Professor Joyce **C3.4 Algebraic Geometry** **MT 2026**

Problem Sheet 4

Section A: introductory questions, not for handing in

1. (a) Show that if $\mathrm{char}(k)$ does not divide d, then the hypersurface

$$\mathbb{V}(x_0^d + \ldots + x_n^d) \subset \mathbb{P}^n$$

is nonsingular.

(b) Under the same assumption on k, find the singular locus of the hypersurface

$$\mathbb{V}(x_0^d + \ldots + x_{n-1}^d) \subset \mathbb{P}^n.$$

Solution: (a) On open affine pieces, the equation of the hypersurface is $1+\sum_{i=1}^n y_i^d = 0$. The Jacobian matrix (vector) has rank one, except at the origin of $\mathbb{A}_k^n$; but the origin is not on the hypersurface.

(b) Similarly, looking on affine pieces, the only singular point is $[0:0:\ldots:0:1] \in \mathbb{P}^n$.

2. Let X be an irreducible affine variety. Show that for any open set $U \subset X$ and point $p \in X$, the rings $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ are subrings of the function field $k(X)$.

Hint. You need to explain first how to include $\mathcal{O}_X(U) \subset k(X)$ and $\mathcal{O}_{X,p} \subset k(X)$.

Solution: By results proved in lectures, given $f \in \mathcal{O}_X(U)$, one can pass to a basic open set $D_h \subset U$ where $f = \frac{g}{h}$ holds for some $h, g \in k[X]$. Thus $\frac{g}{h} \in k(X) \equiv \mathrm{Frac}\, k[X]$. That this gives a well-defined inclusion $\mathcal{O}_X(U) \subset k(X)$ is because if two rational functions $\frac{g}{h}, \frac{g'}{h'} : X \dashrightarrow k$ both equal the rational function $f : X \dashrightarrow k$, then $gh' - g'h$ vanishes on a non-empty open (hence dense, by irreducibility) subset of X and hence everywhere (by continuity). Thus $gh' - g'h = 0 \in k[X]$, thus $\frac{g}{h} = \frac{g'}{h'} \in k(X)$ by definition of $\mathrm{Frac}\, k[X]$. Thus $\mathcal{O}_X(U) \subset k(X)$, and that it is a subring follows immediately: e.g. for the product, given $f = \frac{g}{h}, f' = \frac{g'}{h'}$, then restriction to $D_{hh'}$ gives $f = \frac{gh'}{hh'}, f' = \frac{g'h}{hh'}$ and thus $f \cdot f' = \frac{gh'g'h}{hh'hh'} = \frac{gg'}{hh'}$.

For the second half, recall an element of $\mathcal{O}_{X,p}$ is some open set $p \in U \subset X$ together with a function $f : U \to k$ regular at p. The latter means $f = \frac{g}{h} : W \to k$ on some (possibly smaller) open $p \in W \subset U \subset X$, with $g, h \in k[X]$, with g never vanishing on W (one can in fact shrink W further to ensure that $W = D_h$). Thus $f \in \mathcal{O}_X(W)$ and by above we include this into $k(X)$ by taking $\frac{g}{h}$. This association is well-defined: if we get $f = \frac{g}{h} : W \to k$ and $f' = \frac{g'}{h'} : W' \to k$, then they agree on the non-empty (hence dense) open $p \in W \cap W' \subset X$, and thus as before we deduce $gh' - g'h$ vanishes everywhere so $\frac{g}{h} = \frac{g'}{h'} \in k(X)$. The subring property is analogous to before.

3. Suppose in this exercise that $\mathrm{char}(k) = 0$. Find the singularities of the following curves C in $\mathbb{A}^2_k$. For each singular point $p \in C$ compute the dimension of $\mathfrak{m}_p/\mathfrak{m}_p^2$ as a k-vector space. Here $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{C,p}$.

   (a) $\mathbb{V}(x^6 + y^6 - xy)$.

   (b) $\mathbb{V}(y^2 + x^4 + y^4 - x^3)$.

   You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

   Solution: (a) Note that $\dim C = 1$ by Krull's theorem and by Lecture 9, Theorem 9.13. Thus we need to find the points of C where the gradient of $x^6 + y^6 - xy$ vanishes. The gradient of $x^6 + y^6 - xy$ is $\langle 6x^5 - y, 6y^5 - x\rangle$. Hence we need to solve the equations $x^6 + y^6 - xy = 6x^5 - y = 6y^5 - x = 0$. We have

   $$(x/6)(6x^5 - y) - (y/6)(6y^5 - x) + 2y^6 - xy = x^6 + y^6 - xy$$

   and thus these equations are equivalent to

   $$y(2y^5 - x) = 6x^5 - y = 6y^5 - x = 0.$$

   Now if $y = 0$ then $x = 0$. If $y \neq 0$ then $y^5 = x/2 = x/6$ so $y = 0$, which is a contradiction. So we must have $x = y = 0$. So $\langle 0, 0\rangle$ is the only singular point of C.

