Professor Joyce **C3.4 Algebraic Geometry** **MT 2026**

Problem Sheet 1

Throughout this problem sheet, k denotes an algebraically closed field.

Section A: introductory questions, not for handing in

1. **Zariski topology**

   Verify that arbitrary intersections and finite unions of affine varieties are affine varieties. Deduce that the Zariski topology on an affine variety is indeed a topology.

   Solution: For finite unions, it is of course enough to consider two varieties at a time. If $X_i = \mathbb{V}(I_i) \subset \mathbb{A}^n_k$ are affine varieties for $i = 1, 2$ then $X_1 \cup X_2 = \mathbb{V}(I_1 I_2)$ where the product ideal is $I_1 I_2 = \{a_1 b_1 + \ldots + a_s b_s | a_i \in I_1, b_i \in I_2\}$. To see this, note that $p \in \mathbb{V}(I_1 I_2) \setminus \mathbb{V}(I_1)$, pick $f \in I_1$ with $f(p) \neq 0$, then $fg \in I_1 I_2$ for all $g \in I_2$, so $f(p)g(p) = 0 \in k$, hence $g(p) = 0$ for all $g \in I_2$, so $p \in \mathbb{V}(I_2)$. Another ideal that would work in this situation is $I_1 \cap I_2$. Clearly $I_1 I_2 \subset I_1 \cap I_2$, so $\mathbb{V}(I_1 I_2) \supset \mathbb{V}(I_1 \cap I_2)$. Conversely, if $f \in I_1 \cap I_2$, then $f^2 \in I_1 I_2$, so $f^2(p) = 0$ for $p \in \mathbb{V}(I_1 I_2)$, so $f(p) = 0$, so $p \in \mathbb{V}(I_1 \cap I_2)$. Note that $I_1 \cap I_2 \neq I_1 I_2$ in general: in $\mathbb{Z}$ we have $(4) \cap (6) = (12), (4)(6) = (24)$.

   For intersections, if $X_\alpha = \mathbb{V}(I_\alpha), \alpha \in A$ is a family affine varieties then

   $$\bigcap_{\alpha \in A} X_\alpha = \mathbb{V}\left(\sum_{\alpha \in A} I_\alpha\right)$$

   where the sum of ideals is

   $$\sum_{\alpha \in A} I_\alpha = \left\{\sum a_\alpha \mid a_\alpha \in I_\alpha, a_\alpha = 0 \text{ for all but finite number of } \alpha\right\}.$$

2. **Irreducibility**

   (a) Show that affine n-space $\mathbb{A}^n_k$ is irreducible.

   (b) Show that an affine variety $X \subset \mathbb{A}^n_k$ is irreducible if and only if every non-empty open subset $U \subset X$ is dense in the Zariski topology.

(c) Let X be an irreducible affine variety. Show that any two non-empty open sets intersect in a non-empty open dense set.

Solution: (a) If $\mathbb{A}_k^n = \mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 I_2)$ were true with nontrivial proper ideals $I_1, I_2 \subset k[x_1, \ldots, k_n]$ then $I_1 I_2 = \{0\}$ would hold, but $I_1 I_2$ contains at least one nonzero polynomial, namely the product of nonzero polynomials in I_1 and I_2.

(b) X is reducible if $X = X_1 \cup X_2$ with X_1, X_2 distinct closed, non-empty subsets. But then $X \setminus X_1 \subset X$ is a open not-dense subset. Conversely, if $U \subset X$ is an open not-dense subset then $X = (X \setminus U) \cup \overline{U}$ is a nontrivial cover with closed subsets.

(c) Suppose $U, U' \subset X$ are open with $U \cap U' = \emptyset$. Let $V = X \setminus U$, $V' = X \setminus U'$. These are closed sets and $X = V \cup V'$. As X is irreducible, this forces one of V or V' to be non-proper. So U or U' is empty. Hence for non-empty U, U', the intersection $U \cap U'$ is a non-empty open subset, which is dense by part (b).

Examples of affine varieties

3. Let $X \subseteq \mathbb{A}_k^3$ be the set

$$X := \{(t, t^2, t^3) \,|\, t \in k\}.$$

Show that X is an affine variety and that it is isomorphic to $\mathbb{A}_k^1$ as an affine variety. Provide generators for $\mathbb{I}(X)$.

