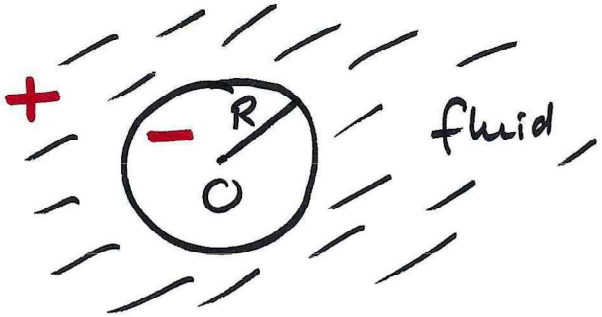
For static configurations, $\underline{u} = \underline{0}$, $\underline{\underline{T}} = -p\underline{\underline{I}}$, $\underline{n}\cdot\underline{\underline{T}} = -p\underline{n}$

Normal stress BC reduces to $p^+ - p^- = \gamma\kappa$

Tangential stress BC

$$0 = -\nabla_s \gamma .$$

Stationary Bubble



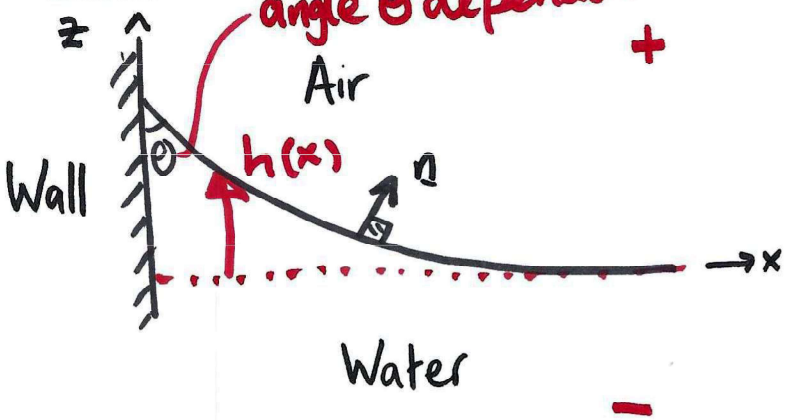
$$\text{Curvature} = \kappa = -\nabla\cdot\underline{n}\,|_{r=R}$$
$$= -\nabla\cdot\left(\frac{\underline{r}}{r}\right)\Big|_{r=R}$$
$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot 1\right)\Big|_{r=R} = -2/R .$$

$$\therefore \quad p^- - p^+ = 2\gamma/R$$

Smaller bubbles have higher internal pressure. Thus they are louder when they burst at a free surface ∴ Champagne fizz louder than beer.

Static meniscus

angle θ depends on water/air/wall properties

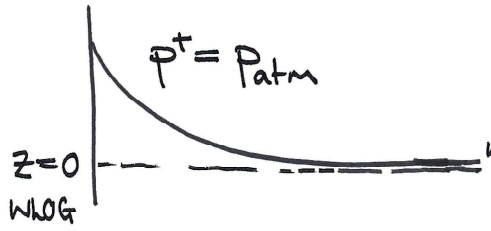


$P_{air} = p^+ = P_{atm}$, constant

atmospheric pressure

Navier-Stokes ($\underline{u}=0$).

$$0 = -\nabla p^- + \rho \underline{g} = -\nabla p^- - \rho g \underline{e}_z \therefore \frac{\partial p^-}{\partial x} \overset{①}{=} 0, \quad \frac{\partial p^-}{\partial z} \overset{②}{=} -\rho g$$



$$p^+ - p^- = \gamma\kappa \simeq 0$$

$$\lim_{x\to\infty} p^-(x,0) = p^{atm}$$

$$\therefore p^-(x,0) \overset{①\text{ by}}{=} p^{atm}, \text{ all } x$$

$$\therefore p^-(x,h) \overset{②}{=} p^{atm} - \rho g h$$

$$\therefore \boxed{\rho g h = \gamma\kappa}$$

Laplace-Young Eqn

Recall $\kappa = \dfrac{h_{xx}}{(1+h_x^2)^{3/2}}$ $\quad\therefore\quad h = \left(\dfrac{\gamma}{\rho g}\right)\dfrac{h_{xx}}{(1+h_x^2)^{3/2}}$

gives shape of meniscus.

BCs $\quad h \to 0$ as $x \to \infty$

$h_x \to -\cot\theta$ as $x \to 0$, for contact angle to be θ.

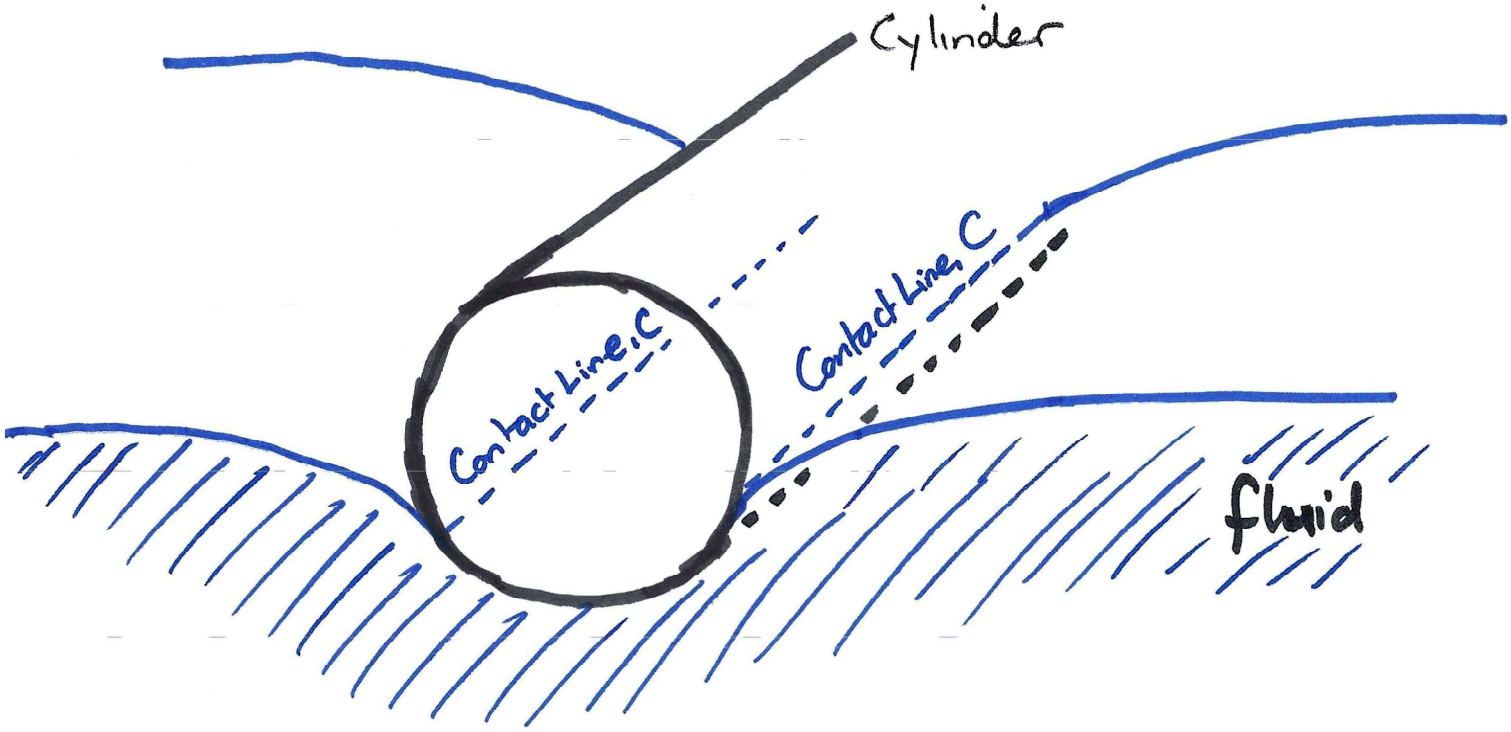
For $\theta \simeq \pi/2$, $|h_x| \ll 1$

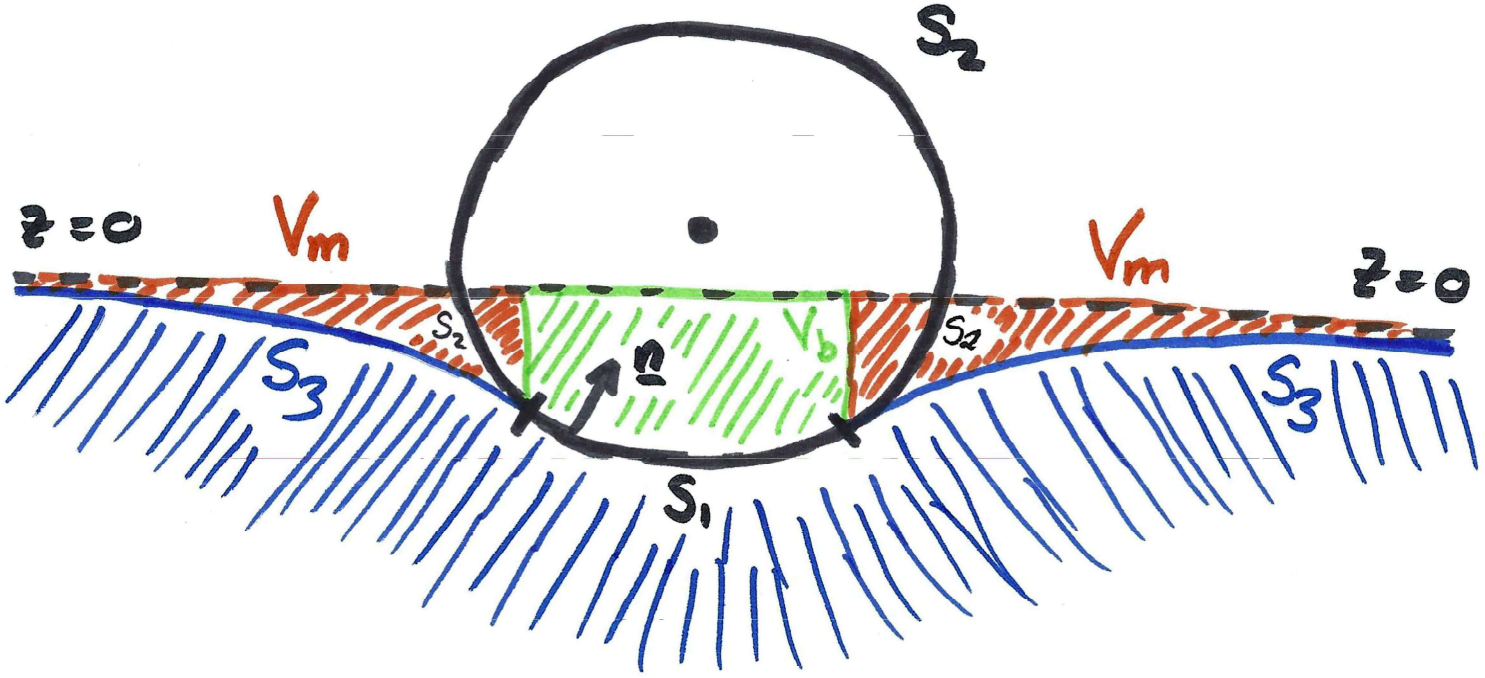
$$\boxed{\begin{aligned} &h = \left(\frac{\gamma}{\rho g}\right) h_{xx} \\ &h \to 0 \text{ as } x \to \infty \\ &h_x \to -\cot\theta \text{ as } x \to 0. \end{aligned}}$$

Solving $\quad h(x) = l_c \cot\theta\, e^{-x/l_c}$, $\quad \underline{l_c = \sqrt{\dfrac{\gamma}{\rho g}}}$

For air-water $l_c \simeq 3$mm $\qquad$ capillary length

Floating Bodies : Supported by buoyancy & surface tension





V_m : Volume displaced by meniscus

V_b : Volume additionally displaced by body

Surface tension between fluid and air, γ, and fluid density, ρ, are constant.

