3. SEQUENCES OF FUNCTIONS

3.1 Convergence

We now consider how sequences of functions converge, which can happen in crucially different ways.

Definition 3.1 *Let $X \subseteq \mathbb{R}$ or $\mathbb{C}$. Let $f_n \colon X \to \mathbb{R}$ or $\mathbb{C}$ be a sequence of functions and $f \colon X \to \mathbb{R}$ or $\mathbb{C}$.*
(a) We say that f_n ***converges pointwise*** *to f on X if*

$$\forall \varepsilon > 0 \quad \forall x \in X \quad \exists N_{\varepsilon,x} \in \mathbb{N} \quad \forall n \geqslant N \quad |f_n(x) - f(x)| < \varepsilon.$$

(b) We say that f_n ***converges uniformly*** *to f on X if*

$$\forall \varepsilon > 0 \quad \exists N_{\varepsilon} \in \mathbb{N} \quad \forall x \in X \quad \forall n \geqslant N \quad |f_n(x) - f(x)| < \varepsilon.$$

Note that (b) is a stronger statement than (a), so that uniform convergence implies pointwise convergence.

Example 3.2 *Let $X = (0,1)$ and for $n \geqslant 1$ define*

$$f_n(x) = \frac{(-1)^n n}{n+x}, \qquad g_n(x) = x^n, \qquad h_n(x) = \frac{x}{n}.$$

(a) $f_n(x)$ does not converge pointwise on $(0,1)$.
(b) $g_n(x)$ converges pointwise to the zero function, but not uniformly.
(c) $h_n(x)$ converges uniformly to the zero function.

Solution. (a) Fix $x \in (0,1)$. By AOL

$$\frac{n}{n+x} = \frac{1}{1+x/n} \to \frac{1}{1+0} = 1.$$

Thus $f_{2n}(x) \to 1$ and $f_{2n+1}(x) \to -1$. So $f_n(x)$ diverges for all $x \in (0,1)$.

(b) Fix $x \in (0,1)$. For this x, as $n \to \infty$, then $x^n \to 0$. This is true for all $x \in (0,1)$, so this shows pointwise convergence to the zero function.

Take $\varepsilon = 1/2$. For any N, there exist $x \in (0,1)$ such that

$$x^N > \frac{1}{2} \quad \Longrightarrow \quad |g_N(x) - 0| > \frac{1}{2}.$$

This means there is no N such that

$$\forall x \in (0,1) \quad |g_N(x) - 0| < \frac{1}{2},$$

so the convergence is not uniform. See Figure 3.1.

(c) Let $\varepsilon > 1$. By the Archimedean property, there is $n \in \mathbb{N}$ such that $n > 1/\varepsilon$. So, for all $x \in (0,1)$, we have

