Professor Joyce **C3.4 Algebraic Geometry** **MT 2026**

Problem Sheet 3

Section A: introductory questions, not for handing in

1. **Degree of a conic, explicitly.** Let $C = \{x_0^2 + x_1^2 + x_2^2 = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2}$. Consider lines

$$L = \{a_0x_0 + a_1x_1 + a_2x_2 = 0\} \subset \mathbb{P}^2_{x_0,x_1,x_2}.$$

   (a) Find the maximum number of intersection points between C and L.

   (b) Find also the set of coefficients (a_0, a_1, a_2) so that the intersection $L \cap C$ *does not* consist of the maximal number of intersection points.

   Solution: (a) Assume that $a_0 \neq 0$. Then we get $x_0 = -a_0^{-1}(a_1x_1 + a_2x_2)$. Substituting into the quadratic equation, we get a quadratic

$$(a_0^2 + a_1^2)x_1^2 + 2a_1a_2x_1x_2 + (a_0^2 + a_2^2)x_2^2 = 0.$$

   This has two distinct roots for $[x_1 : x_2]$ unless the discriminant of the quadratic vanishes.

   (b) The discriminant is, after rearranging,

$$4a_0^4 + 4a_0^2a_1^2 + 4a_0^2a_2^2 = 0$$

   or, since we are working with $a_0 \neq 0$,

$$a_0^2 + a_1^2 + a_2^2 = 0.$$

   This is now symmetric in $[a_0 : a_1 : a_2]$ so this is our condition.

   Remark: The coefficients a_i parametrize the *dual projective plane* $\mathbb{P}^2_{a_0,a_1,a_2}$, which plays the role of the general Grassmannian in the definition of degree. The locus you have just computed is the bad locus inside this Grassmannian; its complement is the generic locus in the definition of degree. The dual conic is isomorphic to the original conic.

2. **Basics on dimension.**

   Show that dimension is an invariant of the isomorphism class of a projective variety.

Solution: Let $f : X \to Y$ be an isomorphism with inverse f^{-1} between $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. If $X_i \subset X$ is irreducible then $f(X_i) \subset Y$ is also irreducible subvariety. Indeed, if we had $f(X_i) = Z_1 \cup Z_2$ then $X_i = f^{-1}(Z_1) \cup f^{-1}(Z_2)$ would give a nontrivial decomposition, contradiction. On the other hand, $f(X_i) = (f^{-1})^{-1}(X_i)$ is closed in Y, so a subvariety. This proves that the chain

$$\emptyset \subsetneq f(\{p\}) \subsetneq f(X_1) \subsetneq \ldots \subsetneq f(X_n) \subset Y$$

is a chain of irreducibles, so $\dim(Y) \leq \dim(X)$. The inverse gives the opposite inequality.

Section B: core questions, to be handed in for marking

3. **Another set of equations for the twisted cubic curve.**

   (a) Let $Q = \{x_0x_2 = x_1^2\} \subset \mathbb{P}^3$ and $F = \{x_0x_3^2 - 2x_1x_2x_3 + x_2^3 = 0\} \subset \mathbb{P}^3$. Prove that $C = Q \cap F \subset \mathbb{P}^3$ is the twisted cubic curve, the image of the third Veronese embedding $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$.
   Hint: multiply the second equation with x_2 and use the first equation to put it in the form of a perfect square.

   (b) Does this mean that the ideal $\mathbb{I}(\nu_3(\mathbb{P}^1))$ can in fact be generated by two elements? (Compare with Sheet 2, Question 3.)