Solution: We have $X = \mathbb{V}((x_2 - x_1^2, x_3 - x_1^3))$ so X is an affine variety. If we let $A : \mathbb{A}_k^1 \to \mathbb{A}_k^3$ be the polynomial map such that $A(t) := (t, t^2, t^3)$ and $B : \mathbb{A}_k^3 \to \mathbb{A}_k^1$ be the polynomial map such that $B(x_1, x_2, x_3) = x_1$, then $A(\mathbb{A}_k^1) \subseteq X$, $B|_X \circ A = \mathrm{id}_{\mathbb{A}_k^1}$, $A \circ B|_X = \mathrm{id}_X$ so A and $B|_X$ are morphisms from $\mathbb{A}_k^1$ to X and from X to $\mathbb{A}_k^1$ respectively, which are inverse to each other. So they give an isomorphism between X and $\mathbb{A}_k^1$. We still have to provide generators for $\mathbb{I}(X)$. For this, consider that $(x_2 - x_1^2, x_3 - x_1^3)$ is prime by the above so $\mathbb{I}(X) = (x_2 - x_1^2, x_3 - x_1^3)$.

4. (a) Let $X \subseteq \mathbb{A}_k^2$ be the set of solutions of the equation $y = x^2$. Show that X is isomorphic to $\mathbb{A}_k^1$ as an affine variety.

   (b) Let $Y \subseteq \mathbb{A}_k^2$ be the set of solutions of the equation $xy = 1$. Show that X is not isomorphic to $\mathbb{A}_k^1$ as an affine variety.

   (c) (Difficult.) Let $P(x, y) \in k[x, y]$ be an irreducible quadratic polynomial and let $Z \subseteq \mathbb{A}_k^2$ be the set of zeroes of $P(x, y)$. Show that Z is isomorphic to one of the affine varieties defined in (a) and (b).

Solution: (a) This is similar to question 3, with $\{(t, t^2) : t \in k\}$ instead of X.

(b) If Y is isomorphic to $\mathbb{A}^1_k$ then $k[x,y]/(xy-1) \simeq k[t]$. Now note that x is a unit in $k[x,y]/(xy-1)$ by construction and that x is not in the image of k in $k[x,y]/(xy-1)$. Since the only units of $k[t]$ lie in the image of k, there is thus no isomorphism $k[x,y]/(xy-1) \simeq k[t]$.

(c) Write $P(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ for some $a, \ldots, f \in k$.

Since k is algebraically closed, we have $P(x,y) = L_1L_2 + L_3 + f$, where $L_3 = dx + ey$ and L_1, L_2 are linear forms in $k[x,y]$ such that $L_1L_2 = ax^2 + bxy + cy^2$.

We split into two cases: (i) $L_1 \sim L_2$ and (ii) $L_1 \not\sim L_2$.

(i) If $L_1 \sim L_2$, then $L_2 = \lambda L_1$ for some $\lambda \in k^\times$, and we have

$$P(x,y) = \lambda L_1^2 + L_3 + f.$$

Since k is algebraically closed and $P(x,y)$ is irreducible in $k[x,y]$, it follows that $L_1 \not\sim L_3$. Therefore

$$(u,v) = \left(-\sqrt{\lambda}\, L_1,\ L_3 + f\right)$$

is an affine change of coordinates such that

$$P(x,y) = v - u^2.$$

(ii) If $L_1 \not\sim L_2$, then there exist $\lambda, \mu \in k$ such that $L_3 = \lambda L_1 + \mu L_2$. As such, we have

$$P(x,y) = (L_1 + \mu)(L_2 + \lambda) - (\lambda\mu - f).$$

Since $P(x,y)$ is irreducible in $k[x,y]$, it follows that $\lambda\mu - f \neq 0$. Therefore

$$(u,v) = \left(\frac{L_1 + \mu}{\sqrt{\lambda\mu - f}}, \frac{L_2 + \lambda}{\sqrt{\lambda\mu - f}}\right)$$

is an affine change of coordinates such that $P(x,y) \sim uv - 1$. If

$$L_1 = \alpha x + \beta y \qquad \text{and} \qquad L_2 = \gamma x + \delta y,$$

then

$$L_1 \sim L_2 \quad \Longleftrightarrow \quad \alpha\delta - \beta\gamma = 0.$$

Moreover, since $L_1L_2 = ax^2 + bxy + cy^2$, we have

$$\begin{aligned}(\alpha\delta - \beta\gamma)^2 &= (\alpha\delta + \beta\gamma)^2 - 4(\alpha\gamma)(\beta\delta) \\ &= b^2 - 4ac.\end{aligned}$$

It thus follows that

$$k[x,y]/(P(x,y)) \cong \begin{cases} k[t], & \text{if } b^2 = 4ac, \\ k[t, t^{-1}], & \text{if } b^2 \neq 4ac. \end{cases}$$

5. Let $X \subseteq \mathbb{A}^3_k$ be the affine variety described by the ideal $(x^2 - yz, xz - x)$. Show that X has three irreducible components. Find generators for the radical (actually prime) ideals associated with these components.

