Professor Joyce **C3.4 Algebraic Geometry** **MT 2026**

Problem Sheet 1

Throughout this problem sheet, k denotes an algebraically closed field.

Section A: introductory questions, not for handing in

1. **Zariski topology**

   Verify that arbitrary intersections and finite unions of affine varieties are affine varieties. Deduce that the Zariski topology on an affine variety is indeed a topology.

2. **Irreducibility**

   (a) Show that affine n-space $\mathbb{A}^n_k$ is irreducible.

   (b) Show that an affine variety $X \subset \mathbb{A}^n_k$ is irreducible if and only if every non-empty open subset $U \subset X$ is dense in the Zariski topology.

   (c) Let X be an irreducible affine variety. Show that any two non-empty open sets intersect in a non-empty open dense set.

   Examples of affine varieties

3. Let $X \subseteq \mathbb{A}^3_k$ be the set

$$X := \{(t, t^2, t^3) \,|\, t \in k\}.$$

   Show that X is an affine variety and that it is isomorphic to $\mathbb{A}^1_k$ as an affine variety. Provide generators for $\mathbb{I}(X)$.

4. (a) Let $X \subseteq \mathbb{A}^2_k$ be the set of solutions of the equation $y = x^2$. Show that X is isomorphic to $\mathbb{A}^1_k$ as an affine variety.

   (b) Let $Y \subseteq \mathbb{A}^2_k$ be the set of solutions of the equation $xy = 1$. Show that X is not isomorphic to $\mathbb{A}^1_k$ as an affine variety.

   (c) (Difficult.) Let $P(x, y) \in k[x, y]$ be an irreducible quadratic polynomial and let $Z \subseteq \mathbb{A}^2_k$ be the set of zeroes of $P(x, y)$. Show that Z is isomorphic to one of the affine varieties defined in (a) and (b).

5. Let $X \subseteq \mathbb{A}^3_k$ be the affine variety described by the ideal $(x^2 - yz, xz - x)$. Show that X has three irreducible components. Find generators for the radical (actually prime) ideals associated with these components.

Section B: core questions, to go over in class (do not hand in)

6. **The Zariski topology in low dimensions**

   (a) List the open and closed subsets of $\mathbb{A}^1_k$ in the Zariski topology.

   (b) Describe carefully the Zariski closed subsets of $\mathbb{A}^2_k$, proving your statements.

   (c) Show that the Zariski topology on $\mathbb{A}^2_k$ is *not* the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$. (Needs k to be an infinite field.)

7. **Reduced algebras as coordinate rings**

   (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for ideals I, J of a finitely generated k-algebra R.

   (b) Show that the ideal $(xy, xz) \subset k[x, y, z]$ is radical but not prime. Sketch the variety it defines in $\mathbb{A}^3_k$.

   (c) Show that a radical ideal I in $k[x_1, \ldots, x_n]$ is the intersection of all the maximal ideals $\mathfrak{m}$ containing it[1].

8. **The pull-back map between coordinate rings.**

   Suppose that $F : X \to Y$ is a morphism of affine varieties over a field k, associated to a map $F^* : k[Y] \to k[X]$ between their coordinate rings.

   (a) Show that F^* is injective if and only if F is dominant, i.e. the image set $F(X)$ is dense in Y.

   (b) Show that F^* is surjective if and only if F defines an isomorphism between X and some algebraic (closed) subvariety of Y.

   (c) Find an example where F is injective but F^* is not surjective.

9. **The affine normal curve.** Consider the homomorphism of rings

$$F^* : k[x_1, \ldots, x_n] \to k[t]$$

   given by $x_i \mapsto t^i$.

   (a) Show that the corresponding morphism of affine varieties $F : \mathbb{A}^1_k \to \mathbb{A}^n_k$ defines an isomorphism between $\mathbb{A}^1_k$ and its image under F.

   (b) Find generators for the radical ideal defining the image of F in $\mathbb{A}^n_k$.

[1] Hint: using methods of this course, it is easier to first translate this into a geometrical statement, and prove that. For an algebraic proof, you might find helpful the following theorem due to Krull: the nilradical $\mathrm{nil}(A) = \{x : x^m = 0 \text{ some } m\}$ of a ring A equals the intersection of all its prime ideals.