4. **Projecting a space curve.**

   Let $Q_0 = \{x_0x_3 = x_1^2\} \subset \mathbb{P}^3$ and $Q_1 = \{x_1x_3 = x_2^2\} \subset \mathbb{P}^3$. Consider the projective variety $C = Q_0 \cap Q_1 \subset \mathbb{P}^3$. Show that the formula $\pi\colon [x_0\colon x_1\colon x_2\colon x_3] \mapsto [x_0\colon x_1\colon x_2]$ defined on a Zariski open subset of C can be extended to a projective morphism

   $$\pi\colon C \to \mathbb{P}^2.$$

   Show that π maps C isomorphically to the projective variety (plane curve)

   $$D = \{x_0x_2^2 = x_1^3\} \subset \mathbb{P}^2.$$

   Hint: to extend π, try to express the ratios $x_0 : x_1 : x_2$ in a different way using the equations of C.

5. **Dimension, degree and Hilbert polynomial.**

   (a) Show carefully that if X is a reducible projective variety with equidimensional irreducible components X_i, then $\deg X = \sum_i \deg X_i$.

   (b) Compute the degree $\deg \nu_d(\mathbb{P}^1)$ directly from the definition.

   (c) Let F be a homogeneous irreducible polynomial of degree d in $k[x_0, \ldots, x_n]$, and let $X = \mathbb{V}(F) \subset \mathbb{P}^n$. Find the Hilbert polynomial of X. Deduce that, as expected, $\dim X = n - 1$ and $\deg X = d$.

6. **Affine and quasi-projective varieties.**

   (a) Find an open affine cover of $\mathbb{A}^2_k \setminus \{(0,0)\}$.

   (b) Show that $\mathrm{GL}(n,k)$ is an affine variety, i.e. that it is isomorphic as a quasi-projective variety to a Zariski closed subset of an affine space.

   (c) Let X be an affine variety and $f \in k[X]$. Show that f vanishes nowhere on X if and only if f is invertible in $k[X]$.

Section C: optional questions

7. **The Veronese image.**

   Find the Hilbert polynomial of the Veronese image $\nu_d(\mathbb{P}^n)$. Find the dimension and degree of this projective variety.

   Solution: Recall from the previous Problem Sheet that $S(\nu_d\mathbb{P}^n) \cong R^d = \oplus_{e\geq 0} R_{de}$. Thus $S(\nu_d(\mathbb{P}^n))_m = k[x_0,\ldots,x_n]_{dm}$, so

$$h_{\nu_d(\mathbb{P}^n)}(m) = \binom{dm+n}{n} = \frac{(dm+n)\cdots(dm+1)}{n!} = \frac{d^n}{n!}m^n + \cdots,$$

   so since $h_X(m) = \frac{\deg(X)}{\dim(X)!}m^{\dim(X)} + \cdots$, we get $\dim(\nu_d(\mathbb{P}^n)) = n, \deg(\nu_d(\mathbb{P}^n)) = d^n$.

8. **Complete intersection of two hypersurfaces**

   (a) Let $f, g \in k[x_0,\ldots,x_n]$ be two essentially distinct irreducible homogeneous polynomials (i.e. not scalar multiples of each other) of degrees d, e such that the homogeneous ideal $I = \langle f, g\rangle$ is radical. Compute the Hilbert function of $X = \mathbb{V}(I) \subset \mathbb{P}^n$. Deduce the dimension and degree of X.

   (b) Conclude that the twisted cubic $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$ is not a *complete intersection*: its (radical) vanishing ideal cannot be generated by two homogeneous polynomials. Compare with Question 3.

   Solution: (a) Write $R = k[x_0,\ldots,x_n]$ and $I = (f,g)$, where $\deg f = d$, $\deg g = e$. We first compute the Hilbert function of $X = \mathbb{V}(I) \subset \mathbb{P}^n$.

   The homogeneous coordinate ring of X is $k[X] = R/(f,g)$.