Solution: Treat x, y, z as variable elements of k. If $x \neq 0$, then $z = 1$ and $y = x^2$. Also we have $\langle 0, 0, 1\rangle \in X$ and hence $\mathbb{V}((x^2 - yz, xz - x)) \supseteq \{\langle x, x^2, 1\rangle \mid x \in k\}$. We have $\{\langle x, x^2, 1\rangle \mid x \in k\} = \mathbb{V}((y - x^2, z - 1))$ and the first projection gives an isomorphism between this affine variety and k. Hence $\{\langle x, x^2, 1\rangle \mid x \in k\}$ is an irreducible affine variety in $\mathbb{A}^3_k$, which is contained in X. If $x = 0$, then the simultaneous solutions of the equation $x^2 = yz$ and $xz = x$ are precisely the solutions of the equation $yz = 0$. The simultaneous solutions of $yz = 0$ and $x = 0$ consist of the union of the z-axis and the y-axis. So X is the union of the z-axis, the y-axis and $\{\langle x, x^2, 1\rangle \mid x \in k\}$, which are all three irreducible. One easily checks that none of these three sets are contained in the union of the two others, so they are the irreducible components of X.

Section B: core questions, to go over in class (do not hand in)

6. **The Zariski topology in low dimensions**

   (a) List the open and closed subsets of $\mathbb{A}^1_k$ in the Zariski topology.

   (b) Describe carefully the Zariski closed subsets of $\mathbb{A}^2_k$, proving your statements.

   (c) Show that the Zariski topology on $\mathbb{A}^2_k$ is *not* the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$. (Needs k to be an infinite field.)

7. **Reduced algebras as coordinate rings**

   (a) Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for ideals I, J of a finitely generated k-algebra R.

   (b) Show that the ideal $(xy, xz) \subset k[x, y, z]$ is radical but not prime. Sketch the variety it defines in $\mathbb{A}^3_k$.

   (c) Show that a radical ideal I in $k[x_1, \ldots, x_n]$ is the intersection of all the maximal ideals $\mathfrak{m}$ containing it[1].

8. **The pull-back map between coordinate rings.**

   Suppose that $F : X \to Y$ is a morphism of affine varieties over a field k, associated to a map $F^* : k[Y] \to k[X]$ between their coordinate rings.

   (a) Show that F^* is injective if and only if F is dominant, i.e. the image set $F(X)$ is dense in Y.

   (b) Show that F^* is surjective if and only if F defines an isomorphism between X and some algebraic (closed) subvariety of Y.

   (c) Find an example where F is injective but F^* is not surjective.

9. **The affine normal curve.** Consider the homomorphism of rings

$$F^* : k[x_1, \ldots, x_n] \to k[t]$$

   given by $x_i \mapsto t^i$.

   (a) Show that the corresponding morphism of affine varieties $F : \mathbb{A}^1_k \to \mathbb{A}^n_k$ defines an isomorphism between $\mathbb{A}^1_k$ and its image under F.

   (b) Find generators for the radical ideal defining the image of F in $\mathbb{A}^n_k$.

[1] Hint: using methods of this course, it is easier to first translate this into a geometrical statement, and prove that. For an algebraic proof, you might find helpful the following theorem due to Krull: the nilradical $\mathrm{nil}(A) = \{x : x^m = 0 \text{ some } m\}$ of a ring A equals the intersection of all its prime ideals.

10. **A reducible variety.** Consider the ideal

$$J = \langle uw - v^2, u^3 - vw \rangle$$

in the ring $k[u, v, w]$, and the corresponding affine variety $X = \mathbb{V}(J) \subset \mathbb{A}^3_k$.

(a) By taking suitable combinations of the generators, show that J is not prime.

(b) Show that X is a reducible variety, which decomposes as

$$X = X_1 \cup X_2$$

with one component, say X_1, isomorphic to the affine line $\mathbb{A}^1_k$.

