B3.6 Topics in Number Theory

Sheet X — MT 26

The purpose of this sheet is to go over some things we will use in the course, but which will not be covered in detail in lectures. You can assume these without proof in the course if you want to, but these exercises take a closer look at them. I may add to this sheet somewhat during the term.

Sums and integrals

We start with various asymptotic questions, many of which can be addressed by comparing sums to integrals or related methods.

1. You should review the meaning of $\ll$ (Section 1.2 of the notes). By comparing sums to integrals, prove the following facts.

   (i) $\sum_{n=1}^{\infty} \frac{\log n}{n^2} < \infty$;

   (ii) $\sum_{n>X} n^{-2} \ll X^{-1}$, uniformly for $X \geqslant 1$;

   (iii) More generally, $\sum_{n>X} n^{-1-\sigma} \ll_\sigma X^{-\sigma}$ for any $\sigma > 0$ and for $X \geqslant 1$. The notation here means that the implied constant is allowed to depend on σ.

   Solution: (i) One relatively painless method is to observe that $\log n \ll \sqrt{n}$ and so it is enough to prove that $\sum_{n=1}^{\infty} n^{-3/2} < \infty$. This is a standard fact and may be proven by observing that

   $$\sum_{n=2}^{\infty} n^{-3/2} \leqslant \int_1^{\infty} \frac{\mathrm{d}t}{t^{3/2}} = [-2t^{-1/2}]_1^{\infty} = 2.$$

   (ii) We only need prove this for integer X, since $\lfloor X \rfloor^{-1} \ll X^{-1}$ because $X \leqslant 2\lfloor X \rfloor$ for $X \geqslant 1$. Replacing X by $\lfloor X \rfloor$ does not change the LHS.

   Now for integer X we have the inequality $\sum_{n>X} \frac{1}{n^2} \leqslant \int_X^{\infty} \frac{\mathrm{d}t}{t^2}$ (draw a picture). The integral evaluates to exactly $1/X$.

   (iii) Exactly the same as (ii), except that now we compare with the integral $\int_X^{\infty} t^{-1-\sigma}\mathrm{d}t = \frac{1}{\sigma}X^{-\sigma}$.

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2. You should review the meaning of $O(\cdot)$ (see Section 1.2 of the notes). Prove the following rather weak version of Stirling's formula: $\sum_{n\leqslant X}\log n = X\log X + O(X)$ uniformly for $X \geqslant 1$.

   Solution: We can assume that X is an integer, since changing X to $X+t$ for some $t\in[0,1]$ does not change either side by more than $O(\log X)$. (To see this for the RHS we use $\log(X+t) = \log X + \log(1+\frac{t}{X}) = \log X + O(1/X)$.)

   Suppose, then, that X is an integer. By drawing the graph of $f(t)=\log t$ and appropriate bars of width 1 and height $\log n$ above it, one may see that

$$\sum_{2\leqslant n\leqslant X}\log n \geqslant \int_1^X \log t\,\mathrm{d}t = [t\log t - t]_1^X = X\log X + O(X).$$

   On the other hand, by drawing bars below the graph we have

$$\sum_{2\leqslant n\leqslant X}\log n \leqslant \int_2^{X+1} \log t\,\mathrm{d}t = [t\log t - t]_2^{X+1} = (X+1)\log(X+1) + O(X).$$

   By the same argument we used to justify replacing X by $X+t$ for $t\in[0,1]$, we have $(X+1)\log(X+1) = X\log X + O(X)$.

   Combining these upper and lower bounds gives the result.

3. Show that if X is a positive integer then

$$\sum_{n\leqslant X} n^{-1/2} = 2X^{1/2} + c + O(X^{-1/2}),$$

   for some constant c. *Hint: you may want to use the Taylor expansion* $(1+x)^{1/2} = 1+\frac{1}{2}x+O(x^2)$*, valid uniformly for* $0\leqslant x\leqslant 1$*, as well as the similar Taylor expansion for* $(1+x)^{-1/2}$.

   Solution: Comparing the sum to the integral of $f(t)=t^{-1/2}$ (by drawing bars of height $f(n)$ on $[n-1,n]$, which sit underneath the graph) we see that

$$\int_1^X t^{-1/2}\mathrm{d}t - \sum_{n=2}^X n^{-1/2} = \sum_{n=1}^{X-1} s_n,$$

   where

$$s_n := \int_n^{n+1} t^{-1/2}\mathrm{d}t - (n+1)^{-1/2}.$$

   Rearranging and evaluating the integral gives

$$\sum_{n\leqslant X} n^{-1/2} = 2(X^{1/2}-1) + 1 - S + \sum_{n\geqslant X} s_n,$$

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where $S = \sum_{n=1}^{\infty} s_n$. To prove the result, it therefore suffices to prove the claim that $s_n \ll n^{-3/2}$, since this implies both that S is finite and that $\sum_{n \geqslant X} s_n \ll X^{-1/2}$.