As previously, the fluid pressure is

$$p = -\rho g z + p_{atm} \quad \text{since } \underline{u} = 0.$$

Force on body from fluid is

$$\underline{F} = \underbrace{\int_{S_1} p\underline{n}\, dS}_{p_{atm}\int_{S_1}\underline{n}\, dS - \rho g\int_{S_1} z\underline{n}\, dS} + p_{atm}\int_{S_2}\underline{n}\, dS + \underbrace{\int_C \gamma \underline{\nu}\, ds}_{\int_{S_1}\gamma\kappa\underline{n}\, dS}$$

$$\therefore \underline{F} = p_{atm}\int_{S_1 \cup S_2}\underline{n}\, dS - \rho g\int_{S_1} z\underline{n}\, dS + \int_{S_1}\gamma\kappa\underline{n}\, dS$$

(the p_{atm} integral over $S_1 \cup S_2$ is 0)

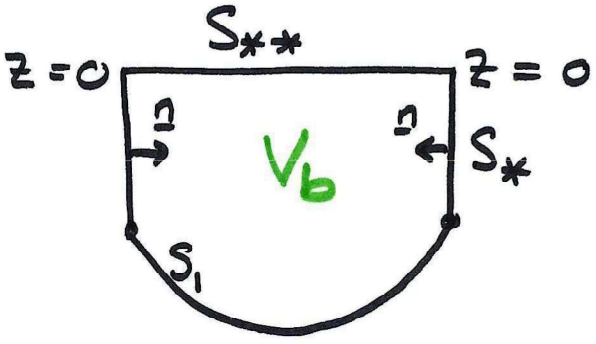
F+Mg=0 as no net force

$$\therefore \quad \underline{F} = -\rho g\int_{S_1} z\underline{n}\, dS + \int_{S_1}\gamma\kappa\underline{n}\, dS = -M\underline{g}$$

, force due to gravity.

g =-ge_z for minus sign

$$\therefore Mg = \underbrace{\underline{e}_z \cdot \left[-\rho g\int_{S_1} z\underline{n}\, dS\right]}_{F_b,\ \text{buoyancy force}} + \underbrace{\underline{e}_z \cdot \left[\int_{S_1}\gamma\kappa\underline{n}\, dS\right]}_{F_c,\ \text{surface tension force}}$$



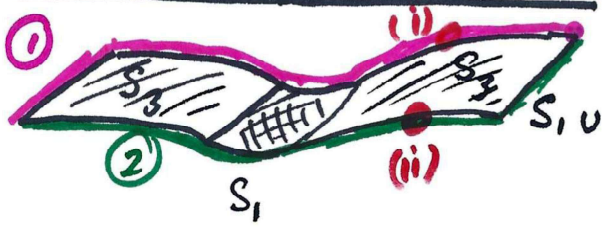
$$\underline{e}_z \cdot \int_{S_{**}} z\,\underline{n}\,dS = \underline{e}_z \cdot \int_{S_*} z\,\underline{n}\,dS = 0$$

Let $S_1^* = S_1 \cup S_* \cup S_{**}$

$$F_b = -\rho g\,\underline{e}_z \cdot \int_{S_1} z\,\underline{n}\,dS = -\rho g\,\underline{e}_z \cdot \int_{S_1^*} z\,\underline{n}\,dS$$

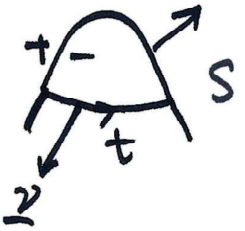
$$= +\rho g \int_{V_b} dV = \left\{ \begin{array}{l} \text{Weight of fluid} \\ \text{displaced from } V_b \end{array} \right\}$$

Surface Tension Force



$\underline{\nu} \perp \underline{t}, \underline{n}$

$\underline{\nu} = \underline{t} \wedge \underline{n}$



Going from point (i) on ① to point (ii) on ② $\left\{ \begin{array}{l} \underline{t}, \underline{n} \to -\underline{t}, \underline{n} \\ \therefore \underline{\nu} \to -\underline{\nu} \end{array} \right.$

$$\therefore \int_{S_1 \cup S_3} \kappa\,\underline{n}\,dS = \int_{\partial(S_1 \cup S_3)} \underline{\nu}\,ds = 0$$

per unit length; per unit length; by cancellation between ① and ②

$$\therefore \text{Noting } \gamma \text{ constant} \quad \int_{S_1} \gamma\kappa\,\underline{n}\,dS = -\int_{S_3} \gamma\kappa\,\underline{n}\,dS$$

$$\therefore F_c = \underline{e}_z \cdot \int_{S_1} \gamma\kappa\,\underline{n}\,dS = -\underline{e}_z \cdot \int_{S_3} \gamma\kappa\,\underline{n}\,dS$$

On S_3 surface tension and buoyancy balance.

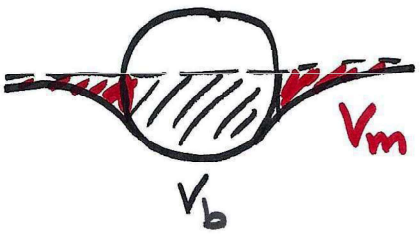
$\therefore$ Laplace Young $\quad p_g z = \gamma K$

$$F_c = -\underline{e}_z \cdot \int_{S_3} p_g \underline{n} z \, dS = p_g \int_{V_m} dV$$

$$= \left\{ \begin{array}{l} \text{Weight of fluid} \\ \text{displaced by meniscus} \end{array} \right\}$$

$\therefore$ Force on body = Weight of displaced fluid

Overall Scales



$V_b \sim \text{depth} \cdot L^2$

$V_m \sim \text{depth} \cdot L \cdot l_c$

where L is the body lengthscale

l_c / meniscus lengthscale

$l_c \sim \sqrt{\frac{\gamma}{p_g}}$

$$\therefore \frac{F_b}{F_c} \sim \frac{V_b}{V_m} \sim L\sqrt{\frac{p_g}{\gamma}}$$

$\therefore$ Once $L \ll \sqrt{\frac{\gamma}{p_g}}$ floating bodies primarily supported by surface tension.

Marangoni Flows

These are flows where surface tension gradient effects dominate.

Examples (see on-line notes) Tears of wine, surfactant flows, ...

Example Thermo-capillary bubble

- A bubble in a temperature gradient can move due to Marangoni effects

Assumptions

- Bubble remains spherical, radius a, constant.
- Temperature in both fluid and bubble given by

$$T = T_\infty + T' z$$

in frame moving with bubble

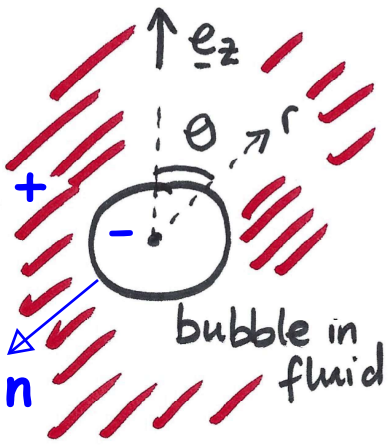
- Surface tension linear in T, hence

$$\gamma = \gamma_0 + \gamma' z = \gamma_0 + \gamma' a \cos\theta$$

- Length scales sufficiently small to ensure

$$Re = \rho \frac{|U|a}{\mu} \ll 1.$$

a is radius of bubble,
U is the speed of the bubble



Bubble radius a

$\underline{u} \sim -U\underline{e}_z$ at infinity, as working in bubble frame

$0 = -\nabla p + \mu\nabla^2\underline{u} + \rho\underline{g}$, Stokes flow

$\underline{u} = u_r\underline{e}_r + u_\theta\underline{e}_\theta$, Spherical polars

$\therefore\quad u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$, $u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$

for streamfunction ψ.

$u_r \sim -U\cos\theta$, $u_\theta \sim U\sin\theta$ as $r\to\infty$

$\therefore$ Seek streamfunction of form

$$\psi = f(r)\sin^2\theta$$

Standard methods for viscous flow [e.g. §3.3 of Ockendon & Ockendon] give

$$0 = -\nabla_\wedge\nabla p + \mu\nabla^2\nabla_\wedge\underline{u} + \nabla_\wedge(\rho\underline{g})$$
$$= \mu\nabla^2(\nabla_\wedge\underline{u}).$$

Thus $\psi = \left[Ar^4 + Br^2 + Cr + D/r\right]\sin^2\theta.$

BCs

- $u_r(r=a) \overset{①}{=} \frac{1}{a^2 \sin\theta} \frac{\partial \psi}{\partial \theta}\Big|_{r=a} = 0$
- $\psi \overset{②}{\to} -\frac{U r^2 \sin^2\theta}{2}$ as $r \to \infty$ to match flow at infinity

Noting bubble gas viscosity negligible compared to fluid viscosity, the stress balance BC gives

$$\underline{e}_r \cdot \underline{\underline{T}}^{fluid} \cdot \underline{e}_\theta = -e_\theta \cdot \nabla\gamma = -\tfrac{1}{a}\frac{\partial \gamma}{\partial \theta}$$

as $\gamma = \gamma_0 + a\gamma' \cos\theta$

$$\therefore \quad \mu\left[-\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]_{r=a} \overset{③}{=} \gamma' \sin\theta$$

From expression for stress $\underline{e}_r \cdot [(\nabla u) + (\nabla u)^T] \cdot \underline{e}_\theta$

see eg Appendix of Batchelor, Introduction to Fluid Dynamics.

② gives $A = 0, \; B = -U/2$

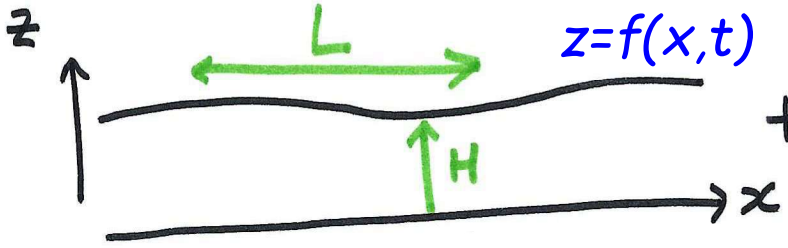
① gives $\frac{U a^2}{2} = Ca + D/a$

③ gives $D = -\frac{1}{6}\frac{\gamma' a^4}{\mu}$ (lots of cancellation)

$$\therefore C = \frac{Ua}{2} + \frac{1}{6}\frac{\gamma' a^2}{\mu},$$

and we thus know ψ, and hence $\underline{u}$.

2.2 Thin films and the lubrication approximation



$$\delta = \frac{\text{Thickness scale}}{\text{lengthscale of horizontal variation}} = \frac{H}{L} \ll 1.$$

Navier-Stokes

$$\rho(\underline{u}_t + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

2D Cartesian

Let $\underline{u} = u\underline{e}_x + w\underline{e}_z$, with velocity scales U, W.

$$0 = \nabla\cdot\underline{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}, \quad \frac{\partial}{\partial x} \sim \frac{1}{L}, \quad \frac{\partial}{\partial z} \sim \frac{1}{H}$$

$$\therefore W \sim H/L\, U \sim \delta U$$

For x-components of the momentum equation

$$\therefore \rho\underline{u}\cdot\nabla u \sim \rho\frac{U^2}{L}, \quad \nabla p \sim P/L, \quad \mu\nabla^2 u \sim \mu\frac{U}{H^2}$$

$$\therefore \text{Can neglect inertia if } \frac{\rho U^2}{L} \ll \frac{\mu U}{H^2}$$

$$\text{i.e. } \left(\frac{\rho}{\mu} UH\right)\frac{H}{L} = \delta Re \ll 1$$

Also, pressure is of the scale

$$p \sim \mu \frac{UL}{H^2}$$

The leading order equation is thus

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$$

z-components

$$\nabla p \sim p/H \sim \mu \frac{UL}{H^3} \quad , \quad \rho \underline{u} \cdot \nabla w \sim \rho \frac{UW}{L} \sim \delta \frac{\rho U^2}{L}$$

$$\sim \delta^3 Re \nabla p$$

$$\mu \nabla^2 w \sim \mu \frac{1}{H^2} W \sim \mu \frac{\delta U}{H^2} \sim \delta^2 \nabla p .$$

∴ Pressure term dominates viscosity and inertia

$$\therefore \quad \frac{\partial p}{\partial z} = \rho \underline{g} \cdot \underline{e}_z \quad \text{at leading order}$$

and $|w| \sim O(\delta)|u|$, neglect vertical velocity at leading order

except in $\nabla \cdot \underline{u} = 0$

BCs

Fixed boundary, $\underline{u} = \underline{0}$.

Free boundary. Typically Air-Liquid interface.

