Topics in Fluid Mechanics, C5.7

1.1 Navier Stokes Equations, recap

Let $\underline{u}(\underline{x}, t)$ be the velocity at location $\underline{x}$, time t.

Then

$$\frac{D\rho}{Dt} + \overset{\text{density}}{\rho}\, \nabla\cdot\underline{u} = 0 \quad , \quad \text{mass conservation}$$

$$\rho\frac{D\underline{u}}{Dt} = -\underbrace{\nabla p}_{\text{pressure}} + \underbrace{\mu}_{\text{viscosity}}\nabla^2\underline{u} + \underbrace{\underline{F}}_{\text{body force, eg. } \underline{F}=\rho\underline{g}} \quad , \quad \text{momentum conservation}$$

where $\dfrac{D}{Dt}(\cdot) = \left[\dfrac{\partial}{\partial t} + \underline{u}\cdot\nabla\right](\cdot)$.

Non-dimensionalise

$$\underline{u}' = U\underline{u} \;,\; t' = \frac{L}{U}t \;,\; \underline{x}' = L\underline{x}$$

$$p' = p^* p \;,\; \underline{F}' = p^*/_L\, \underline{F}$$

$$(Re)\frac{D\underline{u}'}{Dt'} = \left(\frac{p^* L}{\mu U}\right)\left(-\nabla' p' + \underline{F}'\right) + \nabla'^2\underline{u}'$$

Reynolds number $= \dfrac{\rho U^2/L}{\mu U/L^2} = \dfrac{\rho U L}{\mu}$

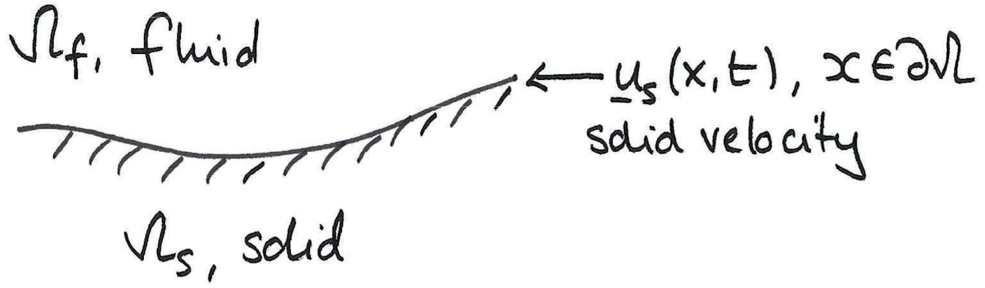
$Re \ll 1$, viscous dominated. Let $p^* = \mu U/L$ for balance

$Re \gg 1$, inertia dominated. Let $p^* = (Re)\dfrac{\mu U}{L}$ for balance.

Drop Primes henceforth

Boundary conditions

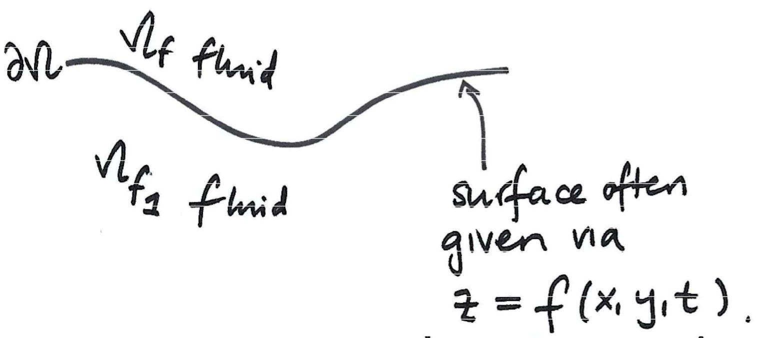
Boundary with a solid, with known motion



Then $\quad \underline{u}(x,t) = \underline{u}_S(x,t), \; x \in \partial\Omega.$

- Tangential components, no slip BC
- Normal Component, kinematic BC

Boundary with a fluid



Continuity of velocity at $\partial\Omega$

$$\underline{u}(x,t) = \underline{u}_{f_1}(x,t), \; x \in \partial\Omega$$

Then kinematic condition often enforced via

$$\underline{u}\cdot\underline{e}_z\Big|_{\Omega} \equiv w\Big|_{\Omega} = \frac{Df}{Dt} = \partial f/\partial t + u\,\partial f/\partial x$$

Stress boundary conditions also required ... will return to this later.