We may evaluate

$$\int_n^{n+1} t^{-1/2} \mathrm{d}t = 2\big((n+1)^{1/2} - n^{1/2}\big) = n^{-1/2} + O(n^{-3/2}), \tag{1}$$

where in the last step we used the Taylor expansion $(1+\frac{1}{n})^{1/2} = 1+\frac{1}{2n}+O(\frac{1}{n^2})$. Similarly,

$$(n+1)^{-1/2} = n^{-1/2}(1+\frac{1}{n})^{-1/2} = n^{-1/2} + O(n^{-3/2}), \tag{2}$$

where we used $(1+\frac{1}{n})^{-1/2} = 1 - \frac{1}{2n} + O(\frac{1}{n^2})$. Combining (1) and (2) gives the claim.

4. Show that

$$\gamma := \lim_{X \to \infty} \sum_{n \leqslant X} \frac{1}{n} - \log X$$

exists. Show that, furthermore, we have

$$\sum_{n \leqslant X} \frac{1}{n} = \log X + \gamma + O(\frac{1}{X}). \tag{3}$$

(The constant γ is called Euler's constant, and it has the value $0.577\ldots$.)

Solution: We proceed analogously to the previous question, but now with $f(t) = t^{-1}$. Now we have

$$s_n = \int_n^{n+1} t^{-1} \mathrm{d}t - \frac{1}{n+1}$$

and we compute

$$s_n = \log(n+1) - \log n - \frac{1}{n+1} = \log(1+\frac{1}{n}) - \frac{1}{n+1} = \frac{1}{n} + O(\frac{1}{n^2}) - \frac{1}{n+1} \ll \frac{1}{n^2}.$$

The rest of the argument is the same as before.

5. The purpose of this question is to show that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. You may assume that if $f : \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ is an integrable function with $\int_0^1 |f(x)|^2 \mathrm{d}x < \infty$, then we have Parseval's identity

$$\int_0^1 |f(x)|^2 \mathrm{d}x = \sum_{n \in \mathbf{Z}} |\hat{f}(n)|^2,$$

where the Fourier coefficient $\hat{f}(n)$ is given by $\int_0^1 f(\theta) e^{-2\pi i \theta n} \mathrm{d}\theta$. Apply this to the function f which is 1-periodic and with $f(x) = x$ for $x \in (-1/2, 1/2]$.

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Solution: We compute the Fourier coefficients of the stated function. We have $\widehat{f}(0) = 0$ since f is odd. For $n \neq 0$, via integration by parts you may confirm that

$$\widehat{f}(n) = \frac{(-1)^{n+1}}{2\pi i n}.$$

In particular,

$$|\widehat{f}(n)|^2 = \frac{1}{4\pi^2 n^2}.$$

On the other hand we have

$$\int_{-1/2}^{1/2} |f(x)|^2 \mathrm{d}x = \int_{-1/2}^{1/2} x^2 \mathrm{d}x = \frac{1}{12}.$$

The result follows using Parseval's identity.

Other inequalities

Miscellaneous useful inequalities.

6. Prove that

$$\log(1+t) \leqslant t$$

for $t \geqslant 0$ and that if $\delta \in [0,1)$ then

$$\log(1-t) \geqslant -\frac{t}{1-\delta}$$

for $0 \leqslant t \leqslant \delta$.

Solution: The first inequality follows from $1 + t \leqslant \sum_{i=0}^{\infty} \frac{t^i}{i!} = e^t$, then taking logs. For the second, one can consider

$$f(t) := \log(1-t) + \frac{t}{1-\delta}.$$

We have

$$f'(t) = \frac{1}{1-\delta} - \frac{1}{1-t},$$

and so $f'(t) \geqslant 0$ for $0 \leqslant t \leqslant \delta$. Since $f(0) = 0$, it follows that $f(t) \geqslant 0$ for $0 \leqslant t \leqslant \delta$.

7. Let C be an arbitrary positive constant. Show $n! > C^n$ for n sufficiently large in terms of C.

Solution: This follows from Stirling's formula, but we also have

$$n! \geqslant n(n-1)\cdots\lfloor\frac{n}{2}\rfloor \geqslant \lfloor\frac{n}{2}\rfloor^{n/2}.$$

This immediately implies the statement if n is large enough that $\lfloor n/2\rfloor > C^2$.

Miscellaneous

8. Prove *Legendre's formula*: for p prime and positive integer m, the exponent of the largest power of p dividing $m!$ is $\sum_{i=1}^{\infty}\lfloor m/p^i\rfloor$.

   Solution: Write the numbers $1, 2, \ldots, m$ in a line. To get the desired exponent, take one for each multiple of p, another one for each multiple of p^2, another for each multiple of p^3, and so on. The number of multiples of r in $\{1, \ldots, m\}$ is $\lfloor m/r\rfloor$.

   Remark. The sum is actually finite since $\lfloor m/p^i\rfloor = 0$ when $p^i > m$, that is to say when $i > \frac{\log m}{\log p}$.

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