   Multiplication by f gives the exact sequence

$$0 \longrightarrow R(-d) \xrightarrow{\cdot f} R \longrightarrow R/(f) \longrightarrow 0.$$

   (This is exact as R has no zero divisors.) Therefore

$$h_{R/(f)}(m) = h_R(m) - h_R(m-d).$$

   Since $h_R(m) = \binom{n+m}{n}$, where we interpret $\binom{a}{n} = 0$ if $a < n$, we obtain

$$h_{R/(f)}(m) = \binom{n+m}{n} - \binom{n+m-d}{n}.$$

   Since g is a non-zero-divisor in $R/(f)$, multiplication by g gives another exact sequence

$$0 \longrightarrow (R/(f))(-e) \xrightarrow{\cdot g} R/(f) \longrightarrow R/(f,g) \longrightarrow 0.$$

(This is exact as $R/(f)$ has no zero divisors, since f is irreducible.) Consequently

$$h_X(m) = h_{R/(f)}(m) - h_{R/(f)}(m-e).$$

Thus

$$h_X(m) = \binom{n+m}{n} - \binom{n+m-d}{n} - \binom{n+m-e}{n} + \binom{n+m-e-d}{n},$$

Equivalently, the Hilbert series is

$$H_X(t) = \frac{(1-t^d)(1-t^e)}{(1-t)^{n+1}}.$$

For $m \gg 0$, the Hilbert function agrees with the Hilbert polynomial

$$p_X(m) = \binom{n+m}{n} - \binom{n+m-d}{n} - \binom{n+m-e}{n} + \binom{n+m-e-d}{n}.$$

where now we define $\binom{a}{n} = \frac{1}{n!}a(a-1)\cdots(a+1-n)$, which in general does *not* have $\binom{a}{n} = 0$ if $a < n$.

To determine its dimension and degree, rewrite the Hilbert series as

$$H_X(t) = \frac{(1+t+\cdots+t^{e-1})(1+t+\cdots+t^{d-1})}{(1-t)^{n-1}}.$$

It follows that the Hilbert polynomial has degree $n-2$. Hence $\dim X = n-2$.

Moreover, if a projective variety of dimension r has degree δ, then its Hilbert polynomial has leading term $\frac{\delta}{r!}m^r$. Here the numerator of the Hilbert series has value

$$(1+\cdots+1)(1+\cdots+1) = de$$

at $t=1$. Therefore the leading term of $p_X(m)$ is $\frac{de}{(n-2)!}m^{n-2}$. Hence $\deg X = de$.

(b) From Question 7, $\nu_3(\mathbb{P}^1)$ has Hilbert polynomial

$$p_{\nu_3(\mathbb{P}^1)}(m) = \binom{3m+1}{1} = 3m+1,$$

and has degree 3. If $\nu_3(\mathbb{P}^1)$ were a complete intersection of hypersurfaces of degrees d, e in $\mathbb{P}^3$ it would have degree de, so the only possibility (up to order) is $d=1$, $e=3$, and then we would have

$$p_{\nu_3(\mathbb{P}^1)}(m) = \binom{3+m}{3} - \binom{3+m-1}{3} - \binom{3+m-3}{3} + \binom{3+m-4}{3} = 3m,$$

a contradiction.

9. **Structure sheaf: computing $\mathcal{O}_{X,p}$ and $\mathcal{O}_X(U)$.**

   (a) Let $X = \mathbb{V}(xy) \subset \mathbb{A}^2_k$ be the union of the two axes. Show algebraically that $\mathcal{O}_{X,(0,1)} \cong k[y]_{(y-1)}$, and then explain geometrically why you are not surprised that this is the local ring of $\mathbb{A}^1_k$ at 1. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Compare, as you vary $p \in X$, the dimensions of the k-vector spaces

   $$\mathfrak{m}_p/\mathfrak{m}_p^2.$$

   Here for an ideal I, by I^2 one means the ideal generated by all $i \cdot j$, for $i, j \in I$. Deduce that the k-algebra $\mathcal{O}_{X,(0,0)}$ cannot be generated by one function.

   (b) Find $\mathcal{O}_X(U)$, where $X = \mathbb{P}^2$ and

   $$U = X \setminus \mathbb{V}(x_0^2 + x_1^2 + x_2^2).$$

   Hint. First compute $\mathcal{O}_X(U \cap U_i)$ for the usual cover of $\mathbb{P}^n$ by affine charts U_i. Then consider the restriction of functions to overlaps $U \cap U_i \cap U_j$.