When $Re \ll 1$, Stokes Equations

$$0 = -\nabla p + \underline{F} + \nabla^2 \underline{u}$$

$Re \gg 1$, Eulers Equations $\quad \dfrac{D\underline{u}}{Dt} = -\nabla p + \underline{F}$

<u>Boussinesq Approximation</u>: changes in density negligible except when coupled to $\underline{g}$ in a buoyancy body force

$$\therefore \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = \rho \nabla \cdot \underline{u} = 0$$ and we have

$\nabla \cdot \underline{u} = 0$, fluid incompressibility

<u>Two dimensional flows</u>: the streamfunction (ψ).

<u>Planar flows</u>

$\underline{u}(x, y) = (u, v, 0)$ and we let $u = \dfrac{\partial \psi}{\partial y}$, $v = -\dfrac{\partial \psi}{\partial x}$

This ensures $\nabla \cdot \underline{u} = 0$.

Analogous streamfunctions for axisymmetric flows are in the on-line notes

1.2 Other conservation laws

- Concentration of a dissolved solute

$$\frac{DC}{Dt} = \nabla \cdot (\kappa \nabla C)$$

diffusion coefficient
$\boxed{\text{(often constant)}}$

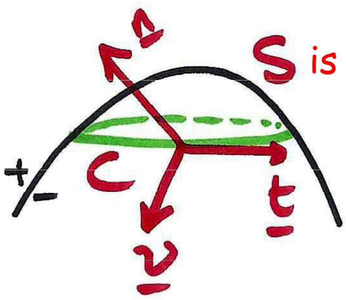
- Thermal energy

$$\frac{DT}{Dt} = \nabla \cdot (\kappa_T \nabla T)$$

neglecting viscous dissipation and any other heat sources (and taking $\nabla \cdot \underline{u} = 0$).

2.1 Surface Tension

Stress conditions at a fluid-fluid interface



S is the surface cap

Fluid interface e.g. water/oil

Surface tension: force at interface in direction $\underline{\nu}$, of magnitude γ per unit length.

Extend S to the interval $(-\epsilon, \epsilon)$ in the normal direction $\underline{n}$ to form the volume V.

Force balance

$\underline{t}^+$ the force per unit area the upper fluid exerts on S

$$\underbrace{\int_V \rho \frac{D\underline{u}}{Dt}\, dV = \int_V \underset{\text{body force}}{\underline{f}}\, dV}_{\to 0 \text{ as } \epsilon \to 0} + \int_S (\underline{t}^+ + \underline{t}^-)\, dS + \int_C \gamma \underline{\nu}\, ds$$

$\boxed{\underline{t}^- \text{ the force per unit area the lower fluid exerts on S}}$

ds: arclength

$\therefore$ Let $\epsilon \to 0$,

$$0 = \int_S (\underline{t}^+ + \underline{t}^-)\, dS + \int_C \gamma \underline{\nu}\, ds$$

The stress tensor is

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \mu\left[\nabla\underline{u} + (\nabla\underline{u})^T\right]$$

Hence $\quad \underline{t}^+ = \underline{n}\cdot\underline{\underline{T}}^+, \quad \underline{t}^- = -\underline{n}\cdot\underline{\underline{T}}^-$

Define γ, $\underline{n}$ off-surface by an invariant extension in direction $\underline{n}$

Then (Q1, Problem Sheet 1)

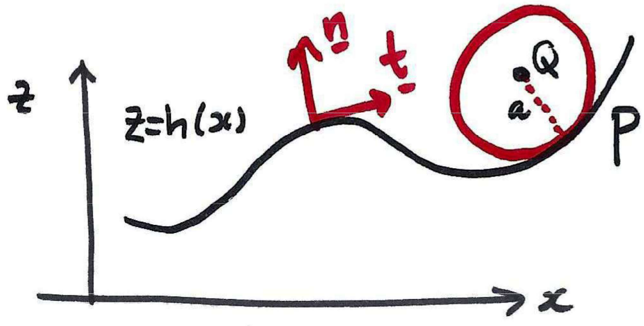
$$\int_C \gamma \underline{\nu}\, ds = \int_S \left[\nabla\gamma - (\nabla\cdot\underline{n})\underline{n}\gamma\right] dS$$

