

Gödel Incompleteness Theorems: Problem sheet 3

A.

1. Show that consistency is strictly weaker than 1-consistency.

2. (i) Show how to construct a sentence, using the Diagonal Lemma, that “says”, “this sentence, when added to PA, results in a system that is ω -inconsistent”.

Hint: We will use a notation (due to Feferman) of putting a dot over a free variable in a formula that occurs within $\ulcorner \urcorner$ to signify that the Gödel number is of a formula that varies with the values of that variable, rather than being the Gödel number of the formula in which that variable occurs as a part (for example, $\forall v' \text{Pr}_{\text{PA}}(\ulcorner \neg 0 = \dot{v}' \urcorner)$ says that for each number n , $\text{Pr}_{\text{PA}} \ulcorner \neg 0 = \overline{n} \urcorner$ holds, while $\text{Pr}_{\text{PA}}(\ulcorner \neg = v' \urcorner)$, with no dot, says that a particular formula $\neg v'^+$ is provable, and says it by referring to its Gödel number; since $\ulcorner \neg v'^+ \urcorner$ is just a number, $\text{Pr}_{\text{PA}}(\ulcorner \neg = v' \urcorner)$ does not contain v' as a free variable, and so $\forall v' \text{Pr}_{\text{PA}}(\ulcorner \neg 0 = v' \urcorner)$ (still with no dot) has exactly the same meaning).

With this notation, show that there is a formula of $\mathcal{L}$ which expresses the following, where $m P n$ is the relation “ m is a part of n in base 13”:

$$\exists v_2 ((\ulcorner v_4 \urcorner P v_2) \wedge \neg(\ulcorner \forall v_4 \urcorner P v_2) \wedge \text{Pr}(\ulcorner (E_{v_1} \rightarrow \exists v_4 E_{v_2}) \urcorner) \wedge \forall v_3 \text{Pr}(\ulcorner (E_{v_1} \rightarrow \neg E_{v_2}[\dot{v}_3]) \urcorner)).$$

Note that v_1 is free in this expression.

B.

3. Show that if a system S is Σ_0 -complete and ω -consistent, then it is Σ_2 -sound.

(i) Prove that the result in the last problem but one is the best possible, in the sense that there exists a system S that is ω -consistent and which proves a false Σ_3 -sentence. (Assume that PA is true in $\mathbb{N}$.)

4. (i) Show that every finite subset of the axioms of R has a finite model.

Hint: consider modular arithmetic.

(ii) Show that R is not finitely axiomatisable.

(iii) Show that Q is a proper extension of R .

(iv) Show that PA is a proper extension of Q .

C.

5. (i) Show that if a theory S is ω -consistent, then at least one of $S \cup \{X\}$ and $S \cup \{\neg X\}$ is ω -consistent.

(ii) Show that there is one and only one complete ω -consistent extension of PA. Take as given that PA is sound.

(This extension will be the theory of $\mathbb{N}$, which we know not to be definable.)

(iii) Explain why the following complete extension S of PA is not ω -consistent. Let $\{X_n : n \in \mathbb{N}\}$ be a listing of all sentences of $\mathcal{L}$. Let K be a sentence such that K is false and $\text{PA} \cup \{K\}$ is ω -consistent, and let S_0 be $\text{PA} \cup \{K\}$. Let S_{n+1} be $S_n \cup \{X_n\}$ if $S_n \cup \{X_n\}$ is ω -consistent, otherwise let S_{n+1} be $S_n \cup \{\neg X_n\}$. For each i , S_i is ω -consistent by part (i). Let $S = \bigcup_{n \in \mathbb{N}} S_n$.