   For $p = \langle 0, 0\rangle$ the dimension of $\mathfrak{m}_p/\mathfrak{m}_p^2$ as a k-vector space cannot be 1, as p is singular. Since $\mathfrak{m}_p$ is generated as a $k[x, y]$-module by the elements x and y, we see that $\mathfrak{m}_p/\mathfrak{m}_p^2$ has dimension at most 2. Hence $\mathfrak{m}_p/\mathfrak{m}_p^2$ has dimension 2.

   (b) The reasoning is similar. Solve $y^2 + x^4 + y^4 - x^3 = 4x^3 - 3x^2 = 2y + 4y^3 = 0$. Combining, we obtain

   $$4(y^2 + x^4 + y^4 - x^3) + (1/4 - x)(4x^3 - 3x^2) - y(2y + 4y^3) = (-3/4)x^2 + 2y^2 = 0.$$

   Now if $x \neq 0$ then $x = 3/4$ since $4x^3 - 3x^2 = 0$ and so $y^2 = 27/128$. But then $y(2y + 4y^3) = 6503409/67108864$ which is a contradiction. So we have $x = 0$ and also $y = 0$. We conclude again that the origin is the only singular point of C. By the same reasoning as above, we see that $\mathfrak{m}_p/\mathfrak{m}_p^2$ has dimension 2.

4. Show that, if $F : X \to Y$ is a surjective morphism of affine algebraic varieties, then the dimension of X is at least as large as the dimension of Y.

Solution: Let $X = X_1 \cup \cdots \cup X_M$ and $Y = Y_1 \cup \cdots \cup Y_N$ be the decompositions of X, Y into irreducible components. Then $\dim X = \max_{i=1}^{M} \dim X_i$ and $\dim Y = \max_{j=1}^{N} \dim Y_j$. Let Y_j have maximal dimension $\dim Y_j = \dim Y$. As F is surjective, there exists $i = 1, \ldots, M$ such that F maps $X_i \to Y_j$ and $F : X_i \to Y_j$ is surjective.

As X_i, Y_j are irreducible their coordinate rings $k[X_i], k[Y_j]$ are integral domains, and have function fields $k(X_i) = \mathrm{Frac}(k[X_i])$, $k(Y_j) = \mathrm{Frac}(k[Y_j])$. By lecture 9, Theorem 9.12, these are finitely generated field extensions of k with

$$\dim X_i = \dim k[X_i] = \mathrm{trdeg}_k\, k(X_i),$$

and similarly for Y_j. As $F : X_i \to Y_j$ is surjective, it is dominant, so $F^* : k[Y_j] \to k[X_i]$ induces $F^* : k(Y_j) \to k(X_i)$ which is a field extension. Therefore $\mathrm{trdeg}_k\, k(X_i) \geq \mathrm{trdeg}_k\, k(Y_j)$, as we may obtain a set of generators for $k(X_i)$ over k by starting with a set of generators for $k(Y_i)$ and adding more. Hence

$$\dim Y = \dim Y_j = \mathrm{trdeg}_k\, k(Y_j) \leq \mathrm{trdeg}_k\, k(X_i) = \dim X_i \leq \dim X.$$

Section B: core questions, to go over in class (do not hand in)

5. Let $X_1 = \mathbb{V}(xy(x-y)) \subset \mathbb{A}^2_k$ and $X_2 = \mathbb{V}(xy, yz, zx) \subset \mathbb{A}^3_k$.
   (a) Decompose these varieties into irreducible components.
   (b) By computing the dimension of tangent spaces at various points, show that X_1 and X_2 are not isomorphic.

6. (a) Let $F : X \dashrightarrow Y$ be a rational map of quasi-projective varieties, with X irreducible. If (U, f) is a representation for F, with U affine, show that
   $$\{(u, f(u)) : u \in U\} \subset U \times Y$$
   is a closed subvariety. Define the *graph* Γ_F of F to be the (Zariski) closure of $\{(u, f(u)) : u \in U\} \subset X \times Y$. Show that the projection from the graph $\Gamma_F \to X$ is a birational equivalence.
   (b) Define $F : \mathbb{A}^2_k \to \mathbb{A}^1_k$ by $F(x, y) = \frac{y}{x}$. Find the equation defining $\Gamma_F \subset \mathbb{A}^3_k$.

7. Recall that the projective varieties $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic. Find a birational equivalence $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

8. In these examples, attempt a resolution of singularities of the given variety by successively blowing up singular points.
   (a) Find a resolution of singularities of the affine curve $C_1 = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2_k$. Deduce that C_1 is rational, in other words birationally equivalent to $\mathbb{P}^1$.
   Hint. Try blowing up the curve at the origin, and consider the map that projects to the exceptional divisor.
   (b) Desingularise the affine curve $C_2 = \mathbb{V}(y^2 - x^4 - x^5) \subset \mathbb{A}^2_k$. Draw a picture of the series of blow-ups.
   (c) Desingularise the affine surface $S_2 = \mathbb{V}(-xy + z^2) \subset \mathbb{A}^3_k$.