Remark. $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the *cotangent space* to X at p (its dual vector space is called *tangent space*). When the dimension of this vector space is not equal to the dimension of X, then p is called a *singularity*. Non-singular points are called *regular*. In algebra, for a Noetherian local ring A with maximal ideal $\mathfrak{m}$, one says that A is *regular* if $\dim A$ equals the vector space dimension $\dim_k \mathfrak{m}/\mathfrak{m}^2$ (in which case, that number equals the minimal number of generators needed to generate the ideal $\mathfrak{m}$).

Solution: (a) Using the lecture notes,

$$\mathcal{O}_{X,(0,0)} = k[X]_{(x,y)} = (k[x,y]/(xy))_{(x,y)}.$$

$$\mathcal{O}_{X,(0,1)} = k[X]_{(x,y-1)} = (k[x,y]/(xy))_{(x,y-1)} = k[y]_{(y-1)}.$$

Here in the last step $k[x,y]/(xy) = k[\overline{x}] + k[\overline{y}]$ where $\overline{x} \cdot \overline{y} = 0$. But $\overline{y}$ is a unit because it does not belong to $(\overline{x}, \overline{y} - 1)$, so $\overline{x} \cdot \overline{y} = 0$ forces $\overline{x} = \overline{x}\,\overline{y}\,\frac{1}{\overline{y}} = 0$.

$k[x,y]/(xy)$ corresponds to the union of the two axes in $\mathbb{A}^2_k$. We localise at $(0,1)$, so we can assume all germs of functions are defined on neighbourhoods of $(0,1)$ contained in the open set $D_y = \mathbb{A}^2_k \setminus \mathbb{V}(y) = (y\text{-axis} \backslash 0)$. This is the same as asking what the germs of functions of $\mathbb{A}^1_k \setminus 0$ are at 1, which is $\mathcal{O}_{\mathbb{A}^1_k,(1)}$.

$\mathfrak{m}_p = (\overline{x}, \overline{y})$, and $\mathfrak{m}_p^2 = (\overline{x}^2, \overline{xy}, \overline{y}^2)$. For $p = (0,1)$, we have $\overline{x} = 0 \in \mathcal{O}_{X,(0,1)}$, so $\mathfrak{m}_p/\mathfrak{m}_p^2 = (\overline{y})/(\overline{y}^2) = k \cdot \overline{y}$ is one-dimensional. A similar argument holds for any $p \neq (0,0)$.

For $p = (0,0)$, we claim $\overline{x}, \overline{y}$ are linearly independent (so the cotangent space has dimension 2). If they were linearly dependent, then for some $a, b \in k$,

$$s \cdot (ax + by) - f \in (xy)$$

for polynomials $f, s \in k[x,y]$ with $s(0,0) \neq 0$ and f vanishing to order≥ 2 at p (so $f \in M_p^2$, with $M_p = \{g \in k[X] : g(p) = 0\}$) – this is unpackaging the condition that $a\overline{x} + b\overline{y} \in \mathfrak{m}_p^2$. This is a contradiction since viewing the left hand side above as a polynomial in two variables x, y, it has a non-zero homogeneous part of degree 1 given by $s(0,0)(ax + by)$, unless $a = b = 0$. From this it follows that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is not one-dimensional (it is two-dimensional, with basis $\overline{x}, \overline{y}$). In particular, this implies that $\mathcal{O}_{X,p}$ cannot be generated by one element f as a k-algebra otherwise (f) would be the maximal ideal (if $k[x] \to \mathcal{O}_{X,p}, x \mapsto f$ is surjective then $k \cong k[x]/x \cong \mathcal{O}_{X,p}$ is a field), and so $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 \leq 1$.