$$|h_n(x) - 0| = \frac{x}{n} < \frac{1}{n} < \varepsilon.$$

This shows uniform convergence to the zero function. See Figure 3.2. ■

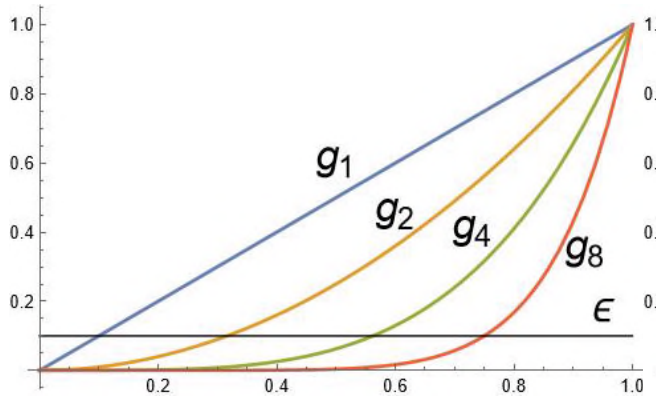


Fig 3.1. – pointwise-only convergence

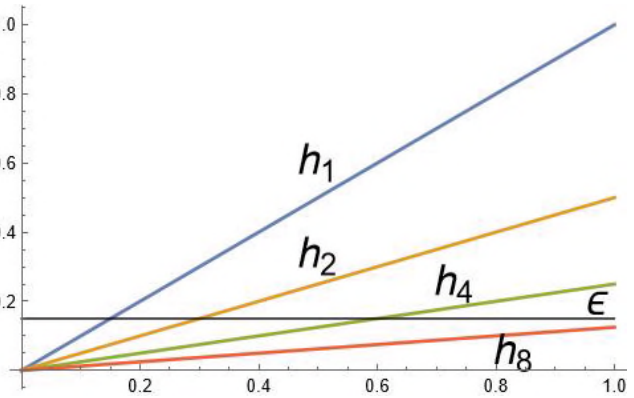


Fig. 3.2. – uniform convergence

Remark 3.3 *Confirming pointwise convergence of a sequence of functions $f_n\colon X \to \mathbb{R}$ or $\mathbb{C}$ simply means checking that for all $x \in X$ the limit $f(x) = \lim f_n(x)$ exists .*

Given pointwise convergence to $f(x)$, we can confirm uniform convergence by defining

$$s_n = \sup_{x \in X} |f_n(x) - f(x)|,$$

understanding this as ∞ if the supremum does not exist. Note that (s_n) is a sequence in $[0, \infty]$.

The functions f_n converge uniformly to f if and only if $s_n \to 0$.

Example 3.4 *Returning to the functions g_n and h_n in the previous example, both of which converge to the zero function, we have:*

(a) $s_n = \sup |x^n - 0| = 1$ for all n and $1 \nrightarrow 0$ as $n \to \infty$. So the convergence is not uniform.

(b) $s_n = \sup |x/n - 0| = 1/n$ for all n and $1/n \to 0$ as $n \to \infty$. So the convergence is uniform.

Example 3.5 *Investigate the convergence of the functions*

$$f_n(x) = n^2(1-x)x^n \qquad 0 \leqslant x \leqslant 1.$$

Solution. Note $f_n(1) = 0$ for all n and for $0 \leqslant x < 1$ we have $f_n(x) \to 0$ as powers dominate polynomials. So f_n converges pointwise to the zero function.

If we (prematurely) use differentiation to find the maximum of f_n we note

$$f_n'(x) = n^2[nx^{n-1} - (n+1)x^n] = n^2x^{n-1}[n - (n+1)x] = 0$$

at $x = 0$, a minimum, and at $x = n/(n+1)$, a maximum. Now

$$f_n\left(\frac{n}{n+1}\right) = n^2\left(1 - \frac{n}{n+1}\right)\left(\frac{n}{n+1}\right)^n = \frac{n^{n+2}}{(n+1)^{n+1}} = \left(\frac{n^2}{n+1}\right) \Big/ \left(1 + \frac{1}{n}\right)^n.$$

Recall that $(1 + n^{-1})^n \to e$ and hence

$$s_n = \sup_{0 \leqslant x \leqslant 1} |f_n(x) - 0| = f_n\left(\frac{n}{n+1}\right) \to \infty.$$

So we have pointwise convergence to 0, but not uniform convergence. ■

Remark 3.6 ***(Importance re integration)*** *Note that the above example is one where the order in which two limit processes – integration and convergence – are performed is significant. We have*

$$\int_0^1 f_n(x)\,\mathrm{d}x = n^2\left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{n^2}{(n+1)(n+2)} \to 1,$$

so

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x = 1, \qquad \textit{yet} \qquad \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0.$$

So pointwise convergence and integration (another limit process) do not 'commute' here.

The reason for this is clear in the diagram below. Whilst (f_n) converges pointwise to the zero function, there remains a narrow region of width $O(1/n)$ but height $O(n)$ near to 1.

