Problem Sheet 2

Section A: introductory questions, not for handing in

1. **Projective space is locally affine.**

   For $0 \leq j \leq n$, consider the map $\phi_j : U_j \to \mathbb{A}^n$ defined in lectures, mapping the j-th affine coordinate patch $U_j \subset \mathbb{P}^n$ to affine n-space. Using the Zariski topology of projective varieties on $\mathbb{P}^n$ restricted to U_j, and the Zariski topology of affine varieties on $\mathbb{A}^n$, show that ϕ_j is a homeomorphism of topological spaces.

   Solution: We need to show that ϕ_i is a continuous bijection of topological spaces with a continuous inverse. We defined

   $$\phi_j : U_j = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n | x_j \neq 0\} \to \mathbb{A}^n$$

   by

   $$[x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_j}, \ldots \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots \frac{x_n}{x_j}\right).$$

   This is continuous: if $X = \mathbb{V}(I) \subset \mathbb{A}^n$ is a closed subset then $\phi_j^{-1}(X) = U_j \cap \mathbb{V}(\hat{I})$ where

   $$\hat{I} = (\hat{f} | f \in I) \subset k[x_0 : \ldots : x_n].$$

   Here $\hat{f}$ stands for the homogenisation of f with respect to the j-th coordinate, that is

   $$\hat{f} = x_j^{\deg f} f\left(\frac{x_0}{x_j}, \ldots \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots \frac{x_n}{x_j}\right).$$

   The inverse is

   $$\phi_j^{-1} : \mathbb{A}^n \to U_j,\ (x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_{j-1} : 1 : x_j : \ldots : x_n]$$

   and this is continuous as for $X = \mathbb{V}(I) \cap U_j \subset \mathbb{P}^n$ where $I \subset k[x_0, \ldots, x_n]$ homogeneous ideal we can define

   $$\tilde{I} = (f(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n)) \subset k[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n],$$

   which after shifting the indices $x_i \to x_{i+1}$ for $i = 0, \ldots, j-1$ defines the ideal $\tilde{I} \subset k[x_1, \ldots, x_n]$ and the inverse $\phi_j^{-1}(X) = \mathbb{V}(\tilde{I})$.

2. **Projective closures and affine cones**

   (a) Let X be the parabola $\mathbb{V}(y - x^2) \subset \mathbb{A}^2$. What is its projective closure $\bar{X} \subset \mathbb{P}^2$? Draw the affine cone $\hat{\bar{X}}$ over $\bar{X}$, in $\mathbb{A}^3$, and identify the line corresponding to the "point at infinity" on $\bar{X}$.

   (b) Show that the affine varieties $\mathbb{V}(y - x^2) \subset \mathbb{A}^2$ and $\mathbb{V}(y - x^3) \subset \mathbb{A}^2$ are isomorphic. Can you give an intuitive explanation why their two projective closures in $\mathbb{P}^2$ are *not* projectively equivalent (isomorphic)? We do not have the tools yet to prove that they are non-isomorphic; try and find an intuitive reason.

Solution: (a) $X = \mathbb{V}(y - x^2) \subset D_z = \{[x : y : 1] | x, y \in k\} \subset \mathbb{P}^3_k$ then $\bar{X} = \mathbb{V}(yz - x^2)$ and the point at infinity is $[0 : 1 : 0]$ which corresponds to the y axis in the cone $\hat{\bar{X}} \subset \mathbb{A}^3$. (b) $f : \mathbb{A}^2 \to \mathbb{A}^2, (x, y) \mapsto (x, x^3)$ restricts to an isomorphism $\mathbb{V}(y - x^2) \cong \mathbb{V}(y - x^3)$. The inverse morphism is

$$f^{-1} : \mathbb{V}(y - x^3) \to \mathbb{V}(y - x^2), (x, y) \mapsto (x, x^2).$$

The projective closures are $\mathbb{V}(yz - x^2), \mathbb{V}(yz^2 - x^3)$ in $\mathbb{P}^2$. They are not projectively equivalent (isomorphic via a global isomorphism of $\mathbb{P}^2$) because a generic line intersects the first in 2, the second in 3 points. Alternatively, they are not isomorphic because the first one is nonsingular (isomorphic to $\mathbb{P}^1$), whereas the second one is singular at $[0 : 1 : 0]$, with affine equation $y^2 = x^3$ at this point.