- Viscosity of air negligible.

$$\underline{\underline{T}}^{air} = -P_{atm}\,\underline{\underline{I}}$$

- Boundary given by $z = f(x,t)$

Kinematic BC $\quad w = f_t + u f_x$

For stress balance, $\underline{\underline{T}} = \begin{pmatrix} -p+2\mu u_x & \mu(u_z+w_x) \\ \mu(u_z+w_x) & -p+2\mu w_z \end{pmatrix}$

Note

$$\frac{|\mu u_x|}{|p|} \sim \frac{\mu U/L}{\mu U L/H^2} \sim \delta^2 \sim \frac{|\mu w_z|}{|p|}$$

$$\frac{|w_x|}{|u_z|} \sim \frac{\delta U/L}{U/H} \sim \delta^2 \qquad \frac{|\mu u_z|}{|p|} \sim \delta$$

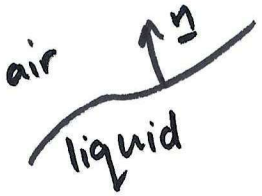
$$\therefore \underline{\underline{T}} = \begin{pmatrix} -p(1+O(\delta^2)) & \mu u_z(1+O(\delta^2)) \\ \mu u_z(1+O(\delta^2)) & -p(1+O(\delta^2)) \end{pmatrix}$$

Similarly $|f_x| \sim H/L \sim \delta$ and

$$\underline{n} = (-f_x, 1)(1+O(\delta^2)), \quad \underline{t} = (1, f_x)(1+O(\delta^2))$$

$$\kappa = f_{xx}\,(1+O(\delta^2))$$

With surface tension constant,



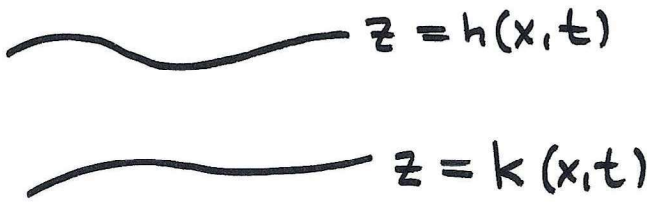
$$\underline{n}\cdot\underline{\underline{T}}^{air} - \underline{n}\cdot\underline{\underline{T}} = -\gamma\kappa\underline{n}$$

Absorbing P_{atm} into p, we have up to $O(\delta^2)$

$$-p = \gamma f_{excx} \qquad \text{(normal component)}$$

$$\frac{\partial u}{\partial z} = 0 \qquad \text{(tangential component)}.$$

Conservation of mass / incompressibility



If $z = k(x,t)$ fixed surface, then $\partial k/\partial t = 0$,

$u = w = 0$ on the surface so that

$$0 = w = \partial k/\partial t + u\partial k/\partial x$$

and kinematic condition still holds

Thus,

$$\frac{\partial}{\partial x}\int_{\bar{z}=k(x,t)}^{\bar{z}=h(x,t)} u(x,\bar{z},t)\,d\bar{z} = \int_k^h \frac{\partial u}{\partial x}\,d\bar{z} + u(x,h,t)\,h_x - u(x,k,t)\,k_x$$

$$= -\int_k^h \frac{\partial w}{\partial \bar{z}}\,d\bar{z} + u(x,h,t)\,h_x - u(x,k,t)\,k_x$$

$$= w(x,k,t) - w(x,h,t) + u(x,h,t)\,h_x - u(x,k,t)\,k_x$$

$$= \frac{\partial k}{\partial t} - \frac{\partial h}{\partial t}\,, \text{ by kinematic condition.}$$

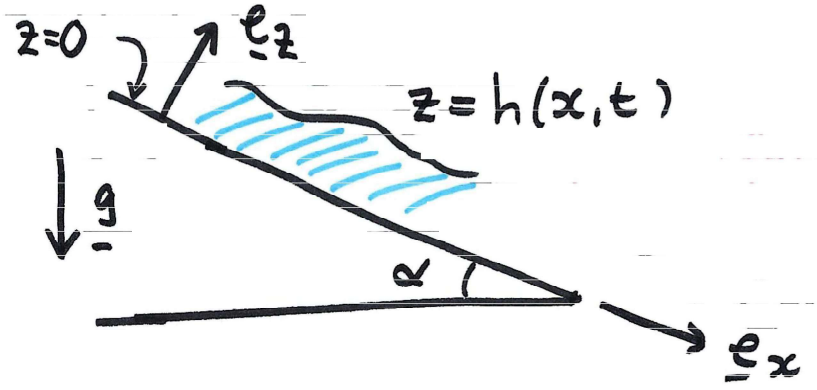
With $\mathcal{H} := h-k$, film thickness, the average velocity is $\bar{u} = \frac{1}{\mathcal{H}}\int_k^h u(x,\bar{z},t)\,d\bar{z}$

and thus

$$\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial}{\partial x}(\mathcal{H}\bar{u}) = 0$$

$\therefore$ We need to find $\bar{u}$ in terms of $\mathcal{H}$ which is context specific.

Free surface flow down an inclined plane



$$\underline{g} = g(\sin\alpha, -\cos\alpha)$$

z - component, momentum

$$\frac{\partial p}{\partial z} = -g\rho\cos\alpha$$

with

$$-p = \gamma h_{xxc}$$

on $z = h$.

$$\therefore \quad p = -\gamma h_{xx} + \rho g(h-z)\cos\alpha.$$

x - component, momentum

$$\mu\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x} - \rho g\sin\alpha$$

$$\therefore \mu\frac{\partial^2 u}{\partial z^2} = -\gamma h_{xxx} + \rho g\cos\alpha\, h_x - \rho g\sin\alpha$$

For α non-zero and a sufficiently shallow film,

$$|h_x| \ll \tan\alpha$$

We rescale using a velocity scale U and lengthscale R

$$\frac{\partial^2 u}{\partial z^2} = -\left[\frac{\gamma}{\mu U}\right] h_{xxx} - \left(\frac{\rho g}{\mu}\frac{R^2}{U}\right)\sin\alpha$$

$$= -\frac{1}{Ca} h_{xxx} - \frac{1}{Ca}\underbrace{\left(\frac{\rho g}{\gamma} R^2\right)}\sin\alpha$$

Ca: Capillary Number

Bo, Bond Number

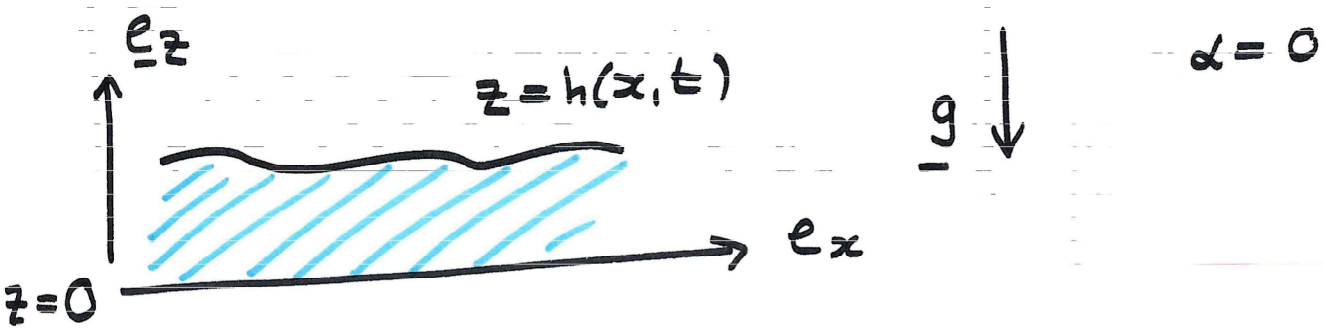
$$\therefore \quad u = \frac{-z(z-2h)}{2Ca}\left(h_{xxx} + Bo\sin\alpha\right)$$

$$\bar{u} = \frac{1}{h}\int_0^h u\,dz = \frac{h^2}{3Ca}\left(h_{xxx} + Bo\sin\alpha\right)$$

$$\therefore \quad 0 = h_t + \frac{\partial}{\partial x}(h\bar{u}) = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3Ca}\left(h_{xxx} + Bo\sin\alpha\right)\right)$$

gives evolution of free surface.

Free surface flow on a horizontal plane



We no longer have $|h_x| \ll \tan\alpha$. Thus,

$$\mu \frac{\partial^2 u}{\partial z^2} = -\gamma h_{xxx} + \rho g h_x$$

After rescaling

$$\therefore \quad u = -\frac{z(z-2h)}{2Ca}\left(h_{xxx} - Bo\, h_x\right)$$

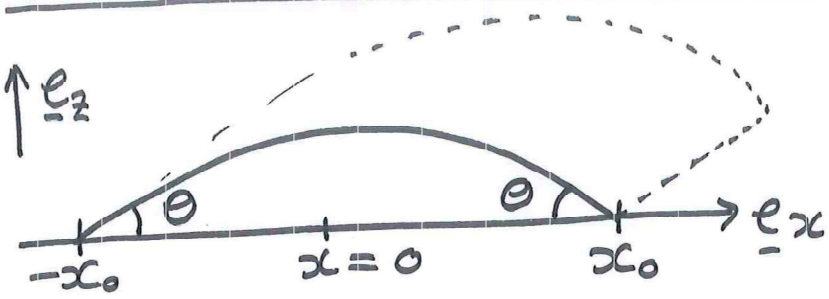
$$\therefore \quad \bar{u} = \frac{h^2}{3Ca}\left(h_{xxx} - Bo\, h_x\right)$$

$$Bo = \frac{\rho g}{\gamma} R^2$$

and hence

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3Ca}\left(h_{xxx} - Bo\, h_x\right)\right) = 0$$

Finite fluid drop on a horizontal plate



2D drop, $\theta \ll 1$

We assume

- Drop is static
- drop is symmetric about $x=0$
- $h(x_0) = h(-x_0) = 0$
- $h_x(x_0) = -\tan\theta \simeq -\theta$, $h_x(-x_0) \simeq \theta$.
- $\int_{-x_0}^{x_0} h\,dx = 2\int_0^{x_0} h\,dx = A$.

Find x_0 given A, θ

Since $h_t = 0$,

$$\frac{h^3}{3Ca}\left(h_{xxx} - B_o h_x\right) = \text{Constant} = 0$$

$$\therefore\ h_{xxx} - B_o h_x = 0$$

$$\therefore\ h = a + \cancel{b}^{\,0}\sinh\left(x/l_c\right) + c\cosh\left(x/l_c\right),$$

by parity

$$l_c = 1/\sqrt{B_o}.$$

$h_x(x_0) = -\theta$ gives

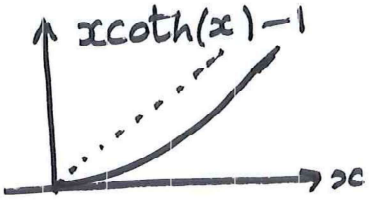
$$c = \frac{-\theta L_c}{\sinh(x_0/L_c)}$$

$h(x_0) = 0$ gives $a = -c\cosh\left(\frac{x_0}{L_c}\right) = \theta L_c \coth\left(\frac{x_0}{L_c}\right)$

Also, $$A/2 = \int_0^{x_0} h\,dx$$

$$= \int_0^{x_0} a + c\cosh(x/L_c)\,dx$$

$$= \theta L_c^2 \left[x_0/L_c \coth(x_0/L_c) - 1 \right]$$



∴ For a given area A, there is a unique drop width, x_0.

Radial drop Static, $Bo \ll 1$. On-line notes for more detail.



$$p = -\kappa \simeq \nabla^2 h$$

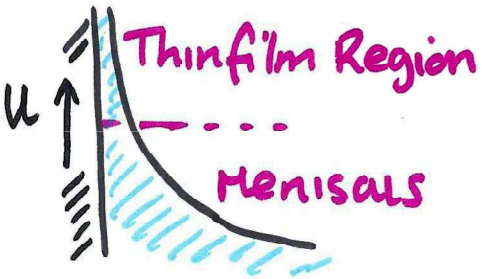
$$u_{zz} = -\frac{1}{Ca}\frac{\partial}{\partial r}(\nabla^2 h)$$

$$\therefore\ h\bar{u} = -\frac{h^3}{3Ca}\frac{\partial}{\partial r}(\nabla^2 h)$$

$\partial h/\partial t = 0$ as static ∴ $\frac{\partial}{\partial r}(\nabla^2 h) = 0$, as previously.

$$\therefore\ h_{rrr} - 1/r^2 h_r + 1/r\, h_{rr} = 0$$ with solution

$$h = A(r_0^2 - r^2).$$

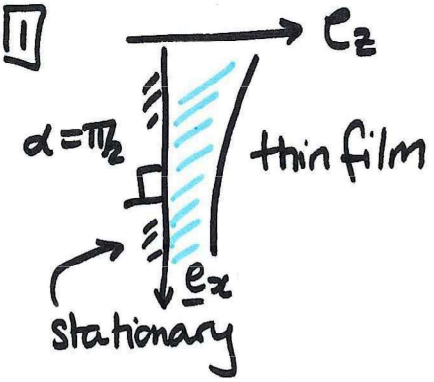


Industrial coating: draw a sheet at constant speed U from a fluid bath.

Non-dimensionalise such that $U=1$.