(c) Show that the other piece X_2 is the image of a map $\mathbb{A}^1_k \to \mathbb{A}^3_k$ defined by $t \mapsto (t^a, t^b, t^c)$ for some positive integers a, b, c. Deduce that X_2 is irreducible.

Section C: optional questions

11. **The disjoint union of affine varieties** Show that a variety $X \subset \mathbb{A}^n_k$ is a union of two disjoint closed subvarieties if and only if its coordinate ring $k[X]$ may be written as the product of two non-trivial finitely generated reduced k-algebras[2].

    Solution: Geometric meaning: the ring of functions on any set is the product of rings of functions on its connected components. To prove the statement, first assume that $X = X_1 \cup X_2, X_1 \cap X_2 = \emptyset$ is the union of disjoint components. Abbreviate $R = k[x_1, \ldots, x_n]$. Then

    $$\mathbb{I}(X_1) + \mathbb{I}(X_2) = \mathbb{I}(X_1 \cap X_2) = \mathbb{I}(\emptyset) = R$$

    by the Nullstellensatz. So $\mathbb{I}(X_1), \mathbb{I}(X_2)$ are coprime ideals and therefore, by the Chinese remainder theorem, $\mathbb{I}(X_1)\mathbb{I}(X_2) = \mathbb{I}(X_1) \cap \mathbb{I}(X_2)$ and

    $$k[X] = R/(\mathbb{I}(X_1)\mathbb{I}(X_2)) \cong (R/\mathbb{I}(X_1)) \times (R/\mathbb{I}(X_2)).$$

    Conversely, assume $k[X] = R/I \cong A_1 \times A_2$ with A_i f.g.reduced k-algebras. Let I_i be the kernel of the composite map

    $$R \to R/I \to A_1 \times A_2 \to A_i$$

    where the last map is the projection. Then we see $I \subset I_i$ and $A_i \cong R/I_i$. The isomorphism $R/I \cong A_1 \times A_2$ corresponds to the natural map

    $$R/I \cong R/I_1 \times R/I_2.$$

    We then see that $I = I_1 \cap I_2$. We also claim $I_1 + I_2 = R$, so these ideals are relatively prime. To see that, note that there are elements $\alpha, \beta \in R/I$ corresponding to $(1, 0)$ and $(0, 1)$ on the right, so $\alpha + \beta$ corresponds to $(1, 1)$, and $\alpha = 0 \bmod I_2$, $\beta = 0 \bmod I_1$, so $1 = \beta + \alpha \in I_1 + I_2$. Now define $X_i = \mathbb{V}(I_i) \subset \mathbb{A}^n$. Then $X = \mathbb{V}(I) = \mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2) = X_1 \cup X_2$ and the X_i are also disjoint as

    $$X_1 \cap X_2 = \mathbb{V}(I_1 + I_2) = \mathbb{V}(R) = \emptyset.$$

12. **The variety of nilpotent matrices** We work in the affine space $\mathbb{A}^4_k$ parametrising 2×2 matrices over k, with variables being the matrix entries x_{ij}.

    (a) Prove that the following conditions are equivalent for a 2×2 matrix A over a field k:

[2]Hint: recall the algebraic form of the Chinese Remainder Theorem: if I_1, I_2 are coprime ideals in a ring R, meaning $I_1 + I_2 = R$, then $I_1 \cap I_2 = I_1 \cdot I_2$ and there is a ring isomorphism $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$ given by $f \mapsto (f + I_1, f + I_2)$.

- A is *nilpotent*: there exists an $n \geq 1$ such that $A^n = 0$;
- $A^2 = 0$;
- $\det A = \operatorname{tr} A = 0$.

Let $I \lhd R = k[x_{11}, x_{12}, x_{21}, x_{22}]$ be the ideal formed by the polynomials $d = \det A, t = \operatorname{tr} A$, viewed as polynomials in the matrix entries. Let $J \lhd R$ be the ideal formed by the entries of A^2, as polynomials in the matrix entries. Show the following.

(b) The ideal J is not radical: it contains a power of t but not t itself.

(c) The ideal I is radical[3].

(d) Deduce that $X = \mathbb{V}(I) = \mathbb{V}(J) \subset \mathbb{A}^4_k$ with $\sqrt{J} = I$, and conversely $\mathbb{I}(X) = I$.

Solution: (a) We use basic results from Linear Algebra. If $A^n = 0$, then the minimal polynomial $m_A(x)$ of A must divide x^n, so it is either x or x^2. By the Cayley–Hamilton theorem, in either case $A^2 = 0$. The converse is automatic. This is also equivalent to the characteristic polynomial $c_A(x) = x^2$, which is equivalent to the vanishing of its nontrivial coefficients, which are $\det A$ and $\operatorname{tr} A$.