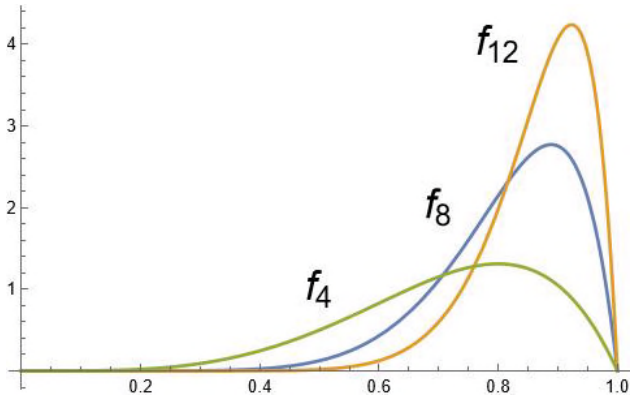


Fig. 3.3. – area and convergence.

This cannot happen with uniform convergence. In fact, in Analysis III it's shown that if (f_n) is a sequence of (Riemann) integrable functions on $[a,b]$ that converge to f, then f is itself integrable and

$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

This should not be that surprising because, if $|f_n - f| < \varepsilon$ on all of the interval $[a,b]$, then it's intuitively clear and straightforward to show that

$$\left|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\right| < \varepsilon(b-a).$$

Notice that uniform convergence is sufficient for the above result, but far from necessary. From Example 3.2, we have $x^n \to 0$ pointwise on $(0,1)$, but not uniformly, yet

$$\lim_{n\to\infty} \int_0^1 x^n \,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n+1} = 0 = \int_0^1 \lim_{n\to\infty} x^n \,\mathrm{d}x.$$

Example 3.7 *Let $D\colon \mathbb{R} \to \mathbb{R}$ denote the Dirichlet function (Example 1.19). We will show*

$$D(x) = \lim_{m\to\infty} \lim_{n\to\infty} \left(\cos\left(m!\pi x\right)\right)^{2n}.$$

For $x \in \mathbb{Q}$, note there is some m such that $\cos(m!\pi x) = 1$ and that this remains true for all greater m. Then $\lim_{n\to\infty} \left(\cos\left(m!\pi x\right)\right)^{2n} = 1$ and this value is retained as m increases, giving a limit of $1 = D(x)$.

For $x \notin \mathbb{Q}$, there is no value of m such that $\cos(m!\pi x) = \pm 1$. So for all m we have $\lim_{n\to\infty} \left(\cos\left(m!\pi x\right)\right)^{2n} = 0$. Letting $m \to \infty$ we obtain a limit of $0 = D(x)$.

Thus the Dirichlet function is the pointwise limit of a double sequence of continuous functions. It can be shown that the Dirichlet function is not the pointwise limit of a single sequence of continuous functions. Were this the case, then the set S of points of continuity of D would be dense in $\mathbb{R}$, that is we'd have $\overline{S} = \mathbb{R}$. However the Dirichlet function is discontinuous everywhere so that $S = \varnothing$.

It is straightforward to see that continuity is not preserved by pointwise convergence. For example, the continuous functions $f_n(x) = x^n$ on $[0,1]$ converge to

$$f(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < 1, \\ 1 & \text{for } x = 1, \end{cases}$$

which is not continuous. But uniform convergence *does* preserve continuity as follows.

Theorem 3.8 *Let $X \subseteq \mathbb{R}$ or $\mathbb{C}$ and let $f_n\colon X \to \mathbb{R}$ or $\mathbb{C}$ be a sequence of continuous functions which uniformly converge to f. Then f is continuous.*

Proof. Let $p \in X$ and $\varepsilon > 0$. As the convergence is uniform

$$\exists N \in \mathbb{N} \quad \forall n \geqslant N \quad \forall x \in X \quad |f_N(x) - f(x)| < \frac{\varepsilon}{3}.$$

As f_N is continuous at p,

$$\exists \delta > 0 \quad \forall x \in X \quad |x - p| < \delta \quad \Longrightarrow \quad |f_N(x) - f_N(p)| < \frac{\varepsilon}{3}$$