With $\quad \nabla_s := \nabla - \underline{n}\dfrac{\partial}{\partial n}, \quad \nabla\gamma = \nabla_s\gamma$

$$\therefore \quad 0 = \int_S \left(\underline{n}\cdot\underline{\underline{T}}^+ - \underline{n}\cdot\underline{\underline{T}}^- + \nabla_s\gamma - \gamma\nabla\cdot\underline{n}\,\underline{n}\right) dS, \quad \text{all } S.$$

$$\therefore \quad \boxed{\underline{n}\cdot\underline{\underline{T}}^+ - \underline{n}\cdot\underline{\underline{T}}^- = \gamma(\nabla\cdot\underline{n})\underline{n} - \nabla_s\gamma} \quad x \in S.$$

Curvature (2D)



In 2D the magnitude of the curvature at P is the reciprocal of the best fit circle radius, a.

With the surface $z = h(x)$, the unit normal and tangent are

$$\underline{n} = \frac{(-h', 1)}{(1+h'^2)^{1/2}} \qquad \underline{t} = \frac{(1, h')}{(1+h'^2)^{1/2}}$$

Recall $\underline{n}$ defined off-surface by invariant normal extension

Then

$$\frac{1}{a} = \frac{d}{dx}\left(\frac{h'}{(1+h'^2)^{1/2}}\right) = -\nabla\cdot\underline{n} = \frac{h''}{(1+h'^2)^{3/2}}$$

[as shown in Appendix of on-line notes]

More generally (see Appendix), in 2D or 3D, the curvature, κ, is given by

$$\kappa = -\nabla\cdot\underline{n}$$

Hence in 3D, for a surface $z = h(x,y)$,

$$\kappa = -\nabla\cdot\underline{n} \quad \text{with} \quad \underline{n} = \frac{\nabla(z-h(x,y))}{|\nabla(z-h(x,y))|}$$

so that $$\kappa = \frac{h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_x h_y h_{xy}}{(1+h_x^2+h_y^2)^{3/2}}.$$

Capillary Statics

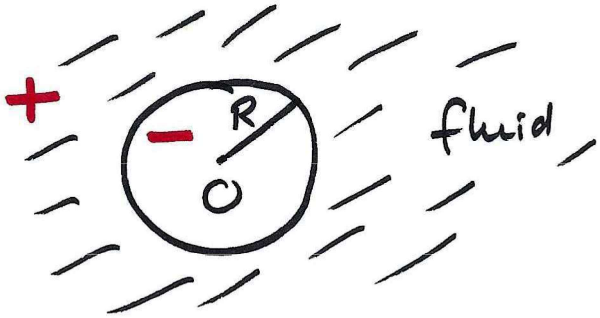
For static configurations, $\underline{u}=\underline{0}$, $\underline{\underline{T}}=-p\underline{\underline{I}}$, $\underline{n}\cdot\underline{\underline{T}}=-p\underline{n}$