(b) We have

$$J = \langle x_{11}^2 + x_{12}x_{21}, x_{11}x_{21} + x_{21}x_{22}, x_{11}x_{12} + x_{12}x_{22}, x_{12}x_{21} + x_{22}^2 \rangle.$$

Clearly $t = x_{11} + x_{22} \notin J$, as J is generated by quadratic polynomials. On the other hand,

$$t^3 = (x_{11} + 3x_{22})(x_{11}^2 + x_{12}x_{21}) - 4x_{12}(x_{11}x_{21} + x_{21}x_{22}) + (3x_{11} + x_{22})(x_{12}x_{21} + x_{22}^2) \in J.$$

This can be obtained by eliminating from t^3 the x_{11}^3 and $x_{11}^2 x_{22}$ terms, then doing the same for the x_{22}^3 and $x_{11}x_{22}^2$ terms, and factoring the resulting expression.

(c) We can use the linear generator $t = x_{11} + x_{22}$ to make a substitution $x_{22} = -x_{11}$ and get a ring isomorphism

$$k[x_{11}, x_{12}, x_{21}, x_{22}]/I \cong k[x_{11}, x_{12}, x_{21}]/\langle q \rangle$$

where q is the substituted form of the determinant d:

$$q = -x_{11}^2 - x_{12}x_{21}.$$

As q is an irreducible quadric in $k[x_{11}, x_{12}, x_{21}]$, it generates a prime ideal, so the right hand ring is an integral domain. Hence I is radical.

(d) By (a), we have $X = \mathbb{V}(I) = \mathbb{V}(J)$: the set of 2×2 matrices where det and tr vanish is the same as the set of matrices with $A^2 = 0$. By the Nullstellensatz, as I is radical, $\mathbb{I}(X) = I$ and $\sqrt{J} = \mathbb{I}(\mathbb{V}(J)) = I$.

[3]Hint: aim to show that I is prime and therefore radical. Show this by mapping R/I to an isomorphic ring using the linear generator in I.

13. Let $X = \mathbb{A}^1_k$, and Y be the affine variety $\{(x, y) \in \mathbb{A}^2 : x^4 - y^3 = 0\}$. Define $F : X \to Y$ by $F(t) = (t^3, t^4)$. Is F an isomorphism?

Solution: The associated morphism of coordinate rings is

$$F^* : k[Y] = k[x, y]/(x^4 - y^3) \longrightarrow k(t) = k[X],$$
$$F^*(x) = t^3, \qquad F^*(y) = t^4.$$

As t is not in the image of F^*, F^* is not an isomorphism, so F is not an isomorphism. In fact $k[X] \cong \langle 1, t, t^2, t^3, \ldots \rangle_k$ and $k[Y] \cong \langle 1, t^3, t^4, t^5, \ldots \rangle_k$.

14. **(C3.4 2022 q. 1.)** Throughout this question, k denotes an algebraically closed field.

(a) Define the notion of an *affine variety* $X \subset \mathbb{A}^n_k$ over k. Define the *coordinate ring* $k[X]$ of $X \subset \mathbb{A}^n_k$.

(b) Define what is meant by an *irreducible* affine variety. Prove that any affine variety can be written as a finite union of irreducible subvarieties.

(c) Fix a positive integer m. Find an explicit ideal $I_m \subset k[x, y]$ such that its vanishing locus $X_m = \mathbb{V}(I_m) \subset \mathbb{A}^2_k$ has exactly m distinct irreducible components.

(d) Let $J = \langle x^2 + yz,\ xz + x \rangle \subset k[x, y, z]$. Decompose the variety $Y = \mathbb{V}(J) \subset \mathbb{A}^3_k$ into irreducible components. Compute the coordinate ring of each component of Y. Describe each component geometrically, both as an abstract affine variety and as a subvariety of $\mathbb{A}^3_k$.

(e) Let $K = \langle x_1x_3 - x_2^2,\ x_2x_4 - x_3^2,\ x_3x_5 - x_4^2 \rangle \subset k[x_1, x_2, x_3, x_4, x_5]$. Decompose the variety $Y = \mathbb{V}(K) \subset \mathbb{A}^5_k$ into irreducible components. Describe each component geometrically.