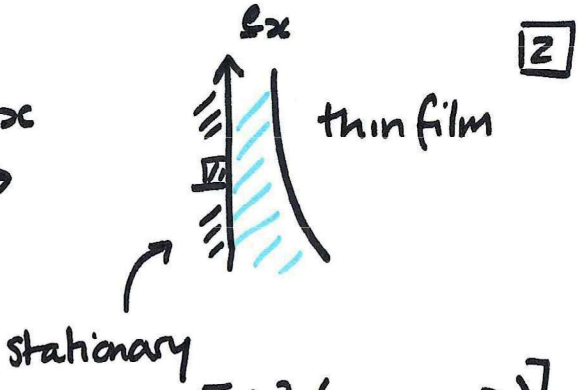
To use previous work, for thin film region

[1]



$x \to -x$

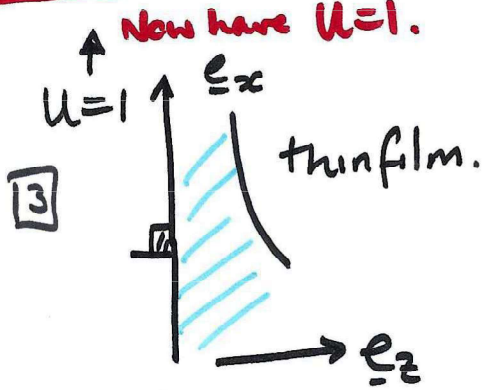
[2]



$$h_t + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}+B_o\right)\right]=0 \longrightarrow h_t + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}-B_o\right)\right]=0.$$

---

[3]



Jump into moving frame,

$$\bar{x} = x-t, \quad \bar{t}=t, \quad \bar{h}(\bar{x},\bar{t}) = h(x,t).$$

Same equation (as still an inertial frame)

$$\therefore \quad \bar{h}_{\bar{t}} + \frac{\partial}{\partial \bar{x}}\left[\frac{\bar{h}^3}{3Ca}\left(\bar{h}_{\bar{x}\bar{x}\bar{x}}-B_o\right)\right]=0$$

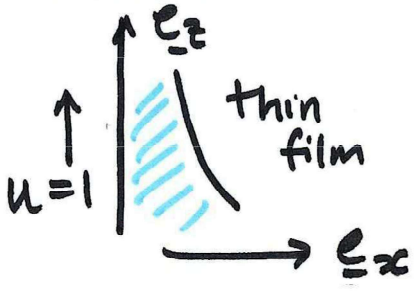
by rewriting in terms of x,t

Transform back to lab frame: $h_t + h_x + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}-B_o\right)\right]=0.$

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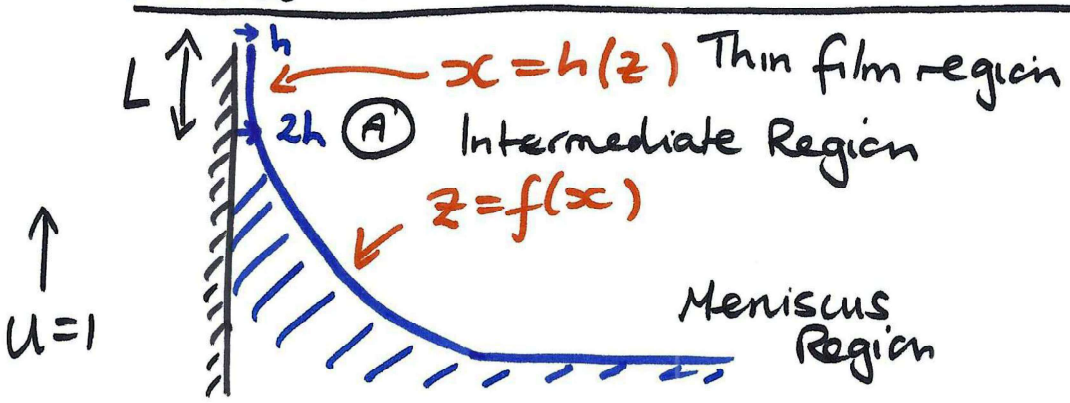
[4] $(x,z) \to (z,x)$

Equations for thin film region



$$h_t + h_z + \frac{\partial}{\partial z}\left[\frac{h^3}{3Ca}\left(h_{zzz}-B_o\right)\right]=0.$$

Scales For Landau-Levich Problem $\frac{\mu U}{\gamma} \ll 1$



Thin Film Equations

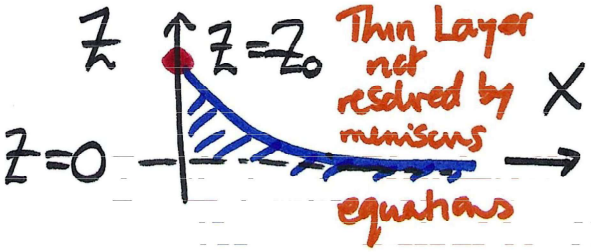
$$\underbrace{0}_{\text{as steady}} \cancel{h_t} + h_z + \partial_z \left[\frac{h^3}{3Ca} \left(h_{zzz} - Bo \right) \right] = 0$$

- h_t: unsteady
- h_z: plate velocity
- Ca: $\mu U/\gamma$
- Bo: gravity / surface tension

Leaving thin film region, h becomes large; h^3 dominate ... suggests meniscus region is a balance between surface tension & gravity

Choose R in Bond Number $Bo = \frac{\rho g}{\gamma} R^2$ to be $R = \sqrt{\frac{\gamma}{\rho g}}$

Then $Bo = \rho g/\gamma \; \gamma/\rho g = 1$.



Meniscus

$$\rho g f = \gamma \kappa$$

Use l_c to non-dim

f, κ now non-dim

$$\rho g l_c f = \gamma \kappa / l_c$$

$$\therefore \quad \kappa = \left(\rho g / \gamma\right) l_c^2 f = f$$

$X = H(Z)$

non-dimensionalised wrt l_c

$$\frac{H_{ZZ}}{(1+H_Z^2)^{3/2}} = Z$$

$H_Z \to -\infty$, $H \to \infty$ as $Z \to 0$

$H \to 0$, $H_Z \to 0$ as $Z \to Z_0$

Z_0 unknown

Meniscus is an outer sdution in the framework of matched asymptotics & the thin film an inner sdution

$$\frac{H_Z}{(1+H_Z^2)^{1/2}} = \frac{Z^2}{2} + \text{const} = \frac{Z^2 - Z_0^2}{2}$$

$$\therefore H_z = \frac{z^2 - z_0^2}{(4 - (z^2 - z_0^2)^2)^{1/2}}$$

As $z \to 0$, $H_z \simeq \dfrac{-z_0^2}{(4 - z_0^4)^{1/2}} \to -\infty$ $\therefore z_0 = \sqrt{2}$

$\therefore H(z_0) = 0$, $H_z(z_0) = 0$

$H_{zz}(z_0) = z_0 (1 + [H_z(z_0)]^2)^{3/2} = \sqrt{2}$.

$\therefore H \simeq \left(\dfrac{z - z_0}{\sqrt{2}}\right)^2 = \left(\dfrac{z - \sqrt{2}}{\sqrt{2}}\right)^2$ as $z \to z_0 = \sqrt{2}$

Back to thin film equations

With $R=\sqrt{\frac{\gamma}{\rho g}}$, $Bo=1$,

$$h_t+\left(\frac{h^3}{3Ca}\left(h_{zzz}-1\right)\right)_z=0$$

Let $z=z_0+\epsilon\bar{z}$ to blow up near z_0

$$H=\frac{(z-z_0)^2}{\sqrt{2}}+\ldots=\frac{\epsilon^2\bar{z}^2}{\sqrt{2}}+\ldots$$

which has been rescaled by $L_c\equiv R$

- L_c: meniscus scaling
- R: thin film scaling

$\therefore$ Can directly equate as they are equal.

$$\therefore \text{Let } h=H=\epsilon^2\bar{h}$$

$$z=Z=z_0+\epsilon\bar{z}$$

- z: thin film variables
- Z: meniscus variables

$$\therefore \quad 0 = \epsilon \bar{h}_{\bar{z}} + \left(\frac{\epsilon^6 \bar{h}^3}{3Ca} \left(\frac{1}{\epsilon} \bar{h}_{\bar{z}\bar{z}\bar{z}} - 1 \right) \right)_{\bar{z}} \frac{1}{\epsilon}$$

$$0 = \bar{h}_{\bar{z}} + \left(\frac{\epsilon^3 \bar{h}^3}{3Ca} \left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \epsilon \right) \right)_{\bar{z}}$$

Let $\epsilon = (Ca)^{1/3}$ to give $0 = \bar{h}_{\bar{z}} + \left(\frac{\bar{h}^3}{3} \left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \epsilon \right) \right)_{\bar{z}}$

Leading order

$$0 = \bar{h}_{\bar{z}} + \partial/\partial\bar{z} \left(\bar{h}^3/3 \, \bar{h}_{\bar{z}\bar{z}\bar{z}} \right)$$

$\bar{h} \to \bar{h}_0$ asymptotic thickness as $\bar{z} \to \infty$

$\bar{h} \sim \frac{1}{\sqrt{2}} \bar{z}^2$ as $H \sim \underbrace{\frac{1}{\sqrt{2}} (z - z_0)^2}_{\text{on meniscal scale}}$ near $z = z_0$

$\therefore \bar{h} \sim \frac{1}{\sqrt{2}} \bar{z}^2$ on a much longer lengthscale

$\therefore \bar{h} \to +1/\sqrt{2} \, \bar{z}^2$ for $\bar{z} < 0$, $|\bar{z}| \gg 1$

$\therefore$ Take $\bar{h} \to \bar{z}^2/\sqrt{2}$ as $\bar{z} \to -\infty$.

Integrate $\bar{h} + \bar{h}^3/3 \, h_{zzz} = \text{const} = \bar{h}_0$

Let $\bar{h} = \bar{h}_0 g$ $\bar{z} = \bar{h}_0 \zeta$ Then

$g + \frac{g^3 g_{\zeta\zeta\zeta}}{3} = 1$ $g \to 1$ as $\zeta \to \infty$

$g_\zeta \to 0$ as $\zeta \to \infty$ (flat)

Solve numerically to find g and find

$$g \sim 0.67\,\zeta^2 \quad \text{as } \zeta \to -\infty$$

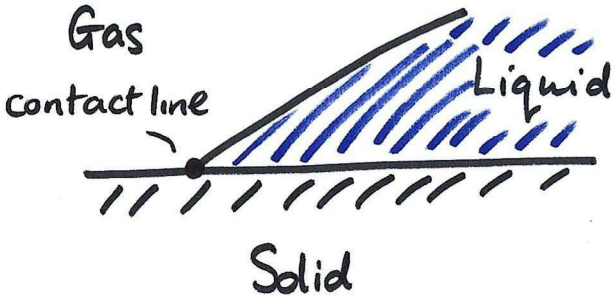
but $g \sim \dfrac{\bar{h}_0 \zeta^2}{\sqrt{2}}$ as $\zeta \to -\infty$

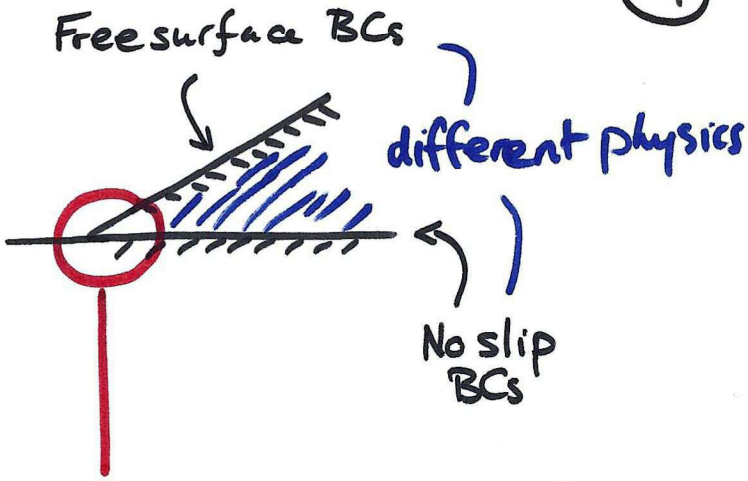
$$\therefore \bar{h}_0 = 0.67\sqrt{2} \simeq 0.948$$

$$\therefore (h_0)_{phys} = R\epsilon^2 \bar{h}_0 = R(Ca)^{2/3}\bar{h}_0$$

$$= \sqrt{\frac{\gamma}{\rho g}}(Ca)^{2/3} \cdot (0.948)$$

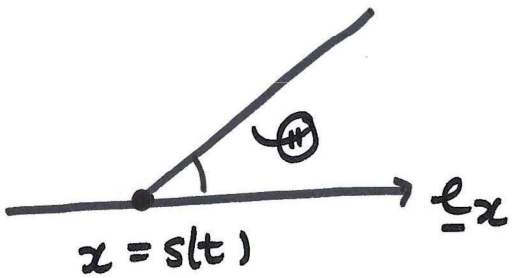
Moving Contact Lines





Close to the contact line, influence of free-surface physics and liquid-solid interface physics present... choice of BC ?

Try BC that previously worked for a static drop, once contact line is moving:



Scales such that

$Bo \ll 1$ $Ca \ll 1$

At $x = s(t)$, $h = 0$, $h_x = \theta$. Also no mass loss at interface.

$$h_t + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}\right)\right] = 0 \qquad Bo \ll 1$$

Let $x = s(t) + \xi$, $h = \theta\xi + f$, $h \geq 0$, $|f| \ll \theta\xi$.

$$h_t - \dot{s}h_\xi + \frac{\partial}{\partial \xi}\left[\frac{h^3}{3Ca} h_{\xi\xi\xi}\right] = 0$$

$$f_t - \dot{s} f_\xi - \dot{s}\,\theta + \frac{\partial}{\partial \xi}\left[\frac{\theta^3 \xi^3}{3Ca} f_{\xi\xi\xi}\right] = 0$$

Scaling $f = Ca\, g$ consistent with $f \ll 1$, we have at leading order

$$\frac{\partial}{\partial \xi}\left[\frac{\theta^3 \xi^3}{3} g_{\xi\xi\xi}\right] = \dot{s}\,\theta$$

$$\therefore \quad g_{\xi\xi\xi} = \frac{3}{\theta^2}\frac{1}{\xi^2}\dot{s} + \frac{\text{constant}}{\xi^3}$$

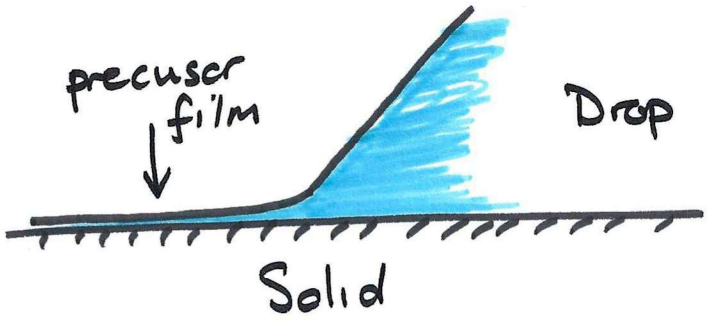
$$g = \frac{3\dot{s}}{\theta^2}(\xi \ln \xi - \xi) + \text{constant} \cdot \ln \xi + A\xi^2 + B\xi + C$$

$$\therefore \quad |Ca\, g| = |f| \not\ll \theta\xi \text{ for } \xi \text{ sufficiently small}$$

No consistent solution on approaching contact line.