Section B: core questions, to be handed in for marking

3. **The Twisted Cubic.** The projective variety $C = \mathbb{V}(F_0, F_1, F_2) \subset \mathbb{P}^3$, where

$$\begin{aligned} F_0(z_0, z_1, z_2, z_3) &= z_0 z_2 - z_1^2 \\ F_1(z_0, z_1, z_2, z_3) &= z_0 z_3 - z_1 z_2 \\ F_2(z_0, z_1, z_2, z_3) &= z_1 z_3 - z_2^2, \end{aligned}$$

is known as the *twisted cubic*.

(a) Show that C is equal to the image of the Veronese map

$$\begin{aligned} \nu : \mathbb{P}^1 &\to \mathbb{P}^3 \\ \nu : [x_0 : x_1] &\mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]. \end{aligned}$$

(b) Restrict to the affine patch $U_0 \subset \mathbb{P}^3$ given by setting $z_0 = 1$. Show that $C \cap U_0$ is equal to $\mathbb{V}(f_0, f_1) \subset \mathbb{A}^3$, where $f_i(z_1, z_2, z_3) := F_i(1, z_1, z_2, z_3)$ for $i = 1, 2$.

(c) For $i = 0, 1, 2$ we write Q_i for the quadric hypersurface $\mathbb{V}(F_i) \subset \mathbb{P}^3$. Show that, for $i \neq j$, the hypersurfaces Q_i and Q_j intersect in the union of C and a line. Therefore no two of them alone may be used to define C. Deduce that the homogenizations of the generators of an affine ideal do not necessarily generate the homogeneous ideal of the projective closure, showing that indeed we need to homogenise *all* elements of the affine ideal.

4. **Veronese varieties**

(a) Show that any projective variety is isomorphic to the intersection of a Veronese variety with a linear space, the projectivisation $\mathbb{P}(V)$ of some k-vector subspace $V \subset k^{n+1}$.

(b) Deduce that any projective variety is isomorphic to an intersection of quadric hypersurfaces.

5. **The ruled surface.** The image of the Segre morphism $\sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) = \Sigma_{1,1} \subset \mathbb{P}^3$ is known as a *ruled surface*.

(a) What equations define $\Sigma_{1,1}$ as a subvariety of $\mathbb{P}^3$?

(b) What are the images in $\Sigma_{1,1}$ of $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$? Show that through any point in $\Sigma_{1,1}$ there are two lines lying in $\Sigma_{1,1}$.

(c) Exhibit some disjoint lines in $\Sigma_{1,1}$. Recall that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \Sigma_{1,1}$. Is this isomorphic to $\mathbb{P}^2$? Draw the "real cartoons" of either surface.

6. **Rational normal curves** Fix a positive integer $d > 1$.

   (a) Let $G(x_0, x_1) = \Pi_{i=1}^{d+1}(b_i x_0 - a_i x_1)$ be a homogeneous degree $(d+1)$ polynomial with distinct roots $[a_i : b_i] \in \mathbb{P}^1$. Show that $H_i(x_0, x_1) = G(x_0, x_1)/(b_i x_0 - a_i x_1)$ form a basis for the space of homogeneous polynomials of degree d.

   (b) Deduce that the image of the map $\mu_d \colon \mathbb{P}^1 \to \mathbb{P}^d$ defined by

   $$[x_0 : x_1] \mapsto [H_1(x_0, x_1) : \cdots : H_{d+1}(x_0, x_1)]$$

   is projectively equivalent to the image of the Veronese embedding, that is, it is a rational normal curve of degree d.

   (c) What is the image of the point $[a_i : b_i]$ under μ_d? If (a_i, b_i) are nonzero for all i, what is the image of $[1 : 0]$ and $[0 : 1]$?

   (d) Deduce that through any $d + 3$ points in general position in $\mathbb{P}^d$, there passes a unique rational normal curve of degree d. (Recall that for $d + 3$ points to be in general position means that no subset of $d + 1$ of these points lies on a hyperplane in $\mathbb{P}^d$.)

Section C: optional questions

7. **Projective variety corresponding to a graded ring.** If $R = \sum_{d \geq 0} R_d$ is a graded ring and $e \geq 1$ is an integer, we define

$$R^{(e)} := \sum_{d \geq 0} R_{de}.$$

We define a grading on $R^{(e)}$ by letting $R^{(e)}_d := R_{de}$.

(a) Find $k[x_0, x_1]^{(2)}$, expressing it in the form $k[z_0, \ldots, z_n]/I$ for some n and I.