Hence if $|x - p| < \delta$, by the triangle inequality

$$|f(x) - f(p)| \leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(p)| + |f_N(p) - f_N(p)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows f is continuous at p. As p was arbitrary then f is continuous on X. ∎

Example 3.9 *Though we have yet to properly discuss differentiability, we note here that the uniform limit of differentiable functions need not be differentiable. See Figure 3.4. Consider* $f_n \colon \mathbb{R} \to \mathbb{R}$ *defined by*

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}} \qquad n \geqslant 1.$$

The pointwise limit as $n \to \infty$ *is* $f(x) = |x|$, *which is not differentiable. The convergence is in fact uniform. By differentiating*

$$e_n(x) = \sqrt{x^2 + \frac{1}{n}} - x \qquad x \geqslant 0,$$

to get

$$e_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}} - 1 < 0,$$

we see that e_n *is decreasing for* $x \geqslant 0$ *and so* e_n *has a maximum at* $x = 0$. *Hence*

$$E_n = \sup\{e_n(x) \mid x \in \mathbb{R}\} = e_n(0) = \frac{1}{\sqrt{n}} \to 0,$$

showing that the convergence is uniform.

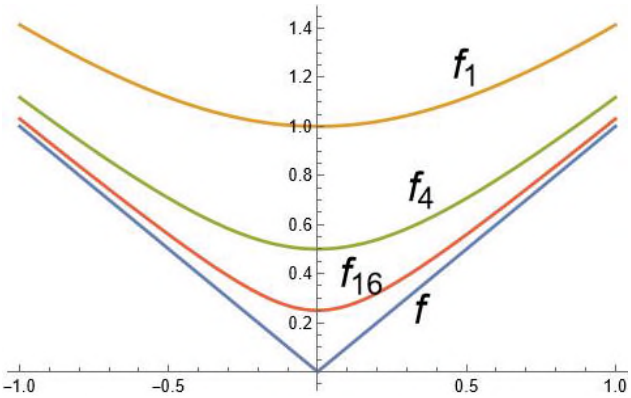


Fig. 3.4. – a uniform non-differentiable limit

You will discover in Part A Complex Analysis that the uniform limit of analytic functions – ones defined locally by a convergent power series – is analytic.

An important *sufficient* condition for the uniform convergence of a series is the following.

Theorem 3.10 ***(Weierstrass M-Test**, c. 1861) Let* $X \subseteq \mathbb{R}$ *or* $\mathbb{C}$. *Let* $f_n \colon X \to \mathbb{R}$ *or* $\mathbb{C}$ *be a sequence of bounded functions such that* $|f_n(x)| \leqslant M_n$ *for all* $x \in X$. *If* $\sum M_n$ *converges, then* $\sum f_n$ *converges uniformly and absolutely.*

Proof. By comparison with $\sum M_n$,

$$S(x) = \sum_{1}^{\infty} f_n(x)$$

converges pointwise, absolutely in fact. Setting $S_N(x) = \sum_1^N f_n(x)$ we note

$$\sup_{x\in X} |S(x) - S_N(x)| = \sup_{x\in X} \left| \sum_{N+1}^{\infty} f_n(x) \right| \leqslant \sum_{N+1}^{\infty} M_n \to 0 \quad \text{as } N \to \infty.$$

Hence the convergence is uniform. ■

Example 3.11 *The converse is not true. Consider the functions $f_n\colon [0,1] \to \mathbb{R}$, where $n \geqslant 1$, defined by*

$$f_n(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n}, \\ 0 & x \neq \frac{1}{n}. \end{cases}$$

Clearly the pointwise sum $s(x) = \sum_1^\infty f_n(x)$ exists and in fact this convergence is uniform as

$$\left| s(x) - \sum_1^N f_n(x) \right| = \left| \sum_{N+1}^{\infty} f_n(x) \right| \leqslant \frac{1}{N+1} \to 0.$$

However the minimal M_n such that $|f_n| \leqslant M_n$ is $M_n = \frac{1}{n}$ and $\sum M_n$ does not converge.