Potential resolutions

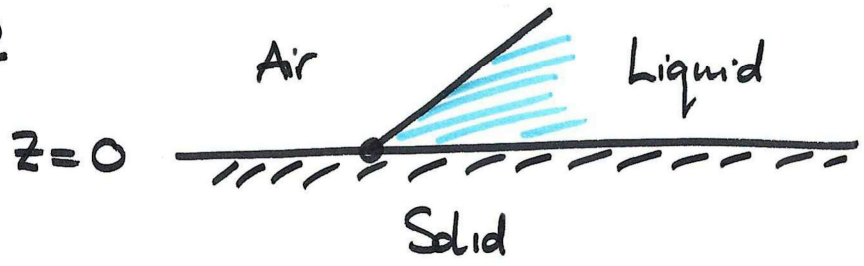
Precursor film



Solid has a precursor film e.g. if drop edge oscillating back and forth.

No contact line
$\therefore$ No problem

Slip



slip length.

Rather than $u=0$ on $z=0$, impose $u = \lambda \frac{\partial u}{\partial z}$.

The thin film equation becomes ($Bo \ll 1$, $Ca \ll 1$)

See example sheets

$$h_t + \left[\frac{1}{Ca} \left(\frac{h^3}{3} + \lambda h^2 \right) h_{xxx} \right]_x = 0 .$$

With $h = \Theta\xi + f = \Theta\xi + Ca\, g$ as previously,

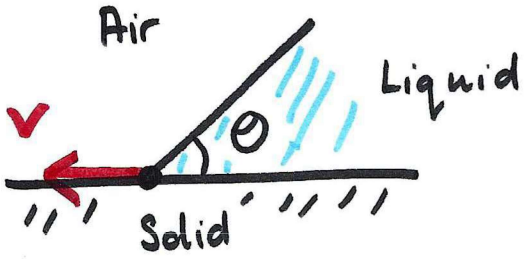
$$\lambda \Theta^2 \xi^2 g_{\xi\xi\xi} \simeq \dot{s}\, \Theta \xi$$

$$\therefore \quad g \simeq \frac{\dot{s}}{\Theta \lambda} \xi^2 \ln \xi$$

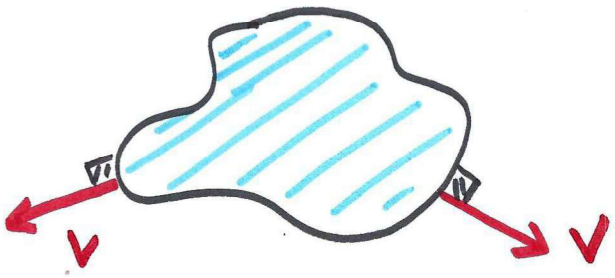
$\therefore |Ca\, g| = |f| \ll \Theta\xi$ for ξ sufficiently small, as required.

Tanner's Law

Empirical, based on observation



From above



Normal velocity, $V = K(\theta^3 - \Theta^3)$

θ: actual contact angle

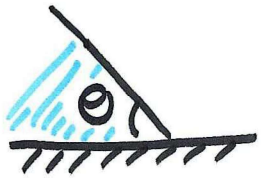
Θ: Static contact angle, i.e. angle when drop stationary

For the thin film approximation

$$v = K\left(h_x^3 - \Theta^3 \right)$$

L for figure.

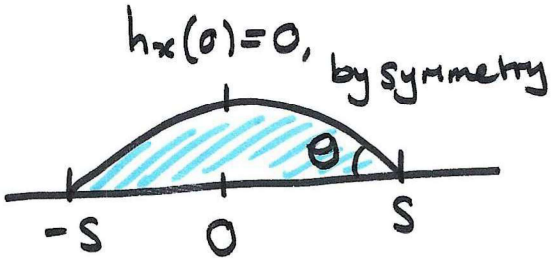
$-h_x^3$ for



Quasi-static evolution with $Bo \ll 1$, $Ca \ll 1$

For a symmetric drop spreading via Tanner's law, where drop transients relaxed, so drop dynamics slow, i.e. $h_t \ll O\left(\frac{1}{Ca}\right)$, i.e. quasi-static.

Then $$\left[\frac{h^3}{3Ca} \; h_{xxx} \right]_x = 0$$



$h(s) = 0$, $h_x(s) = -\Theta$

$h_x(0) = 0$

At s, $\dot{s} = -h_x^3 - \Theta^3$, Tanner's law, after non-dim, use K as velocity scale.

$$A = \int_{-s}^{s} h\,dx = 2\int_0^s h\,dx,$$

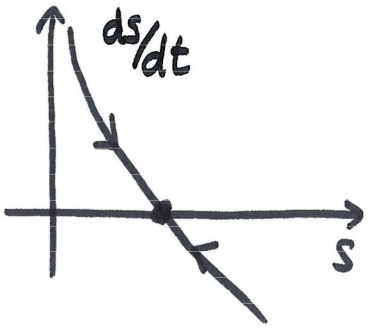
mass conservation

$$\therefore h^3 h_{xxx} = \text{Constant} = 0$$

$$\therefore \quad h = \frac{\theta}{2s}(s^2 - x^2), \qquad A = 2\int_0^s \frac{\theta}{2s}(s^2 - x^2)\,dx$$

$$= \tfrac{2}{3}\theta s^2.$$

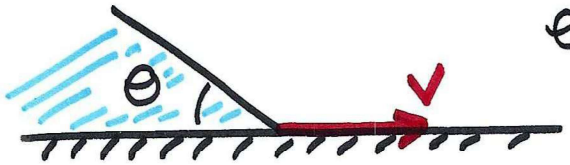
$$\therefore \quad \frac{ds}{dt} = \theta^3 - \theta^3 = \frac{27}{8}\frac{A^3}{s^6} - \theta^3$$



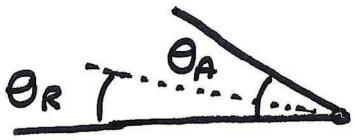
$$\therefore \quad s \to \left(\tfrac{3}{2}\right)^{1/2}\sqrt{\frac{A}{\theta}} \quad \text{as } t \to \infty$$

Advancing and receding contact angles

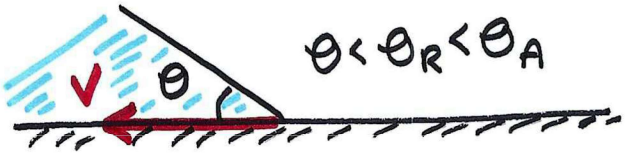
Advancing



$\theta > \theta_A > \theta_R$



Receding

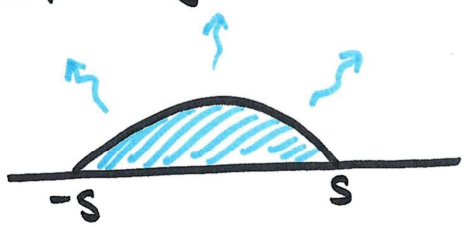


$\theta < \theta_R < \theta_A$

In a generalisation of Tanner's law it is often observed that

$$V = \begin{cases} K_A(\theta^3 - \theta_A^3) & \theta > \theta_A \\ 0 & \theta_R < \theta < \theta_A \\ -K_R(\theta_R^3 - \theta^3) & \theta < \theta_R \end{cases}$$

Evaporating drops and coffee stains



Symmetric drop

- Slowly evaporating drop
- Drop is thin, $|h_x| \ll 1$.
- $Bo \ll 1$, gravity negligible
- $Ca \ll 1$
- Quasi-static ($h_t \simeq 0$).
- Contact line does not move, s constant.

The drop is slowly evaporating:

$$\frac{d}{dt}\int_{-s}^{s} h\,dx = 2\frac{d}{dt}\int_{0}^{s} h\,dx = -2E\int_{0}^{s}(1+h_x^2)^{1/2}\,dx$$

evaporation rate per unit surface area, assumed constant and ord(1)

$$\therefore\ 2\frac{d}{dt}\int_{0}^{s} h\,dx = -2Es, \text{ to leading order.}$$

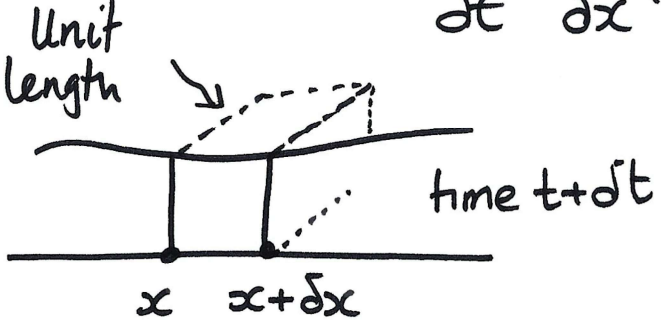
Fluid flow $\quad h(s)=0,\ h_x(s)=-\theta,\ h_x(0)=0$

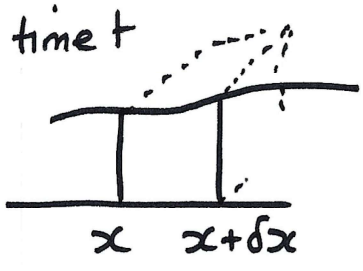
$$h_t + (h\bar{u})_x = h_t + \left(h^3/3Ca\, h_{xxx}\right)_x = -E$$

$\bar{u}$: average velocity

$-E$: sink due to evaporation.

Show $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(\bar{u}h) = -E$





$$\delta x\, h(x,t+\delta t) - \delta x\, h(x,t)$$

$$= \underbrace{-E\,\delta x\,\delta t}_{\substack{\text{evaporation} \\ \text{per unit length} \\ \text{per unit time}}} + \underbrace{\int_0^{h(x,t)} \delta t\, u(x,\bar{z},t)\, d\bar{z}}_{\text{Amount entering/leaving on left}}$$

$$\underbrace{- \int_0^{h(x+\delta x,t)} \delta t\, u(x+\delta x,\bar{z},t)\, d\bar{z}}_{\text{Amount entering/leaving on right}}$$

$$\therefore -E\,\delta x\,\delta t = \delta x\,\delta t \frac{\partial h}{\partial t} + \delta t\,\delta x \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,\bar{z},t)\, d\bar{z} + \text{higher orders}$$

$$\therefore \boxed{-E = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h\bar{u}\right)}$$

Recall $Ca \ll 1$ Let $q = h^2/3\, h_{xxx}$ $\therefore \bar{u} = q/Ca$

Let $h = h_0 + Ca\, h_1 + \cdots$ $q = q_0 + Ca\, q_1 + \cdots$

Leading order $(h_0 q_0)_x = \left(h_0^3/3\, h_{0xxx}\right)_x = 0$

$$\therefore h_0 = \theta/2s\,(s^2 - x^2)$$

As previously $A = 2\int_0^s h\,dx$

$$\simeq 2\int_0^s h_0\,dx = 2\Theta s^2/3$$

We also have $2\,d/dt\int_0^s h\,dx = \dot{A} = -2Es$ (s constant)

$$\therefore \quad h \simeq h_0 = \frac{3A(t)}{4s^3}(s^2-x^2) \qquad h_{0t} = -\frac{3E}{2s^2}(s^2-x^2)$$

$h_0 h_{0xxx} = 0$

Next order $\quad \frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x}(h_0 q_1 + h_1 q_0) = -E$

$-h_{0t}$

$$\therefore \quad h_0 q_1 = \int_0^x -E + \frac{3E}{2s^2}(s^2-x^2)\,dx$$

$$\therefore \quad q_1 = \frac{xE}{2s^2}(s^2-x^2)\frac{1}{h_0} = \frac{xE(s^2-x^2)}{2s^2}\,\frac{1}{\frac{\Theta}{2s}(s^2-x^2)}$$

$$= \frac{xE}{s\Theta} = \frac{2xEs}{3A}$$

$$\therefore \quad \bar{u} \simeq \frac{1}{Ca}(Ca\,q_1) = \frac{2xEs}{3A}$$

∴ Average velocity is from centre to contact line.