(b) Find the homogeneous coordinate rings $S(\mathbb{P}^1)$ and $S(\nu_2(\mathbb{P}^1))$. Comment in the context of part (a).

(c) More generally, show that $S(\nu_e(\mathbb{P}^n)) \cong k[x_0, \ldots, x_n]^{(e)}$, and hence that $k[x_0, \ldots, x_n]^{(e)}$ defines the same projective variety as $k[x_0, \ldots, x_n]$.

(d) Are $k[x_0, \ldots, x_n]^{(e)}$ and $k[x_0, \ldots, x_n]$ isomorphic as graded k-algebras? Are they isomorphic as (ungraded) k-algebras? What does this imply about the affine cones of $\nu_e(\mathbb{P}^n)$ and $\mathbb{P}^n$?

Solution: (a) $k[x_0, x_1]^{(2)} = k[x_0^2, x_0x_1, x_1^2] = k[z_0, z_1, z_2]/(z_0z_2 - z_1^2)$, so the quotient is the affine cone over the Veronese cubic.

(b) $S(\mathbb{P}^1) = k[x_0, x_1]^{(1)}$. Since $\nu_2([x_0 : x_1]) = [x_0^2 : x_0x_1 : x_1^2]$, we have $\mathbb{I}(\nu_2(\mathbb{P}^1)) = (z_0z_2 - z_1^2)$, so $S(\nu_2(\mathbb{P}^1)) = k[z_0, z_1, z_2]/(z_0z_2 - z_1^2) = k[x_0, x_1]^{(2)}$ by (a).

(c) The map $\nu_e([x_0 : \ldots : x_n]) = [\ldots : x^I : \ldots]$, where $I \in \binom{n+d}{d}$, gives the embedding $\nu_e : \mathbb{P}^m \to \mathbb{P}^N$ onto $\nu_e(\mathbb{P}^m)$. We have $\nu_e^*(z_I) = x^I$, giving us an isomorphism from $S(\nu_e(\mathbb{P}^n))$ to $k[\ldots, x^I, \ldots] \subset k[\ldots, x_i, \ldots]$. Therefore $S(\nu_e(\mathbb{P}^n)) \cong k[\ldots, x^I, \ldots] \cong k[x_0, \ldots, x_n]^{(e)}$.

(d) They are not isomorphic as graded k-algebras for $e \geq 2$ since

$$\dim_k k[x_0, \ldots, x_n]^{(e)}_1 = \dim_k k[x_0, \ldots, x_n]_e > n$$

whereas $\dim_k k[x_0, \ldots, x_n]_1 = n$. They are also not isomorphic as k-algebras, because $k[x_0, \ldots, x_n]$ is generated by $x_0, \ldots, x_n$ as a k-algebra, whereas $k[x_0, \ldots, x_n]^{(e)}$ cannot be generated by $n+1$ elements since already $\dim_k k[x_0, \ldots, x_n]^{(e)}_1 > n$, whereas the degree 1 parts of any given $n+1$ elements would only span a k-vector space of dimension at most $n+1$. So although the two projective varieties are isomorphic, $\nu_e(\mathbb{P}^n) \cong \mathbb{P}^n$, their affine cones are not isomorphic affine varieties since their coordinate rings are not isomorphic algebras.

Some further differences: $k[z_0, z_1, z_2]/(z_0z_2 - z_1^2)$ is not a UFD due to the factorisations $z_0z_2 = z_1z_1$, so $S(\nu_2(\mathbb{P}^1))$ is not isomorphic as a k-algebra to $S(\mathbb{P}^1) = k[x_0, x_1]$ which is

a UFD. One can also argue that 0 is a singular point in the affine cone of $\nu_2(\mathbb{P}^1)$ since that has affine coordinate ring $S(\nu_2(\mathbb{P}^1))$, whereas the affine cone of $\mathbb{P}^1$ has no singular points. More generally, $S(\nu_d(\mathbb{P}^n))$ is not a UFD due to factorisations $z_I z_J = z_K z_L$, unlike $S(\mathbb{P}^N) = k[x_0, \ldots, x_N]$ which is always a UFD.

8. **(C3.1a 2011 q. 2.)** Let k be an algebraically closed field.

   (a) What is a *projective variety* over k? What is a *morphism* between projective varieties?

   (b) Define the *Segre morphism* $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$, and show that its image is a projective variety.