Corollary 3.12 *Let $\sum a_n z^n$ be a power series with radius of convergence R.*

(a) The series converges uniformly on every closed disc $|z| \leqslant r$ where $r < R$.

(b) The power series defines a continuous function on the disc $|z| < R$.

Proof. (a) From Analysis I, we know that a power series is absolutely convergent strictly within its radius of convergence. So for $r < R$, the series $\sum |a_n| r^n$ converges. Setting $M_n = |a_n| r^n$, the result follows by the WMT.

(b) As the power series converges uniformly on $|z| < r$, and each partial sum is continuous, then the power series is continuous on $|z| < r$ by Theorem 3.8. Hence it is continuous on the union of all such discs where $r < R$. This union is $|z| < R$. ■

Example 3.13 *A power series need not converge uniformly on its entire disc of convergence. For example, $\sum_0^\infty z^n$ does not converge uniformly on the disc $|z| < 1$.*

The power series converges pointwise to $S(z) = (1-z)^{-1}$, but this convergence is not uniform. Setting $S_N(z) = \sum_0^N z^n$ we see

$$\sup_{|z|<1} |S(z) - S_N(z)| = \sup_{|z|<1} \left| \sum\nolimits_{N+1}^{\infty} z^n \right| = \sup_{|z|<1} \left| \frac{z^{N+1}}{1-z} \right| = \infty,$$

as $S(z) - S_N(z)$ is unbounded near $z = 1$.

3.2 Elementary Transcendental Functions

The *elementary functions,* as opposed to 'special functions' include polynomials, rational functions, trigonometric functions, logarithms and exponentials. The first two would be considered 'algebraic' but those that need to be defined with power series are known as 'transcendental functions'. They were defined naively in *Analysis I* but we did not have the theory or definitions to discuss their continuity or properly define $\log x$ as the inverse of $\exp x$.

Example 3.14 *(**Exponential function**) The exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ *is defined by the infinite series*

$$\exp z = \sum_0^\infty \frac{z^n}{n!}.$$

By the ratio test, the exponential series converges on all of $\mathbb{C}$*, and by Corollary 3.12* $\exp$ *is continuous.*

Proposition 3.15 *(a)* $\exp z \exp(-z) = 1$ *for all* $z \in \mathbb{C}$.
(b) $\exp z \neq 0$ *for all* $z \in \mathbb{C}$.
(b) $\exp x \geqslant 1 + x$ *for all* $x \in \mathbb{R}$.

Proof. (a) Using Cauchy's product of series (from Analysis I), we have

$$\exp z \exp(-z) = \sum_0^\infty c_n z^n,$$

where, for $n > 1$,

$$c_n = \sum_{k=0}^n \frac{(-1)^{n-k}}{k!\,(n-k)!} = \frac{1}{n!}\sum_{k=0}^n (-1)^{n-k}\binom{n}{k} = \frac{(1-1)^n}{n!} = 0,$$

and $c_0 = 1 \times 1 = 1$. So $\exp z \exp(-z) = 1$.
(b) From (a) we see $\exp z \neq 0$ for all z, indeed that $(\exp z)^{-1} = \exp(-z)$.
(c) For $x \geqslant -1$, we argue

$$\exp x = 1 + x + \frac{x^2}{2!}\left(1 + \frac{x}{3}\right) + \frac{x^4}{4!}\left(1 + \frac{x}{5}\right) + \cdots \geqslant 1 + x,$$

as each of the paired summands are non-negative. For $x < -1$ we first note $\exp x = (\exp(-x))^{-1} > 0$ from (a) and the series definition. So

$$1 + x < 0 < \exp x.$$

■

Corollary 3.16 *The real exponential* $\exp\colon \mathbb{R} \to \mathbb{R}$ *has image* $(0, \infty)$ *and has a continuous inverse* $\log\colon (0, \infty) \to \mathbb{R}$.