Normal stress BC reduces to $\quad p^+-p^-=\gamma\kappa$

Tangential stress BC

$$0=-\nabla_s\gamma.$$

Stationary Bubble



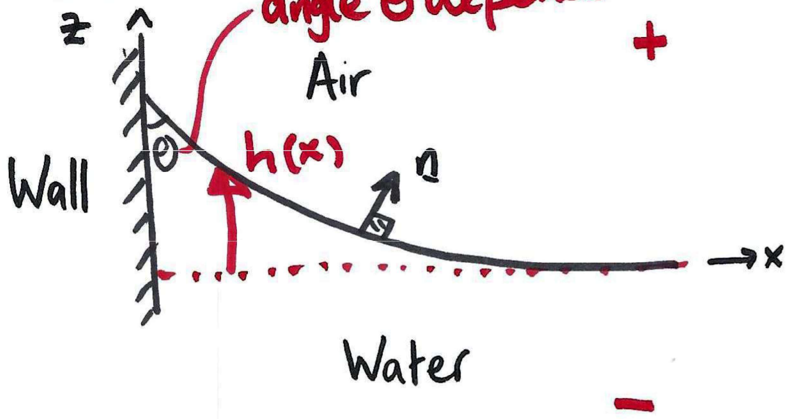
$$\text{Curvature}=\kappa=-\nabla\cdot\underline{n}\big|_{r=R}$$
$$=-\nabla\cdot\left(\frac{\underline{r}}{r}\right)\Big|_{r=R}$$
$$=-\frac{1}{r^2}\frac{d}{dr}\left(r^2\cdot 1\right)\Big|_{r=R}=-2/R.$$

$$\therefore\quad p^--p^+=2\gamma/R$$

Smaller bubbles have higher internal pressure. Thus they are louder when they burst at a free surface ∴ Champagne fizz louder than beer.

Static meniscus

angle θ depends on water/air/wall properties

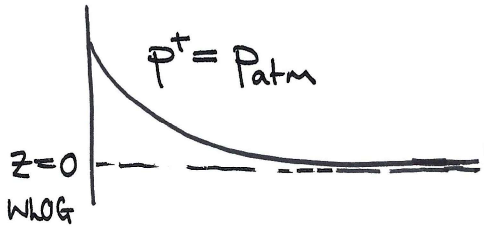


$P_{air}=p^+=P_{atm}$, constant

atmospheric pressure

Navier-Stokes ($\underline{u}=0$).

$$0 = -\nabla p^- + \rho \underline{g} = -\nabla p^- - \rho g \underline{e}_z \therefore \frac{\partial p^-}{\partial x} \overset{①}{=} 0, \quad \frac{\partial p^-}{\partial z} \overset{②}{=} -\rho g$$



here curvature of surface $\simeq 0$

$$p^+ - p^- = \gamma\kappa \simeq 0$$

$$\lim_{x\to\infty} p^-(x,0) = p^{atm}$$

$$\left.\begin{aligned} \therefore\ & p^-(x,0) \overset{\text{by } ①}{=} p^{atm}, \text{ all } x \\ \therefore\ & p^-(x,h) \overset{②}{=} p^{atm} - \rho g h \end{aligned}\right\} \therefore \boxed{\rho g h = \gamma\kappa}$$

Laplace-Young Eqn

Recall $\kappa = \dfrac{h_{xx}}{(1+h_x^2)^{3/2}}$ $\quad\therefore\quad h = \left(\dfrac{\gamma}{\rho g}\right)\dfrac{h_{xx}}{(1+h_x^2)^{3/2}}$

gives shape of meniscus.

BCs $\quad h \to 0$ as $x \to \infty$

$h_x \to -\cot\theta$ as $x \to 0$, for contact angle to be θ.

