Professor Joyce **C3.4 Algebraic Geometry** **MT 2026**

PRELIMINARY VERSION. MAY BE UPDATED.

Problem Sheet 1

Throughout this problem sheet, k denotes an algebraically closed field.

Section A: introductory questions, not for handing in

1. **Zariski topology**

   Verify that arbitrary intersections and finite unions of affine varieties are affine varieties. Deduce that the Zariski topology on an affine variety is indeed a topology.

2. **Irreducibility**

   (a) Show that affine n-space $\mathbb{A}_k^n$ is irreducible.

   (b) Show that an affine variety $X \subset \mathbb{A}_k^n$ is irreducible if and only if every non-empty open subset $U \subset X$ is dense in the Zariski topology.

   (c) Let X be an irreducible affine variety. Show that any two non-empty open sets intersect in a non-empty open dense set.

Section B: core questions, to go over in class (do not hand in)

3. **The Zariski topology in low dimensions**

   (a) List the open and closed subsets of $\mathbb{A}^1_k$ in the Zariski topology.

   (b) Describe carefully the Zariski closed subsets of $\mathbb{A}^2_k$, proving your statements.

   (c) Show that the Zariski topology on $\mathbb{A}^2_k$ is *not* the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$.

4. **Reduced algebras as coordinate rings**

   (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for ideals I, J of a finitely generated k-algebra R.

   (b) Show that the ideal $(xy, xz) \subset k[x, y, z]$ is radical but not prime. Sketch the variety it defines in $\mathbb{A}^3_k$.

   (c) Let $X \subset \mathbb{A}^n_k$ be an affine variety. Show that a radical ideal in $k[X]$ is the intersection of all the maximal ideals containing it[1].

5. **The pull-back map between coordinate rings.**

   Suppose that $F : X \to Y$ is a morphism of affine varieties over a field k, associated to a map $F^* : k[Y] \to k[X]$ between their coordinate rings.

   (a) Show that F^* is injective if and only if F is dominant, i.e. the image set $F(X)$ is dense in Y.

   (b) Show that F^* is surjective if and only if F defines an isomorphism between X and some algebraic subvariety of Y.

   (c) Find an example where F is injective but F^* is not surjective.

6. **The affine normal curve.** Consider the homomorphism of rings

   $$F^* : k[x_0, \ldots, x_{n-1}] \to k[t]$$

   given by $x_i \mapsto t^i$.

   (a) Show that the corresponding morphism of affine varieties $F : \mathbb{A}^1_k \to \mathbb{A}^n_k$ defines an isomorphism between $\mathbb{A}^1_k$ and its image under F.

   (b) Find generators for the ideal defining the image of F in $\mathbb{A}^n_k$.

[1] Hint: using methods of this course, it is easier to first translate this into a geometrical statement, and prove that. For an algebraic proof, you might find helpful the following theorem due to Krull: the nilradical $\mathrm{nil}(A) = \{x : x^m = 0 \text{ some } m\}$ of a ring A equals the intersection of all its prime ideals.

7. **A reducible variety.** Consider the ideal

$$J = \langle uw - v^2, u^3 - vw \rangle$$

in the ring $k[u, v, w]$, and the corresponding affine variety $X = \mathbb{V}(J) \subset \mathbb{A}^3_k$.

(a) By taking suitable combinations of the generators, show that J is not prime.

(b) Show that X is a reducible variety, which decomposes as

$$X = X_1 \cup X_2$$

with one component, say X_1 isomorphic to the affine line $\mathbb{A}^1_k$.

(c) Show that the other piece X_2 is the image of a map $\mathbb{A}^1_k \to \mathbb{A}^3_k$ defined by $t \mapsto (t^a, t^b, t^c)$ for some positive integers a, b, c. Deduce that X_2 is irreducible.

Section C: optional questions

8. **The disjoint union of affine varieties** Show that a variety $X \subset \mathbb{A}^n_k$ is a union of two disjoint closed subvarieties if and only if its coordinate ring $k[X]$ may be written as the product of two non-trivial finitely generated reduced k-algebras[2].

9. **The variety of nilpotent matrices** We work in the affine space $\mathbb{A}^4$ parametrising 2×2 matrices over k, with variables being the matrix entries x_{ij}.

   (a) Prove that the following conditions are equivalent for a 2×2 matrix A over a field k:

   - A is *nilpotent*: there exists an $n \geq 1$ such that $A^n = 0$;
   - $A^2 = 0$;
   - $\det A = \operatorname{tr} A = 0$.

   Let $I \lhd R = k[x_{11}, x_{12}, x_{21}, x_{22}]$ be the ideal formed by the polynomials $d = \det A, t = \operatorname{tr} A$, viewed as polynomials in the matrix entries. Let $J \lhd R$ be the ideal formed by the entries of A^2, as polynomials in the matrix entries. Show the following.

   (b) The ideal J is not radical: it contains a power of t but not t itself.

   (c) The ideal I is radical[3].

   (d) Deduce that $X = \mathbb{V}(I) = \mathbb{V}(J) \subset \mathbb{A}^4$ with $\sqrt{J} = I$, and conversely $\mathbb{I}(X) = I$.

[2]Hint: recall the algebraic form of the Chinese Remainder Theorem: if I_1, I_2 are coprime ideals in a ring R, meaning $I_1 + I_2 = R$, then $I_1 \cap I_2 = I_1 \cdot I_2$ and there is a ring isomorphism $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$ given by $f \mapsto (f + I_1, f + I_2)$.

[3]Hint: aim to show that I is prime and therefore radical. Show this by mapping R/I to an isomorphic ring using the linear generator in I.