[*You may assume without proof that the surjective image of an irreducible affine variety is irreducible.*]

Solution: (a) An affine variety

$$X = \{f_1 = \cdots = f_m = 0\} \subset \mathbb{A}^n$$

is the zero locus of a collection of polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$. (Equivalently, $X = \mathbb{V}(I)$ for $I \lhd k[x_1, \ldots, x_n]$.)

For an affine variety X, define

$$I(X) = \{f \in k[x_i] : f(p) = 0\ \forall p \in X\}$$

and the coordinate ring

$$k[X] = \frac{k[x_i]}{I(X)}.$$

(b) An affine variety $X \subset \mathbb{A}^n$ is *irreducible* if whenever $X = X_1 \cup X_2$ for affine varieties $X_1, X_2 \subseteq \mathbb{A}^n$, we have an i such that $X_i = X$.

Take X, and assume it is not irreducible, so $X = X_1 \cup X_2$. Assume one of these, say X_2, is still not irreducible, then $X = X_1 \cup X_2' \cup X_3$. Assume this goes on indefinitely, then we get a sequence of affine varieties

$$X \supset X_2 \supset X_3 \supset \cdots . \qquad (*)$$

Their vanishing ideals

$$I(X) \subset I(X_2) \subset I(X_3) \subset \cdots$$

eventually stabilise, by the ascending chain condition for ideals of the Noetherian ring $k[x_i]$. Thus the chain $(*)$ eventually stabilises. So a finite decomposition for X exists.

(c) An algebraically closed field k is infinite, so we can choose m distinct scalars $\lambda_1, \ldots, \lambda_m \in k$. Put

$$f_m = \prod_{i=1}^{m} (x - \lambda_i y), \qquad I_m = \langle f_m \rangle.$$

Then

$$X = \mathbb{V}(I_m) = \bigcup_{i=1}^{m} \mathbb{V}(x - \lambda_i y).$$

On the right, we have m distinct lines, all $\cong \mathbb{A}^1$, so irreducible.

(d) On $\mathbb{V}(J)$, either $x = 0$ or $z = -1$, looking at the second generator. In the first case, looking at the first generator, $yz = 0$, so $y = 0$ or $z = 0$. In the second case, $x^2 + y = 0$. So we get

$$\mathbb{V}(J) = \mathbb{V}(x, y) \cup \mathbb{V}(x, z) \cup \mathbb{V}(z + 1, x^2 + y).$$

The first two have coordinate rings

$$\frac{k[x, y, z]}{(x, y)} \cong k[z] \qquad \text{and} \qquad \frac{k[x, y, z]}{(x, z)} \cong k[y].$$

So they are isomorphic to $\mathbb{A}^1$, embedded in $\mathbb{A}^3$ as lines.

The last one has coordinate ring

$$\frac{k[x, y, z]}{(z + 1, x^2 + y)} \cong k[x],$$

where

$$y \longmapsto -x^2, \qquad z \longmapsto -1.$$

So it is isomorphic to $\mathbb{A}^1$, embedded in $\mathbb{A}^3$ as a quadric in the plane $z = -1$.

(e) If $x_2 = 0$, then from the equations, $x_3 = x_4 = 0$ also on Y, and the points $(x_1, 0, 0, 0, x_5) \in Y$ always.

On the other hand, if $x_2 \neq 0$, then we see from the equations that $x_1 \neq 0$ (first equation), then $x_4 \neq 0$ (second equation), and then $x_5 \neq 0$ (last equation).

Hence for these nonzero values,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}. \tag{*}$$

From the equations, from these it follows that the x values satisfy the further equations

$$x_1x_4 - x_2x_3 = x_1x_5 - x_2x_4 = x_2x_5 - x_3x_4 = 0. \tag{**}$$

We get

$$\mathbb{V}(K) = \mathbb{V}(x_2, x_3, x_4) \cup \mathbb{V}(\text{six equations (*) and (**)}). \tag{†}$$

Also, as in the proof of the Veronese embedding theorem, from the six equations (*)–(**) it follows that there are (x, y) such that

$$(x_1, \ldots, x_5) = (x^4, x^3y, x^2y^2, xy^3, y^4).$$

So in (†) above, the first variety

$$\mathbb{V}(x_2, x_3, x_4) \cong \mathbb{A}^2$$

is irreducible, while the second variety is a surjective image of $\mathbb{A}^2$, so is also irreducible. Geometrically, it is the affine cone over the fourth Veronese embedding of $\mathbb{P}^1$.