   (c) Let X and Y be projective varieties. Define $X \times Y$, and prove that $X \times Y$ is a projective variety. Prove that the projection maps $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$, defined by $\pi_X(x, y) = x$, $\pi_Y(x, y) = y$, are morphisms. Prove also that if Z is a projective variety and $\alpha : Z \to X$, $\beta : Z \to Y$ are morphisms, then there exists a morphism $\gamma : Z \to X \times Y$ such that $\pi_X \circ \gamma = \alpha$ and $\pi_Y \circ \gamma = \beta$.

   (d) Define the dimension, degree and Hilbert polynomial of a projective variety $X \subseteq \mathbb{P}^n$.

   (e) Calculate the Hilbert polynomial of $\sigma_{n,m}\big(\mathbb{P}^n \times \mathbb{P}^m\big)$. What is its degree?

Solution: (a) A *projective varietyX* $\subset \mathbb{P}^n$ is the common zero locus

$$\{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid F_\alpha(x_0, \ldots, x_n) = 0 \ \forall \alpha \in A\}$$

of the homogeneous polynomials $\{F_\alpha\} \in k[x_0, \ldots, x_n]$ (not necessarily of the same degree), indexed by a set A.

Let $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ be projective varieties. A map $F : X \to Y$ is a *morphism* of projective varieties if for all $x \in X$ there exists an open neighbourhood $x \in U_x \subseteq X$, and homogeneous polynomials $f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$ of the same degree such that $F|_{U_x} : U_x \to Y$ is given by

$$F([x_0 : \cdots : x_n]) = [f_0(x_0, \ldots, x_n) : \cdots : f_m(x_0, \ldots, x_n)].$$

(b) The *Segre morphism* is defined as $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ such that

$$\sigma_{n,m}\big([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]\big) = [\ldots : x_i y_j : \ldots],$$

where on the right-hand side we list all quadratic monomials with $0 \leq i \leq n$, $0 \leq j \leq m$ in arbitrary order.

Let z_{ij}, $0 \le i \le n$, $0 \le j \le m$, be the coordinates on $\mathbb{P}^{(n+1)(m+1)-1}$, indexed by the entries of an $(n+1) \times (m+1)$ matrix, and let $x_i y_j$ be in the place z_{ij} in the Segre morphism.

The columns of $\sigma_{n,m}([x],[y]) = (x_i y_j)$ are scalar multiples of one another, so all 2×2 minors vanish on $\Sigma_{n,m}$. Let

$$I = (z_{ij}z_{kl} - z_{il}z_{kj} : 0 \le i < k \le n,\ 0 \le j < l \le m)$$

be the ideal generated by the 2×2 minors. Then $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{V}(I)$.

On the other hand, if $(z_{ij}) \in \mathbb{V}(I)$, that is, all 2×2 minors vanish, then it has rank 0 or 1. Rank 0 is impossible, since the coordinates are in projective space. So all columns are scalar multiples of one nonzero column.

Let $z_{i_1 j}, \ldots, z_{i_n j}$ be a nonzero column, and let $\lambda_0, \ldots, \lambda_m$ be these scalars (at least one of them is nonzero). Then

$$[z_{ij}] = \sigma_{n,m}\big([z_{i_1 j}, \ldots, z_{i_n j}], [\lambda_0, \ldots, \lambda_m]\big)$$

is in the image of $\sigma_{n,m}$. Hence $\mathbb{V}(I) \subseteq \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$. Therefore $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{V}(I)$, so its image is a projective variety.

(c) Suppose $X = \mathbb{V}(F_1, \ldots, F_s) \subseteq \mathbb{P}^n$ and $Y = \mathbb{V}(G_1, \ldots, G_r) \subseteq \mathbb{P}^m$. We define

$$X \times Y = \sigma_{n,m}(X \times Y) \subseteq \Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subseteq \mathbb{P}^{(n+1)(m+1)-1},$$

with the induced topology.