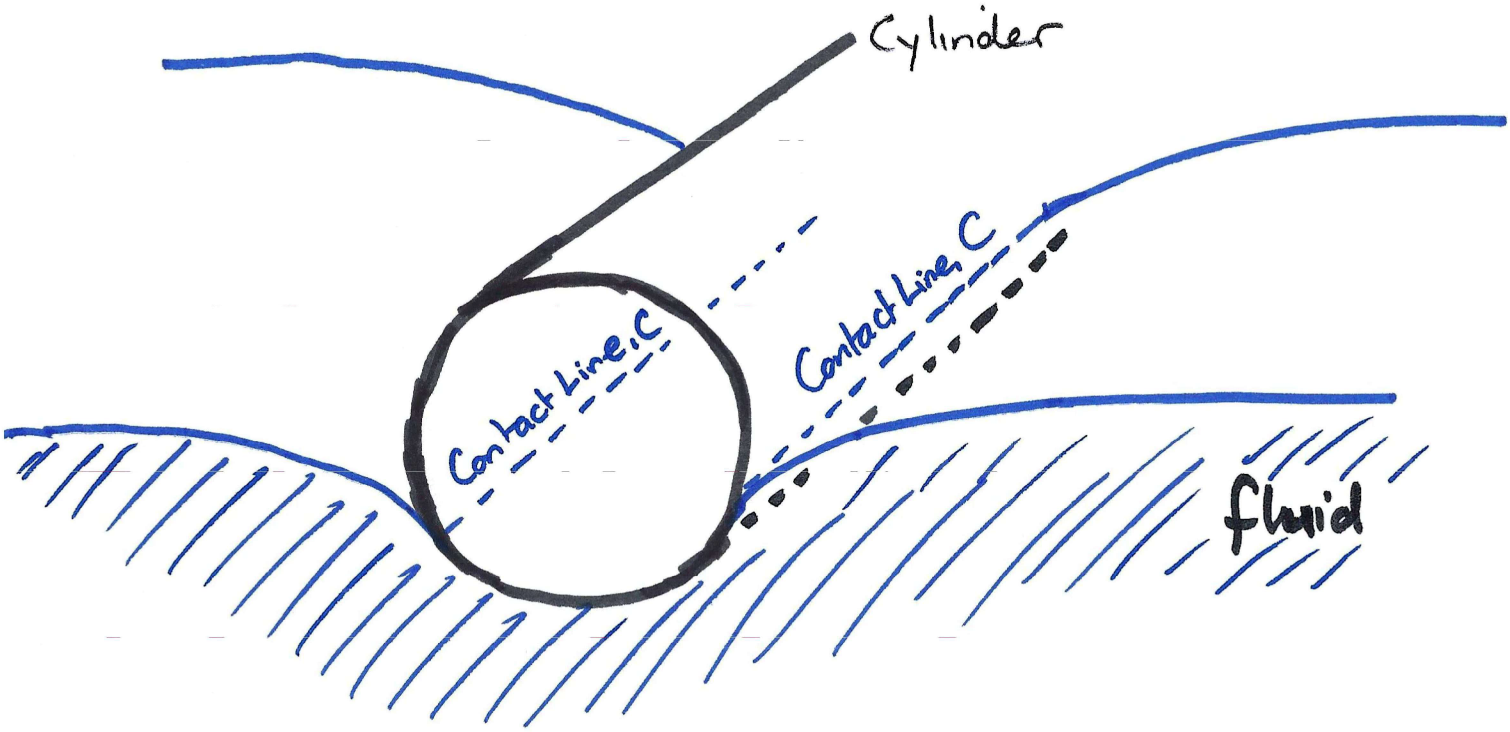
For $\theta \simeq \pi/2$, $|h_x| \ll 1$