Proof. The exponential series on $[0, \infty)$ is clearly strictly increasing and positive. Hence $\exp x = (\exp(-x))^{-1}$is increasing and positive on $(-\infty, 0)$, for if $a < b < 0$ then $0 < -b < -a$ and

$$\exp a = \frac{1}{\exp(-a)} < \frac{1}{\exp(-b)} = \exp b.$$

So $\exp(\mathbb{R}) \subseteq (0, \infty)$ and is a strictly increasing function; in particular this means that $\exp\colon \mathbb{R} \to (0, \infty)$ is injective. Now for $x > 0$, $\exp x > x$ and so exp takes arbitrarily large values of x and similarly $\exp(-x) = 1/\exp(x)$ takes arbitrarily small positive values. Thus, for any $0 < c < d$, there exists a such that $\exp a < c$ and b such that $\exp b > d$. By the IVT $[c, d] \subseteq \exp([a, b])$, showing that $\exp(\mathbb{R}) = (0, \infty)$. Hence

- $\exp\colon \mathbb{R} \to (0,\infty)$ is a bijection and hence invertible.
- The inverse, $\log\colon (0,\infty) \to \mathbb{R}$ is known as the **natural logarithm.**

Finally, for any $y > 0$ with $\log y = x$, we can apply the CIFT to $\exp\colon [x-1, x+1] \to [y/e, ey]$ to show that log is continuous at y. ■

Recall that in *Analysis I* we defined general exponents in terms of the exponential and logarithmic functions as follows.

Definition 3.17 ***(General Exponents)*** *Given $a > 0$ and $x \in \mathbb{R}$, we define*

$$a^x = \exp(x \log a).$$

Note a^x is continuous as it is the composition of continuous functions. With this definition

$$e^x = \exp x \text{ for } x \in \mathbb{R}$$

where $e = \exp 1$.

We then showed for $a, b > 0, x, y \in \mathbb{R}$.

- $\log(ab) = \log a + \log b$.
- $\log(a^x) = x \log a$.
- $a^{x+y} = a^x a^y$.
- $(a^x)^y = a^{(xy)}$.

Example 3.18 *(a) Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x+y) = f(x)f(y)$ for all x, y.*

(b) Find all continuous functions $g\colon \mathbb{R} \to \mathbb{R}$ satisfying $g(xy) = g(x)\,g(y)$ for all x, y.

Solution. (a) Note that $f(x) = f(x/2)^2 \geqslant 0$. If $f(a) = 0$ for some a, then

$$f(x+a) = f(x)f(a) = 0$$

for all x. So assume f is always positive. We have

$$\log f(x+y) = \log f(x) + \log f(y).$$

As $\log f$ is continuous also, then $\log f(x) = cx$ for some c by Sheet 1, Exercise 6. Hence $f(x) = e^{cx}$.

(b) Assume $x, y > 0$ and write $x = e^s$ and $y = e^t$, so that we have

$$g(e^{s+t}) = g(e^s)g(e^t).$$

Setting $h(u) = g(e^u)$ we see $h(s+t) = h(s)h(t)$ and we see $h\colon \mathbb{R} \to \mathbb{R}$ is continuous. Hence $h(u) = e^{cu}$ for all real u implying

$$g(x) = x^c \qquad \text{for } x > 0.$$

Note $g(1) = 1$. For continuity at $x = 0$, we need $c \geqslant 0$. If $c > 0$ we need $g(0) = 0$ for continuity and when $c = 0$ we need $g(0) = 1$.

Now $1 = g(1) = g\left((-1)^2\right) = g(-1)^2$ showing $g(-1) = \pm 1$.

For $c = 0$ we have $g(x) = 1$ for $x \geqslant 0$, so $g(-1) = -1$ leads to discontinuity. Hence $g(x)$ is identically 1 is the only possible solution when $c = 0$.