Remark. For general X, that last argument shows that if $\mathcal{O}_{X,p}$ is generated by $f_1, \ldots, f_N$, as a k-algebra then $(f_1, \ldots, f_N)$ would be a (and hence the) maximal ideal, and so $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 \leq N$. So $\mathcal{O}_{X,p}$ will always require at least as many k-algebra generators as the dimension of the cotangent space. The dimension of X (at p) equals $\dim \mathcal{O}_{X,p}$, which is the maximal number of algebraically independent elements/k (notice in the example above $\overline{x}, \overline{y}$ are not algebraically independent due to $\overline{xy} = 0$). In the Remark above, Krull's height theorem in fact implies that $N \geq \dim A$ where N is the minimal set of generators of $\mathfrak{m} \subset A$ as an ideal. When equality holds, these generators are a *regular system of parameters* and $N = \dim_k \mathfrak{m}/\mathfrak{m}^2$. In general (for a local Noetherian ring A), if $\dim A = d$, one can find $x_1, \ldots, x_d$ (called a *system of parameters*) such that the radical of $I = (x_1, \ldots, x_d)$ is $\mathfrak{m}$, equivalently $\mathfrak{m}$ is a minimal prime ideal of I, equivalently some power of $\mathfrak{m}$ is contained in I, equivalently I is $\mathfrak{m}$-primary. Again by Krull, d is the minimal number of elements x_i needed for these equivalent properties to hold.

(b) $U = (U \cap U_0) \cup (U \cap U_1) \cup (U \cap U_2)$ is a cover with affine charts. By the lecture notes, $\mathcal{O}_X(U \cap U_0) = k[x_1, x_2]_{1+x_1^2+x_2^2}$. But

$$k[x_1, x_2]_{1+x_1^2+x_2^2} = k[x_0, x_1, x_2]_{(x_0^2+x_1^2+x_2^2)},$$

where the right hand side is the degree 0 part of the localised ring $k[x_0, x_1, x_2]_{x_0^2+x_1^2+x_2^2}$, that is

$$k[x_0, x_1, x_2]_{(x_0^2+x_1^2+x_2^2)} =$$
$$\left\{ \frac{f}{(x_0^2 + x_1^2 + x_2^2)^N} \in k[x_0, x_1, x_2]_{x_0^2+x_1^2+x_2^2} : N \in \mathbb{N}, f \text{ homogeneous of degree } \deg f = 2N \right\}.$$

Indeed, for $d = \max(\deg f, \deg g)$ the map

$$k[x_1, x_2]_{1+x_1^2+x_2^2} \to k[x_0, x_1, x_2]_{(x_0^2+x_1^2+x_2^2)}$$

$$\frac{f}{g} \mapsto \frac{x_0^d f(x_1/x_0, x_2/x_0)}{x_0^d g(x_1/x_0, x_2/x_0)}$$

is an isomorphism with inverse

$$k[x_0, x_1, x_2]_{x_0^2+x_1^2+x_2^2} \to k[x_1, x_2]_{(1+x_1^2+x_2^2)}$$

$$\frac{f(x_0, x_1, x_2)}{g(x_0, x_1, x_2)} \mapsto \frac{f(1, x_1, x_2)}{g(1, x_1, x_2)}.$$

The same holds for U_1, U_2, so

$$\mathcal{O}_X(U) = k[x_0, x_1, x_2]_{(x_0^2+x_1^2+x_2^2)}$$

Remark. If f is regular on U then it is regular on $U \cap U_j$, and above we are asking conversely which f_j regular on $U \cap U_j$ extend to a global regular function on U, so what we need to check is compatibility on overlaps: $f_i = f_j$ on $U \cap U_i \cap U_j$. You should compare this with the definition of sheaves in the notes.

10. **Localization.** Let R be a commutative ring.

    (a) Let S be a multiplicative subset of R. Show that the localization $S^{-1}R$ is naturally isomorphic, as an R-algebra, to $R[x_s : s \in S]/\langle sx_s - 1 : s \in S\rangle$.

