**Professor Joyce** **C3.4 Algebraic Geometry** **MT 2026**

**PRELIMINARY VERSION. MAY BE UPDATED.**
**Problem Sheet 2**

## Section A: introductory questions, not for handing in

1. **Projective space is locally affine.**

   For $0 \leq j \leq n$, consider the map $\phi_j : U_j \to \mathbb{A}^n$ defined in lectures, mapping the $j$-th affine coordinate patch $U_j \subset \mathbb{P}^n$ to affine $n$-space. Using the Zariski topology of projective varieties on $\mathbb{P}^n$ restricted to $U_j$, and the Zariski topology of affine varieties on $\mathbb{A}^n$, show that $\phi_j$ is a homeomorphism of topological spaces.

2. **Projective closures and affine cones**

   (a) Let $X$ be the parabola $\mathbb{V}(y - x^2) \subset \mathbb{A}^2$. What is its projective closure $\bar{X} \subset \mathbb{P}^2$? Draw the affine cone $\hat{\bar{X}}$ over $\bar{X}$, in $\mathbb{A}^3$, and identify the line corresponding to the "point at infinity" on $\bar{X}$.

   (b) Show that the affine varieties $\mathbb{V}(y - x^2) \subset \mathbb{A}^2$ and $\mathbb{V}(y - x^3) \subset \mathbb{A}^2$ are isomorphic. Can you give an intuitive explanation why their two projective closures in $\mathbb{P}^2$ are *not* projectively equivalent (isomorphic)? We do not have the tools yet to prove that they are non-isomorphic; try and find an intuitive reason.

## Section B: core questions, to be handed in for marking

3. **The Twisted Cubic.** The projective variety $C = \mathbb{V}(F_0, F_1, F_2) \subset \mathbb{P}^3$, where

$$
\begin{aligned}
F_0(z_0, z_1, z_2, z_3) &= z_0 z_2 - z_1^2 \\
F_1(z_0, z_1, z_2, z_3) &= z_0 z_3 - z_1 z_2 \\
F_2(z_0, z_1, z_2, z_3) &= z_1 z_3 - z_2^2,
\end{aligned}
$$

is known as the *twisted cubic.*

(a) Show that $C$ is equal to the image of the Veronese map

$$
\begin{aligned}
\nu : \mathbb{P}^1 &\to \mathbb{P}^3 \\
\nu : [x_0 : x_1] &\mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3].
\end{aligned}
$$

(b) Restrict to the affine patch $U_0 \subset \mathbb{P}^3$ given by setting $z_0 = 1$. Show that $C \cap U_0$ is equal to $\mathbb{V}(f_0, f_1) \subset \mathbb{A}^3$, where $f_i(z_1, z_2, z_3) := F_i(1, z_1, z_2, z_3)$ for $i = 1, 2$.

(c) For $i = 0, 1, 2$ we write $Q_i$ for the quadric hypersurface $\mathbb{V}(F_i) \subset \mathbb{P}^3$. Show that, for $i \neq j$, the hypersurfaces $Q_i$ and $Q_j$ intersect in the union of $C$ and a line. Therefore no two of them alone may be used to define $C$. Deduce that the homogenizations of the generators of an affine ideal do not necessarily generate the homogeneous ideal of the projective closure, showing that indeed we need to homogenise *all* elements of the affine ideal.

4. **Veronese varieties**

(a) Show that any projective variety is isomorphic to the intersection of a Veronese variety with a linear space, the projectivisation $\mathbb{P}(V)$ of some $k$-vector subspace $V \subset k^{n+1}$.

(b) Deduce that any projective variety is isomorphic to an intersection of quadric hypersurfaces.

5. **The ruled surface.** The image of the Segre morphism $\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \Sigma_{1,1} \subset \mathbb{P}^3$ is known as the *ruled surface.*

(a) What equations define $\Sigma_{1,1}$ as a subvariety of $\mathbb{P}^3$?

(b) What are the images in $\Sigma_{1,1}$ of $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$? Show that through any point in $\Sigma_{1,1}$ there are two lines lying in $\Sigma_{1,1}$.

(c) Exhibit some disjoint lines in $\Sigma_{1,1}$. Recall that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma_{1,1}$. Is this isomorphic to $\mathbb{P}^2$? Draw the "real cartoons" of either surface.

6. **Rational normal curves** Fix a positive integer $d > 1$.

(a) Let $G(x_0, x_1) = \Pi_{i=1}^{d+1}(b_i x_0 - a_i x_1)$ be a homogeneous degree $(d+1)$ polynomial with distinct roots $[a_i : b_i] \in \mathbb{P}^1$. Show that $H_i(x_0, x_1) = G(x_0, x_1)/(b_i x_0 - a_i x_1)$ form a basis for the space of homogeneous polynomials of degree $d$.

(b) Deduce that the image of the map $\mu_d \colon \mathbb{P}^1 \to \mathbb{P}^d$ defined by

$$[x_0 : x_1] \mapsto [H_1(x_0, x_1) : \cdots : H_{d+1}(x_0, x_1)]$$

is projectively equivalent to the image of the Veronese embedding, that is, it is a rational normal curve of degree $d$.

(c) What is the image of the point $[a_i : b_i]$ under $\mu_d$? If $(a_i, b_i)$ are nonzero for all $i$, what is the image of $[1 : 0]$ and $[0 : 1]$?

(d) Deduce that through any $d + 3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve of degree $d$. (Recall that for $d + 3$ points to be in general position means that no subset of $d + 1$ of these points lies on a hyperplane in $\mathbb{P}^d$.)

## Section C: optional questions

7. **Projective variety corresponding to a graded ring.** If $R = \sum_{d \geq 0} R_d$ is a graded ring and $e \geq 1$ is an integer, we define

$$R^{(e)} := \sum_{d \geq 0} R_{de}.$$

We define a grading on $R^{(e)}$ by letting $R^{(e)}_d := R_{de}$.

(a) Find $k[x_0, x_1]^{(2)}$, expressing it in the form $k[z_0, \ldots, z_n]/I$ for some $n$ and $I$.

(b) Find the homogeneous coordinate rings $S(\mathbb{P}^1)$ and $S(\nu_2(\mathbb{P}^1))$. Comment in the context of part (a).

(c) More generally, show that $S(\nu_e(\mathbb{P}^n)) \cong k[x_0, \ldots, x_n]^{(e)}$, and hence that $k[x_0, \ldots, x_n]^{(e)}$ defines the same projective variety as $k[x_0, \ldots, x_n]$.

(d) Are $k[x_0, \ldots, x_n]^{(e)}$ and $k[x_0, \ldots, x_n]$ isomorphic as graded $k$-algebras? Are they isomorphic as (ungraded) $k$-algebras? What does this imply about the affine cones of $\nu_e(\mathbb{P}^n)$ and $\mathbb{P}^n$?