For $c > 0$ we have two possibilities then: for any real x,

$$g(x) = |x|^c \qquad \text{or} \qquad g(x) = \operatorname{sgn}(x)\,|x|^c,$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x = -1. \end{cases}$$

A quick check shows that these are indeed valid solutions of the functional equation. ■

Definition 3.19 ***(Trigonometric functions)*** *Sine and cosine, for $z \in \mathbb{C}$, are defined by*

$$\begin{aligned} \cos z &= \frac{\exp(iz) + \exp(-iz)}{2} = \sum_0^\infty \frac{(-1)^n z^{2n}}{(2n)!}, \\ \sin z &= \frac{\exp(iz) - \exp(-iz)}{2i} = \sum_0^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \end{aligned}$$

By Corollary 3.12, both are continuous functions.

Further:

Proposition 3.20 $\cos^2 z + \sin^2 z = 1$ *for all* $z \in \mathbb{C}$. *So* $|\cos x| \leqslant 1$ *and* $|\sin x| \leqslant 1$ *for all* $x \in \mathbb{R}$.

Proof. We have

$$\begin{aligned} & \cos^2 z + \sin^2 z \\ = \; & \frac{1}{4}\left\{(\exp(iz) + \exp(-iz))^2 - (\exp(iz) - \exp(-iz))^2\right\} \\ = \; & \frac{4\exp(iz)\exp(-iz)}{4} = 1. \end{aligned}$$

■

Proposition 3.21 cos *has a smallest root* α *in the interval* $[0, 2]$. *We define* $\pi = 2\alpha$.

Proof. Note $\cos 0 = 1$ and that

$$\begin{aligned}
\cos 2 &= \sum_{0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \\
&= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \cdots \\
&= \underbrace{1 - 2 + \frac{2}{3}}_{<0} - \underbrace{\frac{2^6}{6!}\left(1 - \frac{2^2}{7 \times 8}\right)}_{>0} - \underbrace{\frac{2^{10}}{10!}\left(1 - \frac{2^2}{11 \times 12}\right)}_{>0} - \cdots \\
&< 0.
\end{aligned}$$

By the IVT, there is at least one solution to $\cos x = 0$ in the interval $[0, 2]$. As cos is continuous, then the infimum of the set $\cos^{-1}(0) \cap [0, 2]$ is in fact a minimum element α. ■

Corollary 3.22 *(a)* exp *has period* $2\pi i$.
(b) $\sin z$ *and* $\cos z$ *has period* 2π.

Proof. (a) We have $\cos \alpha = 0$ and so by the definition of cosine we have

$$\exp(i\alpha) + \exp(-i\alpha) = 0 \quad \Longrightarrow \quad \exp(2i\alpha) = -1$$

and then

$$\exp(z + 4\alpha i) = \exp z \left(\exp(2i\alpha)\right)^2 = \exp z.$$

Thus exp has period $4\alpha i = 2\pi i$.

(b) From the definition of cosine

$$\begin{aligned}
\cos(z + 2\pi) &= \frac{\exp(iz + 2\pi i) + \exp(-iz - 2\pi i)}{2} \\
&= \frac{\exp(iz) + \exp(-iz)}{2} = \cos z.
\end{aligned}$$

A similar argument shows that sine has period 2π. ■

We will see in due course that a power series is differentiable, infinitely so in fact, within the radius of convergence, but we shall leave the details till later as we have yet to formally define differentiability.

3.3 Aside on Fourier Series (Off Syllabus)

As has been previously commented, uniform convergence is particularly helpful when dealing with the convergence of (Riemann) integrals (this will be covered in the Analysis III course).

In particular, the following result is true: if $f_n\colon [a,b] \to \mathbb{R}$ is a sequence of integrable functions uniformly converging to f, then f is integrable and

$$\int_a^b f(x)\,\mathrm{d}x = \lim \int_a^b f_n(x)\,\mathrm{d}x.$$

So, given a integrable function f on $[0,\pi]$ such that the *Fourier cosine series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

converges uniformly to f, we can use the *orthogonality relati*ons

$$\int_0^{\pi} \cos nx \cos mx\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n \geqslant 1 \\ \pi & \text{if } m = n = 0, \end{cases}$$

to argue as follows: when $N \geqslant 1$ we have

$$\begin{aligned} \int_0^{\pi} f(x)\cos Nx\,\mathrm{d}x &= \int_0^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right) \cos Nx\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos nx \cos Nx\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} a_n \frac{\pi}{2} \delta_{nN} \\ &= \frac{\pi}{2} a_N, \end{aligned}$$

showing

$$a_N = \frac{2}{\pi} \int_0^{\pi} f(x)\cos Nx\,\mathrm{d}x$$

with a similar formula holding for $N = 0$.