    Here intuitively we are formally inverting each element of S because we introduce a new abstract variable x_s for each $s \in S$, and we impose the relation $sx_s = 1$. *Hint. To prove you get an isomorphism, it is easiest to try to guess the definition of the inverse map, and show that the two composites give identity maps.*

    (b) Consider the elliptic curve $X \subset \mathbb{A}_k^2$ defined by $y^2 = x(x-1)(x-2)$. Find the localisation of the coordinate ring $k[X]$ at the point $p = (1, 0) \in X$.

    Hint/Definition. You need to study $k[X]_{\mathfrak{m}_p}$, so localize at $S = k[X] \setminus \mathfrak{m}_p$.

    Show that the (unique) maximal ideal of this local ring is generated by y. What does this mean geometrically (i.e. interpret this result in terms of germs of functions at p defined on the curve)?

    (c) Suppose $r \in R$ is zero in each local ring: $\frac{r}{1} = 0 \in R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset R$. Deduce that $r = 0 \in R$.

    Hint. Recall that the annihilator $\operatorname{Ann} r = \{a \in R : ar = 0\}$ *is an ideal in R.*

    (d) Deduce from (c) that for any affine variety X, if two functions $f, g : X \to k$ have

equal local germs (i.e. $f = g$ inside $\mathcal{O}_{X,p}$) at each point of X, then they are globally equal: $f = g$.

Solution: **(a)** A high-tech proof would be to first define the localisation using a universal property, and then checking that the algebra $A = R[x_s : s \in S]/(sx_s - 1 : s \in S)$ satisfies the universal property. More directly: define $\varphi : S^{-1}R \to A$, $\frac{r}{s} \mapsto x_s \cdot r$. Check it is well-defined: if $\frac{r}{s} = \frac{r'}{s'}$ then for some $t \in S$ we have $t(s'r - sr') = 0$. Inside A, multiply that relation by $x_t x_{s'} x_s$ and use the relations, to deduce $x_s r = x_{s'} r' \in A$ as required. Then check φ is an R-algebra homomorphism. Next, define an R-algebra hom $\psi : A \to S^{-1}R$ by the conditions $r \mapsto r$ and $x_s \mapsto \frac{1}{s}$. Again, check it is well-defined. Finally, notice φ, ψ are inverses of each other.

(b) We need to invert elements that are not in the maximal ideal $\mathfrak{m}_p = (x-1, y)$. From lecture notes, the maximal ideal of $A_p = k[X]_{\mathfrak{m}_p}$ is $M_p = A_p \cdot \mathfrak{m}_p = (x-1, y) \subset A_p$. Since $x(x-2) \notin \mathfrak{m}_p$, it is a unit u in $k[X]_{\mathfrak{m}_p}$, thus the equation for X locally becomes $x - 1 = u^{-1}y^2$, so $x - 1 \in (y)$. Thus $M_p = (y) \subset A_p$. Geometrically, A_p is the ring of germs at p of functions defined on X, and M_p are those germs which vanish at p. This result says that all germs vanishing at p are multiples of the coordinate function y.

(c) If r is non-zero, consider the annihilator $\mathrm{Ann}(r) \subset R$ (which is a proper ideal, since $1 \notin \mathrm{Ann}(r)$). Pick any maximal ideal $\mathfrak{m} \subset R$ which contains $\mathrm{Ann}(r)$. By assumption, $r/1 = 0$ in $R_{\mathfrak{m}}$, therefore $ar = 0$ for some $a \in R \setminus \mathfrak{m}$. This contradicts the fact that $a \in \mathrm{Ann}(r) \subset \mathfrak{m}$.

(d) Follows from (c), taking $R = k[X]$ and $r = f - g$ (and recall from lectures that any maximal ideal of R arises as $\mathfrak{m}_p$ for some $p \in X$, since this holds for $k[x_1, \ldots, x_n]$ and $\mathbb{A}^n_k$).

