B3.6 Topics in Number Theory

Sheet X — MT 26

The purpose of this sheet is to go over some things we will use in the course, but which will not be covered in detail in lectures. You can assume these without proof in the course if you want to, but these exercises take a closer look at them. I may add to this sheet somewhat during the term.

Sums and integrals

We start with various asymptotic questions, many of which can be addressed by comparing sums to integrals or related methods.

1. You should review the meaning of $\ll$ (Section 1.2 of the notes). By comparing sums to integrals, prove the following facts.

   (i) $\sum_{n=1}^{\infty} \frac{\log n}{n^2} < \infty$;

   (ii) $\sum_{n>X} n^{-2} \ll X^{-1}$, uniformly for $X \geqslant 1$;

   (iii) More generally, $\sum_{n>X} n^{-1-\sigma} \ll_\sigma X^{-\sigma}$ for any $\sigma > 0$ and for $X \geqslant 1$. The notation here means that the implied constant is allowed to depend on σ.

2. You should review the meaning of $O(\cdot)$ (see Section 1.2 of the notes). Prove the following rather weak version of Stirling's formula: $\sum_{n\leqslant X} \log n = X \log X + O(X)$ uniformly for $X \geqslant 2$.

3. Show that if X is a positive integer then

$$\sum_{n\leqslant X} n^{-1/2} = 2X^{1/2} + c + O(X^{-1/2}),$$

   for some constant c. *Hint: you may want to use the Taylor expansion* $(1+x)^{1/2} = 1 + \frac{1}{2}x + O(x^2)$*, valid uniformly for* $0 \leqslant x \leqslant 1$*, as well as the similar Taylor expansion for* $(1+x)^{-1/2}$.

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4. Show that
$$\gamma := \lim_{X\to\infty} \sum_{n\leqslant X} \frac{1}{n} - \log X$$
exists. Show that, furthermore, we have
$$\sum_{n\leqslant X} \frac{1}{n} = \log X + \gamma + O(\frac{1}{X}). \tag{3}$$
(The constant γ is called Euler's constant, and it has the value $0.577\ldots$.)

5. The purpose of this question is to show that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. You may assume that if $f : \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ is an integrable function with $\int_0^1 |f(x)|^2 \mathrm{d}x < \infty$, then we have Parseval's identity
$$\int_0^1 |f(x)|^2 \mathrm{d}x = \sum_{n\in\mathbf{Z}} |\hat{f}(n)|^2,$$
where the Fourier coefficient $\hat{f}(n)$ is given by $\int_0^1 f(\theta)e^{-2\pi i\theta n}\mathrm{d}\theta$. Apply this to the function f which is 1-periodic and with $f(x) = x$ for $x \in (-1/2, 1/2]$.

Other inequalities

Miscellaneous useful inequalities.

6. Prove that
$$\log(1+t) \leqslant t$$
for $t \geqslant 0$ and that if $\delta \in [0, 1)$ then
$$\log(1-t) \geqslant -\frac{t}{1-\delta}$$
for $0 \leqslant t \leqslant \delta$.

7. Let C be an arbitrary positive constant. Show $n! > C^n$ for n sufficiently large in terms of C.

Miscellaneous

8. Prove *Legendre's formula*: for p prime and positive integer m, the exponent of the largest power of p dividing $m!$ is $\sum_{i=1}^{\infty} \lfloor m/p^i \rfloor$.

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