The homogeneous coordinates on $\mathbb{P}^{(n+1)(m+1)-1}$ have the form z_{ij}, $0 \le i \le n$, $0 \le j \le m$. Let $F_{kj} \in k[z_{ij}]_{i=0,\ldots,n}$ be the polynomial

$$F_{kj}(z_{00}, \ldots, z_{nm}) = F_k(z_{0j}, \ldots, z_{nj}).$$

Similarly,

$$G_{il}(z_{00}, \ldots, z_{nm}) = G_l(z_{i0}, \ldots, z_{im}).$$

Then

$$\sigma_{n,m}(X \times Y) = \Sigma_{n,m} \cap \mathbb{V}\left(\{F_{kj}\}_{\substack{k=1,\ldots,s \\ j=0,\ldots,m}}, \{G_{il}\}_{\substack{i=0,\ldots,n \\ l=1,\ldots,r}}\right),$$

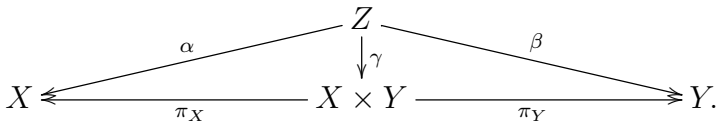
so by definition $X \times Y$ is closed.

Define the morphism $\pi_n : \Sigma_{n,m} \to \mathbb{P}^n$ such that π_n takes $[z_{00}, \ldots, z_{nm}]$ to a nonzero column of the matrix $[z_{0j} : \cdots : z_{nj}] \in \mathbb{P}^n$, the point whose homogeneous coordinates are that column. Likewise define π_m using the rows.

These are well-defined projections, since at least one column is nonzero, and they are scalar multiples of each other, so the map agrees on the intersection of the standard

affine charts. This implies the second statement, because π_X, π_Y are the restrictions of the projections $\pi_n : \Sigma_{n,m} \to \mathbb{P}^n$, $\pi_m : \Sigma_{n,m} \to \mathbb{P}^m$ to $\sigma_{n,m}(X \times Y)$.

Suppose α, β are given morphisms of varieties. Since $X \times Y$ is set-theoretically the categorical product, there exists set-theoretically a map $\gamma : Z \longrightarrow X \times Y$ such that it is the only possible one to make the following diagram commute:



It remains to check that γ is a morphism of projective varieties. If $p \in Z$, let $U \subseteq Z$ be an open set containing p, and suppose that $\alpha|_U$ is given by $\alpha : p \longmapsto [f_0(p) : \cdots : f_n(p)]$ and $\beta|_U$ is given by $\beta : p \longmapsto [g_0(p) : \cdots : g_m(p)]$. Then on U, γ is given by $\gamma : p \longmapsto [\ldots : f_i(p)g_j(p) : \ldots]$. This is a morphism of projective varieties.

(d) The dimension of the projective variety $X \subseteq \mathbb{P}^m$ is the maximal length n of a chain

$$\varnothing \subsetneq X_0 \subsetneq \cdots \subsetneq X_n \subseteq X,$$

where the X_i are irreducible projective subvarieties of $\mathbb{P}^m$.

The degree of $X \subseteq \mathbb{P}^m$ is the maximal number of points of the intersection $X \cap L$, where $L \subseteq \mathbb{P}^m$ is a linear subspace with $\dim(L) + \dim(X) = m$.

The Hilbert function of $X \subseteq \mathbb{P}^m$ is a function $h_X : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, defined by $h_X(d) = \dim S(X)_d$, where $S(X)_d$ is the degree-d part of the graded coordinate ring $S(X)$. This is a polynomial function for $d \gg 0$, and the Hilbert polynomial p_X is defined as $p_X(d) = h_X(d)$ for $d \gg 0$.

(e) The degree-d part of the graded coordinate ring of $\Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m)$ is

$$S(\Sigma_{n,m})_d = k[x_i y_j]/(2 \times 2 \text{ minors}) = k[x_i y_j]_{2d}.$$

So

$$h_{\Sigma_{n,m}}(d) = \#\{\text{monomials } x_0^{i_0} \cdots x_n^{i_n} y_0^{j_0} \cdots y_m^{j_m} : i_0 + \cdots + i_n = j_0 + \cdots + j_m = d\}.$$

Hence

$$h_{\Sigma_{n,m}}(d) = \binom{n+d}{d}\binom{m+d}{d}.$$

This is already a polynomial in d, so this is the Hilbert polynomial. We know that its leading term is

$$p_X(d) = \frac{\deg(X)}{\dim(X)!} d^{\dim(X)} + \cdots,$$

so since

$$p_{\Sigma_{n,m}}(d) = \frac{1}{n!m!} d^{n+m} + \cdots = \frac{\binom{n+m}{n}}{(n+m)!} d^{n+m} + \cdots,$$

the degree is $\deg(\Sigma_{n,m}) = \binom{n+m}{n}$.