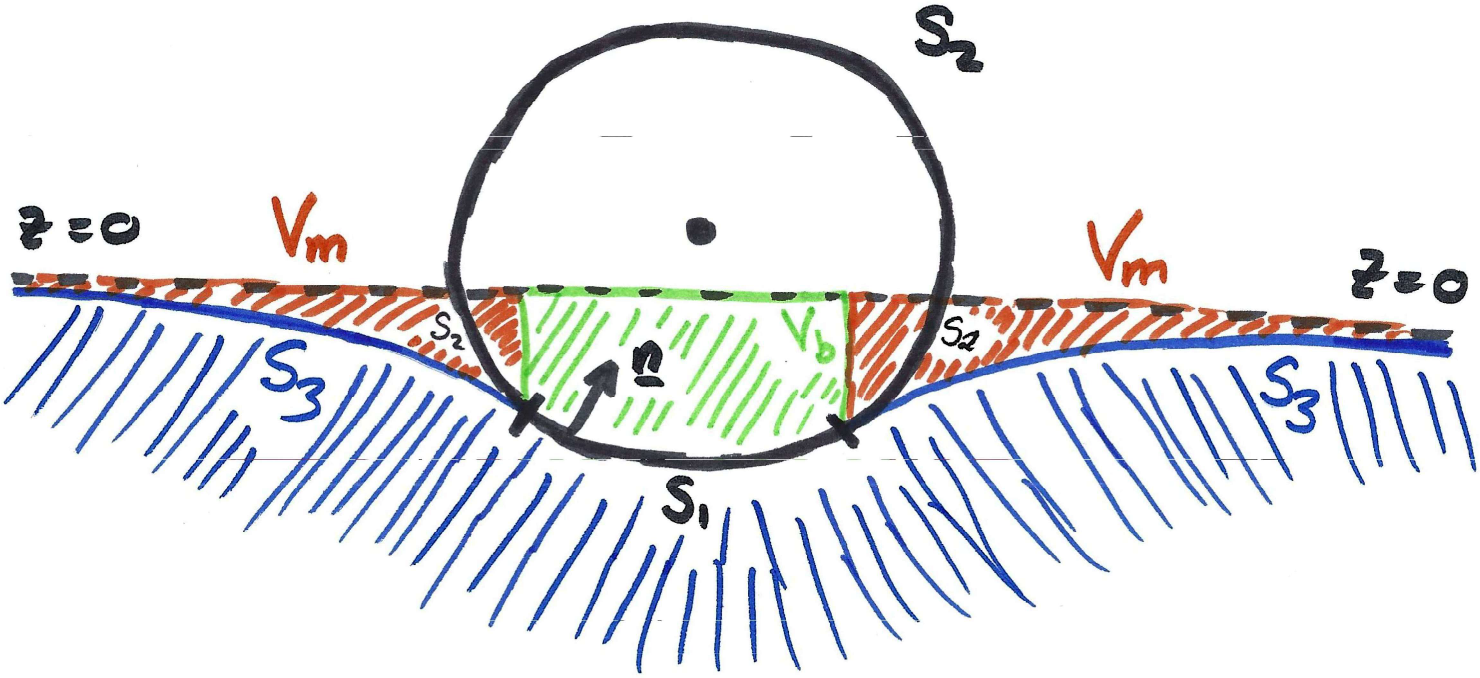
$$\boxed{\begin{aligned} & h = \left(\frac{\gamma}{\rho g}\right) h_{xx} \\ & h \to 0 \text{ as } x \to \infty \\ & h_x \to -\cot\theta \text{ as } x \to 0. \end{aligned}}$$

Solving $\quad h(x) = l_c \cot\theta \, e^{-x/l_c}$, $\quad \underbrace{l_c = \sqrt{\frac{\gamma}{\rho g}}}_{\text{capillary length}}$

For air-water $l_c \doteq 3\text{mm}$

Floating Bodies : Supported by buoyancy & surface tension





V_m : Volume displaced by meniscus

V_b : Volume additionally displaced by body

Surface tension between fluid and air, γ, and fluid density, ρ, are constant.

As previously, the fluid pressure is

$$p = -\rho g z + P_{atm} \quad \text{since } \underline{u} = 0.$$

Force on body from fluid is

$$\underline{F} = \underbrace{\int_{S_1} p\underline{n}\, dS}_{P_{atm}\int_{S_1}\underline{n}\, dS - \rho g\int_{S_1} z\underline{n}\, dS} + P_{atm}\int_{S_2}\underline{n}\, dS + \underbrace{\int_C \gamma \underline{\nu}\, ds}_{\int_{S_1}\gamma\kappa\underline{n}\, dS}$$

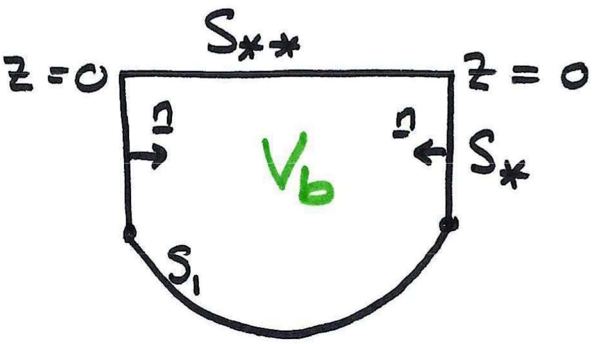
$$\therefore \underline{F} = \cancel{P_{atm}\int_{S_1\cup S_2}} \xrightarrow{} 0 \ \underline{n}\, dS - \rho g\int_{S_1} z\underline{n}\, dS + \int_{S_1}\gamma\kappa\underline{n}\, dS$$