Remark. The statement is obvious, since equal germs implies $f(p) = g(p)$ for all p, so $f = g$.

11. **(C3.4 2016 question 3.)** Let k be an algebraically closed field.

    (a) State the definitions of a *projective variety* $X \subset \mathbb{P}^n$, the *dimension* of X, and the *degree* of X.

    (b) Define the *Hilbert polynomial* of a projective variety X, and state how we can recover the degree and the dimension of X from this polynomial.

    (c) Show that the hyperplanes in $\mathbb{P}^n$ are in 1:1 correspondence with the points of another projective space.

    (d) Show that the Veronese map $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$, $[x, y, z] \mapsto [x^2, xy, y^2, xz, yz, z^2]$ is an isomorphism onto its image.

(e) An *affine conic section* is a variety of the form

$$X = \mathbb{V}(ax^2 + bxy + cy^2 + dx + ey + f) \subset \mathbb{A}^2_k$$

for scalars $a, b, c, d, e, f \in k$, with a, b, c not all zero. State the equation defining the projective closure $\overline{X} \subset \mathbb{P}^2$. Show that the conic sections $X \subset \mathbb{P}^2$ are in 1:1 correspondence with the points in an open subset of $\mathbb{P}^5$. Using ν_2, briefly outline an argument to show that there is a unique conic section through five points in general position.

(f) Consider the Segre embedding $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $([x_0, x_1], [y_0, y_1]) \mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. Show that $\sigma_{1,1}$ is a bijection onto $X = \mathbb{V}(z_0z_3 - z_1z_2) \subset \mathbb{P}^3$. Define the Zariski topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (using $\sigma_{1,1}$). Briefly explain why this is not the product Zariski topology on $\mathbb{P}^1 \times \mathbb{P}^1$.

(g) Compute the Hilbert polynomial of $X = \sigma_{1,1}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$, and deduce that a general line in $\mathbb{P}^3$ will intersect X twice.

Solution: **(a)** Projective variety:

$$\mathbb{V}(I) = \{[a] \in \mathbb{P}^n : f(a) = 0 \text{ for all } f \in I\}$$

where $I \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal (i.e. an ideal generated by homogeneous polynomials). $\dim X$ is the maximal length m of a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_m$ of irreducible closed subvarieties. $\deg X$ is the number of intersections of X with a general linear subspace of $\mathbb{P}^n$ of dimension $n - \dim X$.

(b) Let $S(X) = \bigoplus_{d \geq 0} S(X)_d$ be the homogeneous coordinate ring of X. The Hilbert polynomial p_X is determined by the fact that $p_X(d)$ equals the function $d \mapsto \dim_k S(X)_d$ for large d. The leading term in $p_X(d) = \frac{\deg X}{(\dim X)!} d^{\dim X} + \cdots$ determines the degree and dimension.

(c) A hyperplane is defined by a linear homogeneous polynomial $a_0x_0 + \cdots + a_nx_n = 0$, so it defines a point $[a_0, \ldots, a_n] \in \mathbb{P}^n$. Rescaling the polynomial by a scalar $\lambda \in k^*$ does not affect the hyperplane, analogously to the rescaling action on homogeneous coordinates $[a] = [\lambda a] \in \mathbb{P}^n$.

(d) It is a polynomial map defined by homogeneous polynomials of equal degree, so it is a morphism. If $x \neq 0$, we may rescale so that $x = 1$, then the image becomes $[1, y, y^2, z, yz, z^2]$ so one can recover y, z: the inverse map is $[a, b, c, d, e, f] \longmapsto [a, b, d]$.

Similarly for $y = 1$, we get $[x^2, x, 1, xz, z, z^2]$ and inverse map $[a, b, c, d, e, f] \longmapsto [b, c, e]$.