The convergence or divergence of Fourier series, and the rate of any convergence, became important questions of analysis during the nineteenth century. The details are subtle, often technical and well beyond the scope of this course. There are continuous functions whose Fourier series diverge. A positive first result is:

- Let f be 2π-periodic, continuous, piecewise-differentiable, with left- and right-sided derivatives. The Fourier cosine series of f converges uniformly to f on $[0,\pi]$.

Example 3.23 *(a) Show that the function*

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2},$$

is well-defined and continuous.

(b) Let

$$g(x) = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}.$$

Show, for $n \geqslant 1$, *that*

$$a_n = \frac{2}{\pi}\int_0^\pi g(x)\cos nx\,\mathrm{d}x = \frac{1}{n^2}$$

and find a_0.

(c) Deduce that $f(x) = g(x)$ *for* $0 \leqslant x \leqslant \pi$. *Sketch a graph of* $f(x)$.

(d) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. (a) Setting $M_n = n^{-2}$ and noting $\sum_0^\infty M_n$ converges, we see that the sum converges uniformly by WMT. As each summand is continuous, then $f\,(x)$ is continuous.

(b) For $n = 0$ we have

$$\frac{\pi}{2}a_0 = \int_0^\pi \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}\,\mathrm{d}x = \frac{\pi^3}{6} - \frac{\pi^3}{4} + \frac{\pi^3}{12} = 0,$$

so that $a_0 = 0$. For $n \geqslant 1$, IBP twice shows

$$\begin{aligned}
\frac{\pi}{2}a_n &= \int_0^\pi \left(\frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}\right)\cos nx\,\mathrm{d}x \\
&= \left[\left(\frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}\right)\frac{\sin nx}{n}\right]_0^\pi + \frac{1}{2n}\int_0^\pi (\pi - x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{2n}\left[(\pi - x)\,\frac{(-\cos nx)}{n}\right]_0^\pi + \frac{1}{2n}\int_0^\pi \frac{\cos nx}{n}\,\mathrm{d}x \\
&= \frac{\pi}{2n^2}
\end{aligned}$$

and hence $a_n = 1/n^2$.

(c) $g(x)$ has Fourier coefficients $a_n = 1/n^2$ which agree with those of $f(x)$. If we extend g evenly and periodically we note the extended $g(x)$ is continuous, piecewise differentiable with left- and right-sided derivatives.By the quoted result above,$f(x)$ converges to $g(x)$ on $[0, \pi]$.

A sketch of $f(x)$ is given below:

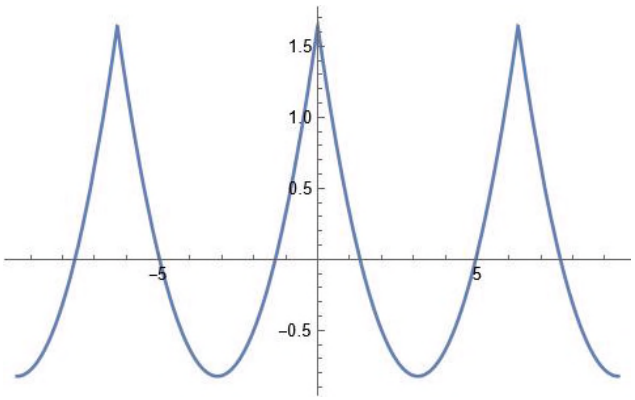


Fig. 3.5 – Fourier series limit

.(d) Set $x = 0$ in the Fourier cosine series. ■