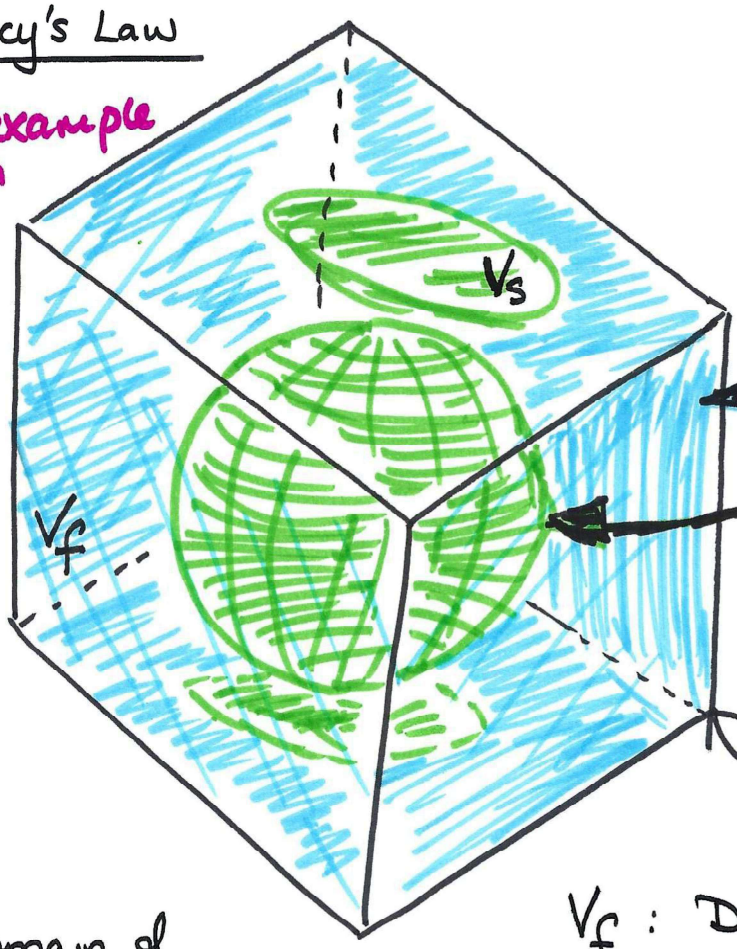
∴ For a drop of coffee, dissolved coffee carried to rim of drop, where it is deposited as liquid evaporates

∴ Coffee rings.

3 Porous Media & Thermal Convection

3.1 Darcy's Law

Typical example Flow in Porous Rock



Porous medium:

2 phases,
1 fluid,
1 rigid solid

Geometry: Repeating periodic, cuboidal unit, V.

V_s : Domain of solid within the unit

V_f : Domain of fluid within the unit.

Objective: Show that

For small Reynolds number, and fixed density ρ, the volume averaged flow, $\underline{u}$, satisfies

(will define in detail below, but velocity field averaged over repeating unit.)

$$\underline{u} = -k/\mu \nabla p - \frac{k}{\mu} \rho g \underline{e}_z .$$ Darcy's law

k: permeability

μ: viscosity

p: volume averaged pressure.

We also have $0 = \nabla \cdot \underline{u} = -k/\mu \nabla^2 p$.

We use homogenisation, starting from

$$\rho \underline{u}_t + \rho \underline{u}.\nabla \underline{u} = -\nabla(\overbrace{p+\rho g z}^{p^*}) + \mu \nabla^2 \underline{u}$$

with $\underline{u}$ redefined to be the microscale velocity field (no averaging).

Similarly p is redefined analogously.

Boundary conditions $\underline{u} = 0$ on ∂V_s.

On cuboidal faces $\notin \partial V_s$, we have periodic boundary conditions.

Scales r_p = Pore lengthscale ~ $[\text{Volume}(V)]^{1/3}$

L = lengthscale of pressure gradient driving flow

$\epsilon = r_p/L \ll 1$, typically

Rescale p^* by P. For a balance

$$\frac{P}{L} \sim \mu U / r_p^2$$

With this pressure scaling, non-dimensionalisation with lengthscale r_p, timescale r_p/U, gives

$$Re(\underline{u}_t + \underline{u}.\nabla \underline{u}) = -\frac{1}{\epsilon}\nabla p^* + \nabla^2 \underline{u},$$

$$Re = \frac{\rho U r_p}{\mu}.$$

We have two lengthscales, r_p and L.

$\underline{x}$ is on the porescale. Let $\underline{X} = \epsilon \underline{x}$, macroscale.

Write $\underline{u} = \underline{u}(\underline{x}, \underline{X})$, $p^* = p^*(\underline{x}, \underline{X})$

$$\nabla_i \rightarrow \frac{\partial}{\partial x_i} + \frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial X_j} = \frac{\partial}{\partial x_i} + \epsilon \frac{\partial}{\partial X_i}$$

With $\underline{u} = \underline{u}^0 + \epsilon \underline{u}^1 + \cdots$, $p^* = p^{*0} + \epsilon p^{*1} + \cdots$

and $Re \sim O(\epsilon)$

$$\therefore \quad \frac{\partial p^{*0}}{\partial x_i} = 0 \quad \text{i.e.} \quad p^{*0} = p^{*0}(\underline{X})$$

$$\frac{\partial p^{*0}}{\partial X_i} \overset{①}{=} -\frac{\partial p^{*1}}{\partial x_i} + \frac{\partial^2 u_i^0}{\partial x_\alpha \partial x_\beta} \quad \text{with} \quad \frac{\partial u_i^0}{\partial x_i} \overset{②}{=} 0 \quad \text{from} \quad \nabla \cdot \underline{u} = 0$$

$\therefore$ By linearity

$$u_i^\circ = - \underbrace{K_{ij}(x, X) \frac{\partial p^{\circ *}}{\partial X_j}}_{\text{linear in forcing}}$$

With $\quad \langle f \rangle := \frac{1}{|V|} \int_{V_f} f \, dV_{\underline{x}}$

$$\langle u_i^\circ \rangle = - \langle K_{ij} \rangle \frac{\partial p^{\circ *}}{\partial X_j}$$

Incompressibility $\quad \nabla \cdot \underline{u} = 0$

$$\therefore \quad \frac{\partial u_i^\circ}{\partial x_i} = 0 \qquad \frac{\partial u_i^\circ}{\partial X_i} + \frac{\partial u_i^1}{\partial x_i} = 0$$

$$\therefore \quad \nabla \cdot \langle \underline{u}^\circ \rangle = \frac{\partial}{\partial X_i} \left[\frac{1}{|V|} \int_{V_f} u_i^\circ(\underline{x}, \underline{X}) \, dV_{\underline{x}} \right]$$

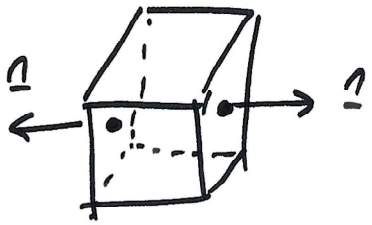
$$= \frac{1}{|V|} \int_{V_f} \frac{\partial u_i^\circ}{\partial X_i} \, dV_{\underline{x}}$$

$$\therefore \nabla \cdot \langle \underline{u}^0 \rangle = -\frac{1}{|V|} \int_{V_f} \frac{\partial u_i^1}{\partial x_i} \, dV_{\underline{x}}$$

$$= -\frac{1}{|V|} \left[\underbrace{\int_{\partial V_f \cap \partial V_s} n_i u_i^1 \, dS}_{①} + \overbrace{\int_{\partial V_f \cap \partial \overline{V}_s} n_i u_i^1 \, dS}^{②} \right]$$

① = 0 as $\underline{u} = 0$ on $\partial V_f \cap \partial V_s$

② = 0 by periodicity. On boundary $\underline{u}$ same at opposite point, by periodicity, but $\underline{n}$ changes sign. Hence no contribution.



$$\therefore \quad \nabla \cdot \langle \underline{u}^0 \rangle = 0$$

For $\langle K_{ij}(\underline{x}, \underline{X}) \rangle$ isotropic, we have $\langle K_{ij}(\underline{x}, \underline{X}) \rangle = \delta_{ij} K(\underline{X})$, for some $K(\underline{X})$.

Redefining $\underline{u} = \langle \underline{u}^0 \rangle$ and $p^* = p^{*0}(\underline{X})$ we have

$\underline{u} = -K(\underline{X}) \nabla p^*$ where $\nabla \equiv \nabla_{\underline{x}}$, the macroscale gradient

Redimensionalising

$$\underline{u} = -\frac{k}{\mu}\nabla p^* = -\frac{k}{\mu}\nabla(p+\rho g z).$$ Darcy's Law

Also $\nabla\cdot\underline{u} = 0$ and hence $0 = -\frac{k}{\mu}\nabla^2 p$

Finally,

$$\underline{u} = \langle \underline{u}^0 \rangle = \frac{1}{|V|}\int_{V_f} \underline{u}^{(0)}\, dV_{\underline{x}}$$

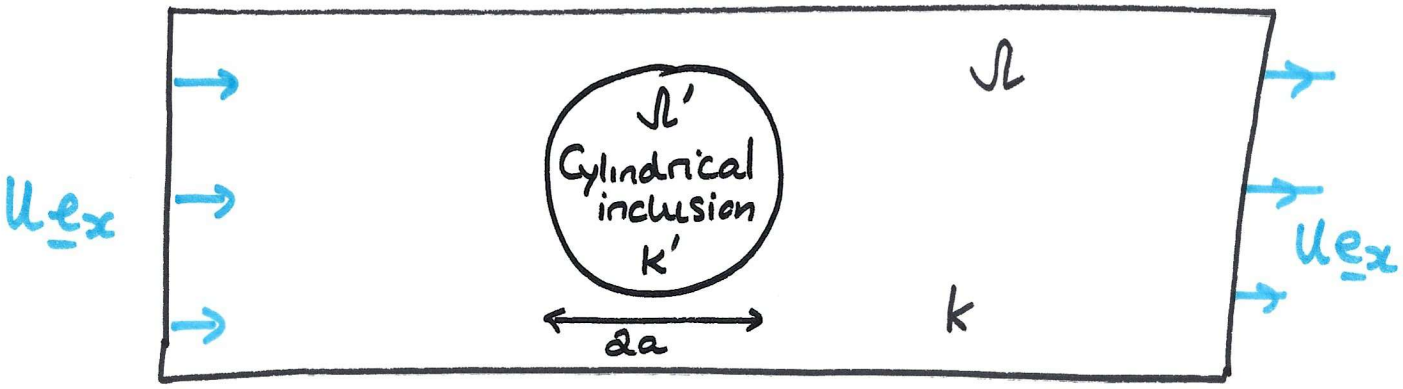
$\underline{u}$: u macroscale; $\underline{u}^0$: microscale

If $\underline{u}^{(0)}$ constant $\quad \underline{u}_{\text{macroscale}} = \frac{|V_f|}{|V|}\underline{u}^{(0)} = \varphi\,\underline{u}^{(0)}$

$$= \varphi\,\underline{u}_{\text{microscale}}.$$

Useful later.

Flow focussing



Consider a porous medium, permeability k, with a cylindrical inclusion, Ω', permeability k', radius a.

We assume

- k, k' constant
- No gravity
- $\underline{u} = U\underline{e}_x$ far from inclusion
- Problem is steady
- Darcy's law holds

$$\therefore \quad \underline{u} = -k/\mu \nabla p = -\nabla\left(\frac{kp}{\mu}\right), \qquad 0 = \nabla\cdot\underline{u} = -\nabla^2\left(\frac{kp}{\mu}\right).$$

At the boundary $\partial\Omega'$, pressure p and normal flux $\underline{n}\cdot\underline{u} = -\underline{n}\cdot\nabla(kp/\mu)$ are continuous.

r, θ polars

Let $\varphi = -kp/\mu$. $\quad \underline{u} = \nabla\varphi \rightarrow U\underline{e}_x = U\cos\theta\,\underline{e}_r - U\sin\theta\,\underline{e}_\theta$

$$\therefore \varphi \sim \cos\theta \quad \text{as } r \to \infty$$

$$\therefore \varphi = \left(Ar + B/r\right)\cos\theta$$

$A = U$ as $\underline{u} = \nabla\varphi \rightarrow U\underline{e}_x$ at ∞

since $\nabla^2\varphi = 0$, and WLOG origin at centre of inclusion.

$$\therefore \varphi = \begin{cases} (Ur + B/r)\cos\theta & r > a \\ (A_< r + B_</r)\cos\theta & r < a \end{cases}$$

$B_< = 0$ as $\varphi(r=0)$ finite.

$$0 = [p]_{a^-}^{a^+} = -\frac{\mu}{k}\varphi(a^+) + \frac{\mu}{k'}\varphi(a^-)$$

$$\therefore \quad \frac{1}{k}(Ua + B/a) = \frac{A_<}{k'}a$$

$$0 = [\underline{n}\cdot\nabla\varphi]_{a^-}^{a^+} = \left[\frac{d\varphi}{dr}\right]_{a^-}^{a^+} = \cos\theta\left[(U - B/a^2) - (A_<)\right]$$

$$\therefore \quad U - B/a^2 = A_<.$$

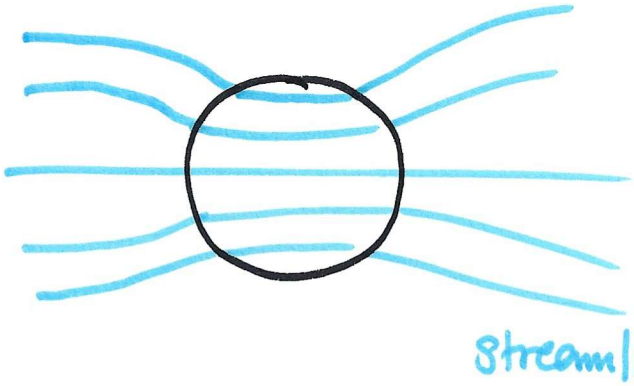
Solving for the unknowns $A_<$, B gives

$$\varphi = \begin{cases} -kp/\mu & r > a \\ -k'p/\mu & r < a \end{cases} = U\cos\theta \begin{cases} r + \dfrac{(k-k')a^2}{(k+k')r}, & r > a \\ \dfrac{2k'}{k+k'}r, & r < a \end{cases}$$

As $k' \to 0$, $\varphi \to U\cos\theta\,(r^2 + a^2/r)$, $r > a$,

the solution for flow around a cylinder.