Section C: optional questions

9. (a) Find an inductive procedure to desingularise the affine surface

$$S_n = \mathbb{V}(-xy + z^n) \subset \mathbb{A}^3_k$$

for a positive integer $n > 2$.

(b) By successively blowing up singular points, desingularise the surface

$$T_1 = \{x^2 + y^3 + z^3 = 0\} \subset \mathbb{A}^3_k.$$

(c) Investigate whether it is possible to desingularise the affine surface

$$T_2 = \{x^2 - y^2 z = 0\} \subset \mathbb{A}^3_k$$

by a procedure which begins by blowing up the origin $(0, 0, 0) \in T_2$. How is this example different from all the earlier examples of singular affine surfaces?

Solution: (a) The blowup of $\mathbb{A}^3_k$ at $(0, 0, 0)$ (the unique singular point) can be covered by three charts $\mathbb{A}^3_k$:

(i) Over $\{[x, y, z] \in \mathbb{P}^3 : x \neq 0\}$ with coordinates x, $u = y/x$, $v = z/x$ on the blowup of $\mathbb{A}^3_k$, the blow up of S_n has equation $-u + x^{n-2}v^n = 0$, and is nonsingular.

(ii) Over $\{[x, y, z] \in \mathbb{P}^3 : y \neq 0\}$ with coordinates y, $u = x/y$, $v = z/y$, the blow up of S_n has equation $-u + x^{n-2}v^n = 0$, and is nonsingular.

(iii) Over $\{[x, y, z] \in \mathbb{P}^3 : z \neq 0\}$ with coordinates z, $u = x/z$, $v = y/z$, the blow up of S_n has equation $-uv + z^{n-2} = 0$, which is isomorphic to S_{n-2}.

So $\lfloor \frac{n}{2} \rfloor$ blowups are needed.

(b) The only singular point is $(0, 0, 0)$. Using charts (i)–(iii) as in (a), for (i) we get $1 + xu^3 + xv^3 = 0$ which is nonsingular. For (ii) we get $u^2 + y + yv^3 = 0$, which is singular where $y = u = 0$ and $1 + v^3 = 0$, giving three points $(0, 0, \sqrt[3]{-1})$. For (iii) we get $u^2 + z + zv^3 = 0$, which is singular where $z = u = 0$ and $1 + v^3 = 0$, giving the same three points as in (ii), since they lie in both charts.

To blow up $u^2 + y + yv^3 = 0$ at $(y, u, v) = (0, 0, -1)$, change coordinates to $(r, s, t) = (y, u, v + 1)$, giving the equation $s^2 + r(t^3 - 3t^2 - 3t) = 0$. If $\mathrm{char}(k) \neq 3$, the blowup of this at $(r, s, t) = (0, 0, 0)$ is nonsingular near $(r, s, t) = (0, 0, 0)$, in all three charts. (If you consider singularities not near $(r, s, t) = (0, 0, 0)$, you will find the other two

singular points of $u^2 + y + yv^3 = 0$, but these are not a problem.) The same happens for blowing up the other two singular points.

(c) The blowup at the origin returns a surface isomorphic to the original one. So this does not lead to a desingularisation process. The difference between this example and the earlier ones is that $\mathrm{Sing}(T_2)$ is the line $x = y = 0$, whereas the earlier examples only had a single point as singular locus.

10. Assume that the base field $k = \mathbb{C}$. Show that the projective surface

$$X = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\} \subset \mathbb{P}^3$$

contains at least 27 lines.

Hint. Move two of the summands to the other side, and factorise both sides of the resulting equality.

Comment. It is possible to show that there exists a union of six disjoint lines

$$E = L_1 \cup \ldots \cup L_6 \subset X$$

and a birational morphism $\pi\colon X \to \mathbb{P}^2$ such that π is the blowup of six points in $\mathbb{P}^2$ and $E \subset \mathbb{P}^2$ is the exceptional locus of π. For details, see the last chapter of Reid: Undergraduate Algebraic Geometry.

Solution: We may write

$$X = \{(x_0 + x_1)(x_0^2 - x_0x_1 + x_1^2) + (x_2 + x_3)(x_2^2 - x_2x_2 + x_3^2) = 0\}$$

Hence we get a line L in X

$$L = \{x_0 + x_1 = 0, \quad x_2 + x_3 = 0\} \subseteq X.$$

More generally, if α, β are third roots of unity, we may write

$$X = \{(x_0 + \alpha x_1)(x_0^2 - \alpha x_0x_1 + \alpha^2 x_1^2) + (x_2 + \beta x_3)(x_2^2 - \beta x_2x_2 + \beta^2 x_3^2) = 0\},$$

so we get the following 9 lines in X

$$L = \{x_0 + \alpha x_1 = 0, \quad x_2 + \beta x_3 = 0\} \subseteq X.$$

Repeating this for the groupings $(x_0, x_2), (x_1, x_3)$ and $(x_0, x_3), (x_1, x_2)$ gives $3 \times 9 = 27$ distinct lines in X.