10. **A reducible variety.** Consider the ideal

$$J = \langle uw - v^2, u^3 - vw \rangle$$

in the ring $k[u, v, w]$, and the corresponding affine variety $X = \mathbb{V}(J) \subset \mathbb{A}^3_k$.

(a) By taking suitable combinations of the generators, show that J is not prime.

(b) Show that X is a reducible variety, which decomposes as

$$X = X_1 \cup X_2$$

with one component, say X_1, isomorphic to the affine line $\mathbb{A}^1_k$.

(c) Show that the other piece X_2 is the image of a map $\mathbb{A}^1_k \to \mathbb{A}^3_k$ defined by $t \mapsto (t^a, t^b, t^c)$ for some positive integers a, b, c. Deduce that X_2 is irreducible.

Section C: optional questions

11. **The disjoint union of affine varieties** Show that a variety $X \subset \mathbb{A}^n_k$ is a union of two disjoint closed subvarieties if and only if its coordinate ring $k[X]$ may be written as the product of two non-trivial finitely generated reduced k-algebras[2].

12. **The variety of nilpotent matrices** We work in the affine space $\mathbb{A}^4_k$ parametrising 2×2 matrices over k, with variables being the matrix entries x_{ij}.

    (a) Prove that the following conditions are equivalent for a 2×2 matrix A over a field k:

    - A is *nilpotent*: there exists an $n \geq 1$ such that $A^n = 0$;
    - $A^2 = 0$;
    - $\det A = \operatorname{tr} A = 0$.

    Let $I \lhd R = k[x_{11}, x_{12}, x_{21}, x_{22}]$ be the ideal formed by the polynomials $d = \det A, t = \operatorname{tr} A$, viewed as polynomials in the matrix entries. Let $J \lhd R$ be the ideal formed by the entries of A^2, as polynomials in the matrix entries. Show the following.

    (b) The ideal J is not radical: it contains a power of t but not t itself.

    (c) The ideal I is radical[3].

    (d) Deduce that $X = \mathbb{V}(I) = \mathbb{V}(J) \subset \mathbb{A}^4_k$ with $\sqrt{J} = I$, and conversely $\mathbb{I}(X) = I$.

13. Let $X = \mathbb{A}^1_k$, and Y be the affine variety $\{(x, y) \in \mathbb{A}^2 : x^4 - y^3 = 0\}$. Define $F : X \to Y$ by $F(t) = (t^3, t^4)$. Is F an isomorphism?

14. **(C3.4 2022 q. 1.)** Throughout this question, k denotes an algebraically closed field.

    (a) Define the notion of an *affine variety* $X \subset \mathbb{A}^n_k$ over k. Define the *coordinate ring* $k[X]$ of $X \subset \mathbb{A}^n_k$.

    (b) Define what is meant by an *irreducible* affine variety. Prove that any affine variety can be written as a finite union of irreducible subvarieties.

    (c) Fix a positive integer m. Find an explicit ideal $I_m \subset k[x, y]$ such that its vanishing locus $X_m = \mathbb{V}(I_m) \subset \mathbb{A}^2_k$ has exactly m distinct irreducible components.

    (d) Let $J = \langle x^2 + yz,\ xz + x \rangle \subset k[x, y, z]$. Decompose the variety $Y = \mathbb{V}(J) \subset \mathbb{A}^3_k$ into irreducible components. Compute the coordinate ring of each component of Y. Describe each component geometrically, both as an abstract affine variety and as a subvariety of $\mathbb{A}^3_k$.

[2]Hint: recall the algebraic form of the Chinese Remainder Theorem: if I_1, I_2 are coprime ideals in a ring R, meaning $I_1 + I_2 = R$, then $I_1 \cap I_2 = I_1 \cdot I_2$ and there is a ring isomorphism $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$ given by $f \mapsto (f + I_1, f + I_2)$.

[3]Hint: aim to show that I is prime and therefore radical. Show this by mapping R/I to an isomorphic ring using the linear generator in I.

(e) Let $K = \langle x_1x_3 - x_2^2,\ x_2x_4 - x_3^2,\ x_3x_5 - x_4^2 \rangle \subset k[x_1, x_2, x_3, x_4, x_5]$. Decompose the variety $Y = \mathbb{V}(K) \subset \mathbb{A}_k^5$ into irreducible components. Describe each component geometrically.

[*You may assume without proof that the surjective image of an irreducible affine variety is irreducible.*]