For $k' \gg k$,
flow focussed by
inclusion.



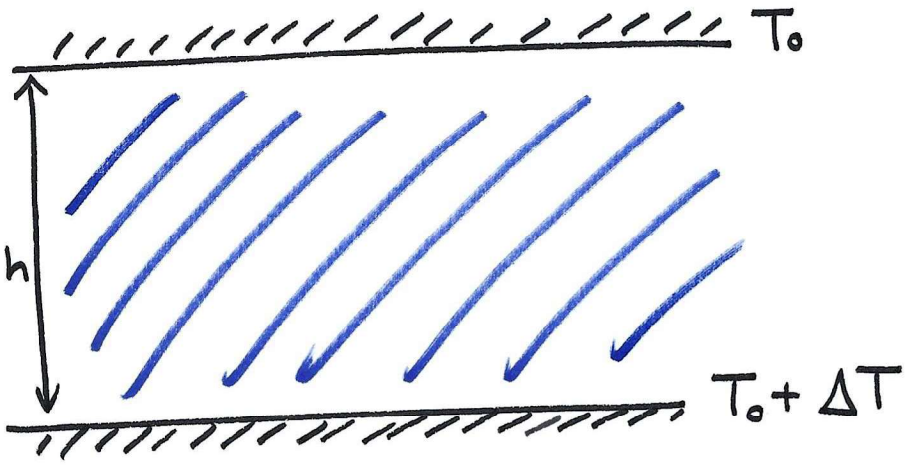
Streamlines

3.2 Thermal Convection

· We assume

- the Boussinesq approx (variations in density only important for buoyancy)
- $\rho = \rho_0 (1 - \beta(T - T_0))$ for the fluid density, $\beta > 0$
- specific heat capacities of solid and fluid in porous medium the same. Similarly for density

· We have



Porous layer, depth h, heated from below.

ΔT small enough to allow equilibrium, $\underline{u} = 0$, $\partial/\partial t = 0$.

$\therefore$ Heat equation $\frac{DT}{Dt} = \partial T/\partial t + \underline{u}.\nabla T = \kappa_T \nabla^2 T$ reduces

to $\nabla^2 T_s = 0$, T_s for steady state.

$\therefore \quad T_s = T_0 + \Delta T (1 - z/h)$.

$$\therefore \rho_s = \rho_0 \left[1 - \beta \Delta T (1 - z/h)\right], \quad \text{steady state.}$$

As $0 = -\frac{\partial p}{\partial z} - \rho g$ at steady state

$$p_s = p_0 - \rho_0 g \left[z - \beta \Delta T (z - z^2/2h)\right].$$

Linear instability Perturb the steady state:

$$\underline{u} = \underline{0} + \underline{u}' \qquad T = T_s + T' \qquad p = p_s + p' \qquad \rho = \rho_s + \rho'$$

We thus have $\nabla \cdot \underline{u}' = 0$ and from D'Arcy's law

$$\text{i.e. } \underline{u} = -\frac{k}{\mu}\nabla p - \frac{k}{\mu}\rho g \underline{e}_z$$

we have $$\underline{u}' = -\frac{k}{\mu}\nabla p' - \frac{k}{\mu}\rho' g \underline{e}_z = -\frac{k}{\mu}\left(\nabla p' - \beta \rho_0 g T' \underline{e}_z\right)$$

as $\rho' = -\beta \rho_0 T'$.

Heat Eqn $$\frac{\partial T'}{\partial t} + \underline{u}' \cdot \nabla T_s + \underline{0} \cdot \nabla T' = \kappa_T \nabla^2 T'$$

$$\therefore \frac{\partial T'}{\partial t} - w' \Delta T/h = \kappa_T \nabla^2 T$$

With $\underline{x} = h\underline{X}$, $\underline{u}' = \frac{\kappa_T}{h}\underline{U}$, $t = \frac{h^2}{\kappa_T}\tau$, $p' = \frac{\mu \kappa_T}{k} P$

$T' = \Delta T \Theta$,

$$\nabla \cdot \underline{U} \overset{(1)}{=} 0, \qquad \underline{U} \overset{(2)}{=} -\nabla P + Ra\, \Theta \underline{e}_z$$

$$\frac{\partial \Theta}{\partial \tau} - W \overset{(3)}{=} \nabla^2 \Theta, \quad \text{with } Ra = \frac{\beta \rho_0 g \Delta T k h}{\mu \kappa_T}$$

Rayleigh Number

From (2)

$$\nabla_h(\nabla_h \underline{u}) = Ra\, \nabla_h(\nabla_h \Theta\, \underline{e}_z)$$

as $\nabla \cdot \underline{u} = 0$ ↘

$$-\nabla^2 \underline{u} = Ra\left[\Theta_{xz}\,\underline{e}_x + \Theta_{yz}\,\underline{e}_y - \nabla_H^2 \Theta\, \underline{e}_z\right]$$

dot with $\underline{e}_z$ where $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$\therefore \quad \nabla^2 W = Ra\, \nabla_H^2 \Theta$$

<u>BCs</u> $W = 0 = \Theta$ at $z = 0, 1$ as perturbations zero there.

$\therefore$ Try $W = \sin(m\pi z)\, e^{\sigma t}\, e^{ipx} e^{iqy}$, $m \in \{1, 2, \ldots\}$

$$\Theta = \Theta_m \sin(m\pi z) e^{\sigma t} e^{ipx} e^{iqy}.$$

$$\therefore \quad \sigma \Theta_m - 1 = -(m^2\pi^2 + \underbrace{p^2+q^2}_{\alpha^2})\,\Theta_m$$

$$-(m^2\pi^2 + \alpha^2) = -Ra\,\alpha^2 \Theta_m$$

$$\therefore \quad \Theta_m = \frac{m^2\pi^2 + \alpha^2}{Ra\,\alpha^2},$$

growth rate: $\sigma = \frac{1}{\Theta_m} - (m^2\pi^2 + \alpha^2) = \frac{Ra\,\alpha^2}{m^2\pi^2 + \alpha^2} - (m^2\pi^2 + \alpha^2)$

Instability $\sigma > 0$. $\sigma \downarrow$ as $m \uparrow$ $\therefore$ $m = 1$ most unstable Fourier mode.

For $m=1$, $\sigma = \dfrac{Ra\,\alpha^2}{\pi^2+\alpha^2} - (\pi^2+\alpha^2)$

For α fixed, unstable if $Ra > Ra_c(\alpha)$ where

$$0 = Ra_c(\alpha)\left[\frac{\alpha^2}{\pi^2+\alpha^2}\right] - (\pi^2+\alpha^2)$$

$\therefore$ Varying over all α, unstable if

$$Ra > \min_{\alpha} Ra_c(\alpha) = \min_{\alpha}\left[\frac{(\pi^2+\alpha^2)^2}{\alpha^2}\right] = 4\pi^2$$

Double-diffusive convection

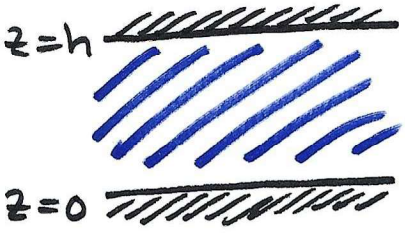
* Buoyancy where density depends on two quantities, e.g. salt and temperature

We consider

$$\rho = \rho_0\left[1-\beta_T(T-T_0)+\beta_c(C-C_0)\right]$$

salt concentration, higher concentration, fluid denser.

Salt balance $\dfrac{\partial C}{\partial t} + \underline{u}\cdot\nabla C = \kappa_c \nabla^2 C$.



$C = C_0,\ T = T_0.$

$C = C_0 + \Delta C$
$T = T_0 + \Delta T$

Static solutions

$$T_s = T_0 + \Delta T(1 - z/h)$$
$$C_s = C_0 + \Delta C(1 - z/h)$$

Hence $\rho_s = \rho_0 \left[1 + \xi \beta_c \Delta C - \beta_T \Delta T \xi (1 - z/h) \right]$

with $0 = -\frac{\partial p}{\partial z} - \rho g$

Linear stability $\underline{u} = 0 + \underline{u}'$

$$\begin{pmatrix} T \\ C \\ \rho \end{pmatrix} = \begin{pmatrix} T_s \\ C_s \\ \rho_s \end{pmatrix} + \begin{pmatrix} T' \\ C' \\ p' \end{pmatrix}$$

so that $\nabla \cdot \underline{u}' = 0$ $\quad \underline{u}' = -\frac{k}{\mu} \left[\nabla p' + \rho_0 g \left(\beta_c C' - \beta_T T' \right) \underline{e}_z \right]$.

Analogous to previously $\frac{\partial}{\partial t} \begin{pmatrix} T' \\ C' \end{pmatrix} - \frac{\omega'}{h} \begin{pmatrix} \Delta T \\ \Delta C \end{pmatrix} = \begin{pmatrix} \kappa_T \nabla^2 T' \\ \kappa_C \nabla^2 C' \end{pmatrix}$.

Non-dimensionalise, as previously, with $C' = \Delta C \cdot \chi$.

$$\nabla \cdot \underline{u} = 0, \qquad \underline{u} = -\nabla P + \left(Ra\, \Theta - \frac{Ra_s}{Le} \chi \right) \underline{e}_z$$

where $Ra_s = \frac{\beta_c \rho_0 g \Delta C k h}{\mu \kappa_c}$, $Le = \kappa_T / \kappa_c$

Solutal Rayleigh number

Lewis Number

As previously

$$\frac{\partial \Theta}{\partial \tau} - W \overset{1}{=} \nabla^2 \Theta, \quad Le\left(\frac{\partial \chi}{\partial \tau} - W\right) \overset{2}{=} \nabla^2 \chi.$$

Taking the curl twice of $\underline{u} = -\nabla P + \left(Ra\,\Theta - \frac{Ra_s \chi}{Le}\right)\underline{e}_z$,

using $\nabla \cdot \underline{u} = 0$, gives

$$\nabla^2 W \overset{3}{=} Ra\left(\nabla_H^2 \Theta - N \nabla_H^2 \chi\right)$$

$$\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \quad N = \frac{\beta_C \Delta C}{\beta_T \Delta T}.$$

Also, $W = \Theta = \chi = 0$ at $z = 0, 1$.

Let $f_m := \sin(m\pi z)\exp(\sigma t)\exp(ipx + iqy)$.

As before, let $W = f_m$ (to find Fourier mode)

$$\Theta = \theta_m f_m, \quad \chi = \chi_m f_m.$$

From 1 $\quad \sigma\theta_m - 1 = -g_m \theta_m, \quad g_m := m^2\pi^2 + p^2 + q^2 = m^2\pi^2 + \alpha^2.$

$$\therefore \theta_m = \frac{1}{\sigma + g_m}$$

From 2 $\quad Le(\sigma\chi_m - 1) = -g_m \chi_m$

$$\therefore \chi_m = \frac{Le}{\sigma Le + g_m}$$

From 3 $\quad -g_m = Ra\left(\alpha^2 \theta_m + N\alpha^2 \chi_m\right)$

$$= -\frac{Ra\,\alpha^2}{\sigma + g_m} + \frac{Ra\,N\,Le\,\alpha^2}{\sigma Le + g_m}$$

Ra_s

$$\therefore g_m(\sigma+g_m)(\sigma Le+g_m) = Ra\alpha^2(\sigma Le+g_m) - Ra_S\alpha^2(\sigma+g_m)$$

$$\underbrace{Le\, g_m}_{A}\sigma^2 + B\sigma + \underbrace{C}_{g_m^3 - Ra\, g_m\alpha^2 + Ra_S g_m\alpha^2} = 0$$

Note $A = Le\, g_m = Le(m^2\pi^2+\alpha^2) > 0$.

$$2A\,\sigma_\pm = -B \pm (B^2-4AC)^{1/2}.$$

Guaranteed instability if $C<0$, as then $\sigma_+ > 0$.

$$\therefore \quad g_m(g_m^2 - (Ra-Ra_S)\alpha^2) < 0$$

$$\therefore \quad (Ra-Ra_S) > g_m^2/\alpha^2 = \frac{(\pi^2m^2+\alpha^2)^2}{\alpha^2}.$$

Once $(Ra-Ra_S) > \min_{\alpha}\ \min_{m\in\{1,2,\ldots\}} \frac{(\pi^2m^2+\alpha^2)^2}{\alpha^2}$

$$= \min_{\alpha} \frac{(\pi^2+\alpha^2)^2}{\alpha^2} = 4\pi^2, \text{ as before}$$

guaranteed an instability.