Similarly for $z = 1$, we get $[x^2, xy, y^2, x, y, 1]$ and inverse map $[a, b, c, d, e, f] \longmapsto [d, e, f]$. These three inverse maps agree on overlaps, for example if $x \neq 0$, $y \neq 0$, then $[a, b, d] =$

$[x^2, xy, xz] = [x, y, z]$ and $[b, c, e] = [xy, y^2, yz] = [x, y, z]$. So we get a well-defined inverse, which is a morphism since it is locally defined by homogeneous polynomials of equal degree.

(e) $X = \mathbb{V}(ax^2 + bxy + cy^2 + dxz + eyz + fz^2)$.

Conics are in 1:1 correspondence with points $[a : b : c : d : e : f] \in \mathbb{P}^5$ in the open $\mathbb{P}^5 \setminus \mathbb{V}(a, b, c)$.

A conic defined by $[a, b, \ldots, f]$ passes through $p = [x_0, y_0, z_0] \in \mathbb{P}^2$ iff $\nu_2(p) = [z_0, z_1, \ldots, z_5]$ satisfies $az_0 + bz_1 + \cdots + fz_5 = 0$, that is iff $[a, b, \ldots, f]$ lies on the hyperplane defined by $[z_0, z_1, \ldots, z_5]$. So the conic sections passing through five general points $p_1, \ldots, p_5 \in \mathbb{P}^2$ is the intersection of five hyperplanes in $\mathbb{P}^5$ corresponding to $\nu_2(p_1), \ldots, \nu_2(p_5)$ (which one checks are in general position). These intersect in one point, corresponding to the required conic.

(f) On the open $x_0 \neq 0$, $y_0 \neq 0$, we may assume $x_0 = 1$, $y_0 = 1$, so $\sigma_{1,1} = [1, y_1, x_1, x_1y_1]$ so we recover x_1, y_1. A similar argument holds in all four cases $x_i \neq 0$, $y_j \neq 0$. So $\sigma_{1,1}$ is injective.

The image of $\sigma_{1,1}$ satisfies

$$z_0z_3 - z_1z_2 = x_0y_0x_1y_1 - x_0y_1x_1y_0 = 0.$$

To show surjectivity, we assume $z_0 \neq 0$ (a similar argument holds in the four cases $z_j \neq 0$). We solve for x_1, y_1 as we can assume $x_0 = 1$, $y_0 = 1$. Thus we need to solve $[1, y_1, x_1, x_1y_1] = [1, z_1, z_2, z_3]$, with $z_3 - z_1z_2 = 0$, so $x_1 = z_2$ and $y_1 = z_1$ works. So $\sigma_{1,1}$ is surjective.

The Zariski topology on $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by the open sets $\sigma_{1,1}^{-1}(U)$, for all open $U \subset \mathbb{P}^3$.

This is not the product topology: the proper closed sets in $\mathbb{P}^1$ are finite collections of points, so the proper closed sets of the product topology are either finite collections of points or a product $\mathbb{P}^1 \times S$ or $S \times \mathbb{P}^1$ where S is finite. But in the Zariski topology the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ is closed: $\mathbb{V}(x_0 - y_0, x_1 - y_1)$.

(g) $S(X) = k[z_0, z_1, z_2, z_3]/(z_0z_3 - z_1z_2)$. The degree d part of the ideal is $f \cdot (z_0z_3 - z_1z_2)$ for any degree $d - 2$ homogeneous polynomial f. The dimension of the vector space of homogeneous polynomials in $n + 1$ variables of degree d is

$$\binom{n+d}{d}.$$

So

$$\begin{aligned} p_X(d) &= \binom{3+d}{d} - \binom{3+d-2}{d-2} \\ &= \binom{3+d}{d} - \binom{1+d}{d-2} \\ &= \frac{(3+d)(2+d)(1+d) - (1+d)d(d-1)}{6} \\ &= (d+1)\frac{d^2+5d+6-(d^2-d)}{6} \\ &= (d+1)^2 \\ &= \frac{2}{2!}d^2 + 2d + 1. \end{aligned}$$

So the degree is 2, which proves the last claim.