F+Mg=0 as no net force

$$\therefore \quad \underline{F} = -\rho g\int_{S_1} z\underline{n}\, dS + \int_{S_1}\gamma\kappa\underline{n}\, dS = -M\underline{g},$$ force due to gravity.

g =-ge_z for minus sign

$$\therefore Mg = \underbrace{\underline{e}_z\cdot\left[-\rho g\int_{S_1} z\underline{n}\, dS\right]}_{F_b,\ \text{buoyancy force}} + \underbrace{\underline{e}_z\cdot\left[\int_{S_1}\gamma\kappa\underline{n}\, dS\right]}_{F_c,\ \text{surface tension force}}$$



$$\underline{e}_z\cdot\int_{S_{**}} z\,\underline{n}\,dS = \underline{e}_z\cdot\int_{S_*} z\,\underline{n}\,dS = 0$$

Let $S_1^* = S_1 \cup S_* \cup S_{**}$

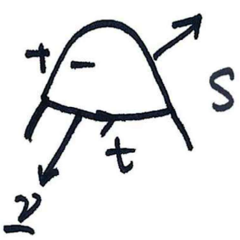
$$F_b = -\rho g\,\underline{e}_z\cdot\int_{S_1} z\,\underline{n}\,dS = -\rho g\,\underline{e}_z\cdot\int_{S_1^*} z\,\underline{n}\,dS$$

$$= +\rho g\int_{V_b} dV = \left\{\begin{array}{l}\text{Weight of fluid}\\ \text{displaced from } V_b\end{array}\right\}$$

Surface Tension Force

$\underline{\nu} \perp \underline{t}, \underline{n}$

$\underline{\nu} = \underline{t} \wedge \underline{n}$



Going from point (i) on (1) to point (ii) on (2) $\Big\}$ $\underline{t},\underline{n} \to -\underline{t},\underline{n}$ $\therefore\ \underline{\nu} \to -\underline{\nu}$

$$\therefore \underbrace{\int_{S_1\cup S_3}}_{\text{per unit length}} \kappa\,\underline{n}\,dS = \underbrace{\int_{\partial(S_1\cup S_3)}}_{\text{per unit length}} \underline{\nu}\,ds = 0$$

by cancellation between (1) and (2)

$\therefore$ Noting γ constant $\quad \int_{S_1}\gamma\kappa\,\underline{n}\,dS = -\int_{S_3}\gamma\kappa\,\underline{n}\,dS$

$$\therefore\ F_c = \underline{e}_z\cdot\int_{S_1}\gamma\kappa\,\underline{n}\,dS = -\underline{e}_z\cdot\int_{S_3}\gamma\kappa\,\underline{n}\,dS$$

On S_3 surface tension and buoyancy balance.

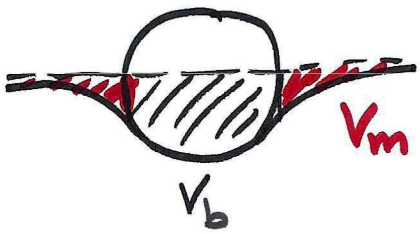
$\therefore$ Laplace Young $\rho g z = \gamma \kappa$

$$F_c = -\underline{e}_z \cdot \int_{S_3} \rho g \underline{n} z \, dS = \rho g \int_{V_m} dV$$

$$= \left\{ \begin{array}{l} \text{Weight of fluid} \\ \text{displaced by meniscus} \end{array} \right\}$$

$\therefore$ Force on body = Weight of displaced fluid

Overall Scales



$V_b \sim \text{depth} \cdot L^2$

$V_m \sim \text{depth} \cdot L \cdot l_c$

where L is the body lengthscale

meniscus lengthscale $l_c \sim \sqrt{\dfrac{\gamma}{\rho g}}$

$$\therefore \frac{F_b}{F_c} \sim \frac{V_b}{V_m} \sim L\sqrt{\frac{\rho g}{\gamma}}$$

$\therefore$ Once $L \ll \sqrt{\dfrac{\gamma}{\rho g}}$ floating bodies primarily supported by surface tension.