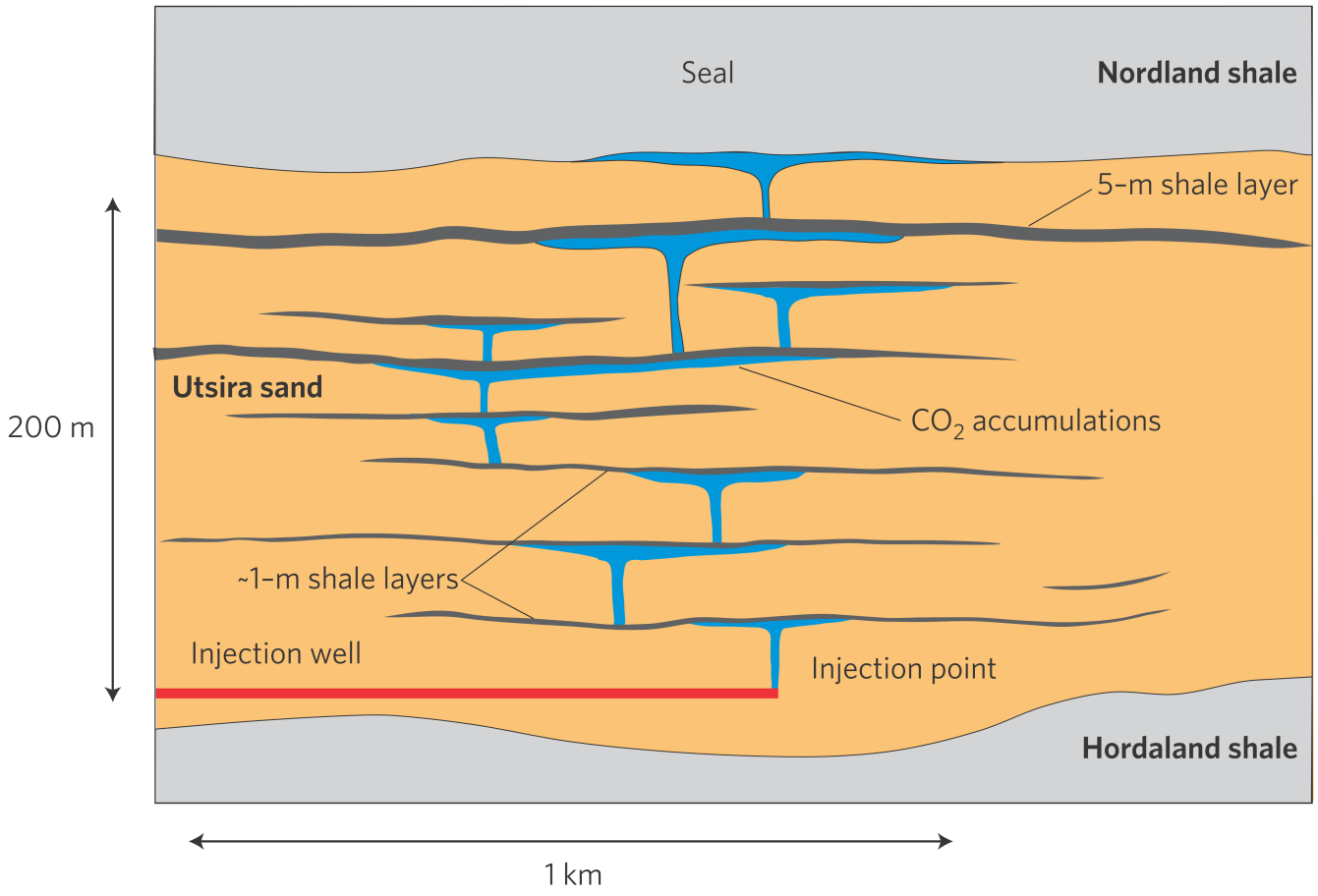
Other instabilities possible depending on parameter values.

See example sheet for details

Geological carbon storage

Mike J. Bickle

Storage of the carbon dioxide that is produced by burning fossil fuels is one way to avoid the damaging consequences of climate change. A range of observations suggests that geological carbon storage is much less risky than unabated carbon emissions to the atmosphere.

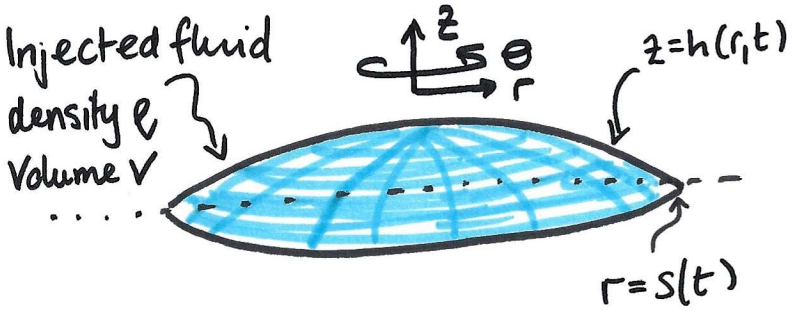


Horizontal gravity driven waves

* Finite volume of fluid, V, released above an impermeable boundary within a porous medium.

Porous medium analogue of spreading drop (no surface tension). Related to carbon storage.

Assumptions and scalings



- Axisymmetric flow
- Vertical scale H
- Horizontal scale R
- $\delta = H/R \ll 1$

- Velocity field $\underline{u} = u\underline{e}_r + w\underline{e}_z$, $u \sim U$, $w \sim H$.
- Conservation of mass $U/R \sim W/H \therefore W \sim \delta U$
- D'Arcy (horizontal) $\frac{\partial p}{\partial r} = -\frac{\mu}{k} u \therefore P \sim \mu/k \, RU.$ pressure scaling
- D'Arcy (vertical)

$$\frac{\mu W/k}{\partial p/\partial z} \sim \frac{\mu W/k}{P/H} \sim \frac{\mu/k \, U\delta}{\mu/k \, UR/H} \sim \delta^2$$

$$\therefore \quad \frac{\partial p}{\partial z} = -\rho g \quad \text{at leading order.}$$

Evolution of the flow profile

$$p(h,r) - p(z,r) = \int_z^h \frac{\partial p}{\partial z_*} dz_* = \int_z^h -\rho g \, dz_* = -\rho g (h-z)$$

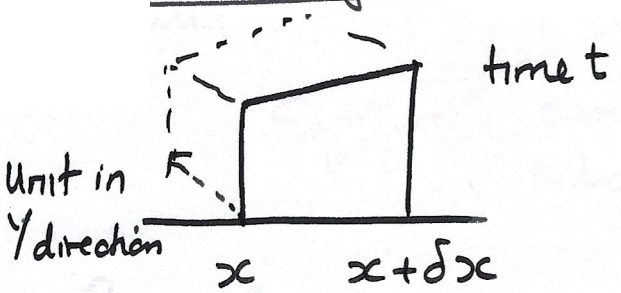
Take $p(h,r) \simeq P_0$ (const, a simplifying approximation)

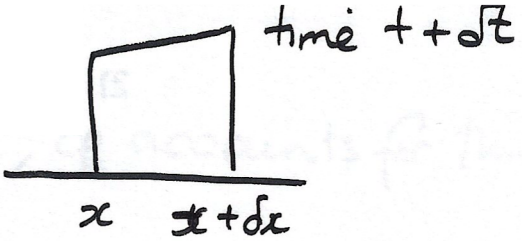
$$p(z,r) = P_0 + \rho g (h-z) .$$

Radial Darcy's Law

$$u = u_{\text{macroscale}} = -k/\mu \frac{\partial p}{\partial r} = -k/\mu \rho g \partial h/\partial r \text{ , independent of } z .$$

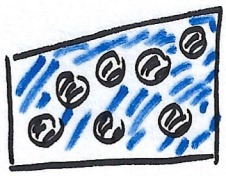
Conservation of mass





$$\left.\begin{array}{l}\text{Volume of fluid} \\ \text{entering } (x, x+\delta x) \\ \text{in time } (t, t+\delta t)\end{array}\right\} = \frac{V_f}{|V|}\left[h(x,t+\delta t)\delta x - h(x,t)\delta x\right]$$

$$= \varphi \left[\frac{\partial h}{\partial t} \delta x \delta t\right]$$



Volume of fluid entering $\} = \delta t \int_0^{h(x,t)} u_{macro}(x,t,\bar{z})\,d\bar{z}$

$$- \delta t \int_0^{h(x+\delta x,\,t)} u_{macro}(x+\delta x,\,t,\bar{z})\,d\bar{z}$$

(as previously)

$$= -\delta t \delta x \frac{\partial}{\partial x} \int_0^h u_{macro}(x,t,\bar{z})\,d\bar{z}$$

$$\therefore \quad \varphi \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h \left\{ \frac{1}{h} \int_0^h \underbrace{u_{macro}(x,t,\bar{z})\,d\bar{z}}_{-k/\mu\, \rho g\, \partial h/\partial x \text{ independent of } z} \right\} \right) = 0$$

Poiseuille ↓

$$\therefore \quad \varphi \frac{\partial h}{\partial t} = -\frac{k\rho g}{\mu} \frac{1}{r} \frac{\partial}{\partial r}\left(r h\, \partial h/\partial r \right)$$

Volume of fluid

$$V = 2\pi c \varphi \int_0^{s(t)} r h\, dr$$

(some of volume is solid phase, φ accounts for this.)

Rescale with lengthscale $(V/\varphi)^{1/3}$

timescale $(\varphi^2 V)^{1/3} \frac{\mu}{k\rho g}$

Then

$$\frac{\partial H}{\partial T} = \frac{1}{R} \frac{\partial}{\partial R}\left(R H \frac{\partial H}{\partial R} \right)$$

$$2\pi\int_0^{S(T)} R+1\, dR = 1$$

$$s(t) = \left(\frac{V}{Q}\right)^{1/3} S(T), \qquad H(S(T),T) = 0.$$

We seek long time behaviour, hence look for a similarity solution.

Consider $R = \eta T^{\alpha}$ $\qquad H(R,T) = T^{-\beta}\chi(\eta)$

Similarity variable, WLOG scaling with R as taking more R derivatives

$$\therefore \frac{\partial\chi}{\partial T} = \frac{\partial\eta}{\partial T}\frac{d\chi}{d\eta} = -\frac{\alpha}{T}\eta\frac{d\chi}{d\eta}, \quad \frac{\partial\chi}{\partial R} = \frac{d\eta}{\partial R}\frac{d\chi}{d\eta} = \frac{1}{T^{\alpha}}\frac{d\chi}{d\eta}.$$

Volume constraint

$$\frac{1}{2\pi} = \int_0^{S(T)} R+H\, dR = \int_0^{\eta_s} \eta T^{\alpha}\frac{1}{T^{\beta}}\chi(\eta)\, T^{\alpha}\, d\eta$$

with $S(T) = \eta_s T^{\alpha}$.

RHS must be independent of time, T $\therefore$ $2\alpha-\beta=0$, $\underline{\underline{2\alpha=\beta}}$.

$$\therefore R=\eta T^{\alpha}, \quad H = T^{-2\alpha}\chi$$

$$\frac{\partial H}{\partial T} = -\frac{2\alpha}{T^{1+2\alpha}}\chi - \frac{\alpha}{T^{1+2\alpha}}\eta\frac{d\chi}{d\eta}$$

$$R+\frac{\partial H}{\partial R} = (\eta T^{\alpha})\left(\frac{1}{T^{2\alpha}}\chi\right)\left(\frac{1}{T^{2\alpha}}\frac{1}{T^{\alpha}}\frac{d\chi}{d\eta}\right) = \frac{1}{T^{4\alpha}}\eta\chi\frac{d\chi}{d\eta}$$

$$\frac{\frac{\partial}{\partial R}\left(R+\frac{\partial H}{\partial R}\right)}{R} = \frac{\frac{1}{T^{4\alpha}}\frac{1}{T^{\alpha}}\frac{d}{d\eta}\left(\eta\chi\frac{d\chi}{d\eta}\right)}{\eta T^{\alpha}}$$

$$\therefore \quad -\left[2\alpha\chi + \alpha\eta\chi'\right]\frac{1}{T^{2\alpha+1}} = \frac{1}{T^{6\alpha}}\frac{1}{\eta}(\eta\chi\chi')'$$

$$\therefore \quad 2\alpha+1 = 6\alpha \quad \therefore \boxed{\alpha = 1/4,\ \beta = 1/2}$$

$$\therefore \quad -\frac{1}{4}\underbrace{\left[2\eta\chi + \eta^2\chi'\right]}_{(\eta^2\chi)'} = (\eta\chi\chi')'$$

$$\therefore \quad -\tfrac{1}{4}\eta^2\chi = \eta\chi\chi' + \text{Constant}$$

When $\eta = \eta_s$, $\chi = 0$ $\therefore$ Constant is zero

$$\therefore \quad \chi' = -\eta/4$$

$$\therefore \quad \chi = \tfrac{1}{8}(\eta_s^2 - \eta^2)$$

$$1 = 2\pi\int_0^{\eta_s}\eta\chi\,d\eta = \pi/16\,\eta_s^4 \quad \therefore \quad \eta_s = 2/\pi^{1/4}$$

$$\therefore S(T) = \left(2/\pi^{1/4}\right)T^{1/4}$$ gives extent of the fluid at large time.