Homological Algebra C2.2 Lecture 1 15/10/2026

(L5)

Overview

Homological Algebra

Algebra: Linear algebra

- Vector spaces / field
- [• Modules / rings.
  - ↳ More complicated e.g. not every module is free / has a basis.

Ex: $R = \mathbb{Z}$ $M = \mathbb{Z}/n\mathbb{Z}$ $\quad$ $R = \mathbb{C}[x]$ $M = \mathbb{C}[x]/(x-\alpha)$ $\alpha \in \mathbb{C}$

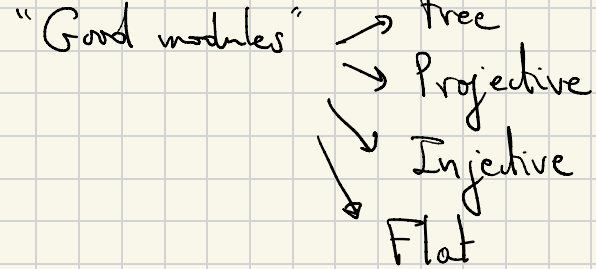
↳ Algebraic operations behave in more complicated way.

Homological algebra: toolbox to describe what happens.

Basic idea: "replace bad modules by good ones"

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

"resolution" Work with complexes of modules.



"Measure of bad": ---- Ext & Tor groups.

Toolbox actually more general than for modules/rings.

General context: "abelian categories"

↳ Can be applied in many other context.

Plan → Categories [see C2.7. Category theory]
Abelian categories
→ Resolutions, chain complexes
→ Derived functors
→ Ext, Tor, Koszul complexes
Group homology

Historical remark. Homological algebra

Homology first introduced in the context of algebraic topology

General ideal: attach algebraic data to a space "Homology of a space"

Poincaré: 1895

~ 1940's: Purely algebraic applications "group homology"

"Lie algebra homology"

....

Unified into homological algebra

"Homological algebra": book Cartan-Eilenberg (1956)

"Sur quelques points d'algèbre homologique" Grothendieck (1957)

[Motivation: sheaves of modules]

Define "abelian categories"

Category theory: Eilenberg-Maclane (1945)

After that, homological algebra invaded many parts of mathematics:

- Algebraic number theory (eg group coh. for Galois groups ...)
- Complex geometry / Algebraic geometry ["Sheaves of modules"]

.....

- Further developments in hom. algebra: derived categories ...

→ Toolbox useful in many context.

Remaining of today: Towards category theory.

Tensor products.

k: field Vec_k: category of k-vector spaces → Vector spaces; → linear maps between them

$V, W, L \in \mathrm{Vec}_k$ $\mathrm{Bilin}(V, W; L) := \{$bilinear maps $V \times W \to L\}$

Can we understand bilinear maps using only the notion of linear map?

$\mathrm{Hom}(V, W)$: set of linear maps from V to W.

↑ k-vector space $\underline{\mathrm{Hom}}(V, W)$ (new notation)

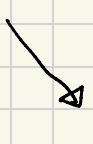
$\mathrm{Bilin}(V, W; L)$ $T(v, w) = T(v)(w) = T(w)(v)$

$= \mathrm{Hom}(V, \underline{\mathrm{Hom}}(W, L))$

$= \mathrm{Hom}(W, \underline{\mathrm{Hom}}(V, L))$

Q° Is $\mathrm{Bilin}(V, W; -) : \mathrm{Vec}_k \to \mathrm{Set}$ functor representable? In other words, is there a k-vector space $V \otimes W$ s.t. $\mathrm{Bilin}(V, W; -) = \mathrm{Hom}(V \otimes W, -)$?

This is equivalent to asking for the universal bilinear map: vector space $V \otimes W$ with a bilinear map $V \times W \to V \otimes W$ s.t. all others factorize in a unique way:



This is also equivalent to asking that the functor

$\underline{\mathrm{Hom}}(W, -) : \mathrm{Vec}_k \to \mathrm{Vec}_k$ has a left adjoint $- \otimes W$:

$$\mathrm{Hom}(- \otimes W, -) = \mathrm{Hom}(-, \mathrm{Hom}(W, -))$$
$$= \mathrm{Bilin}(-, W; -)$$

$V \otimes W$ exists: look at the vector space spanned by $V \times W$:

$$k[V \times W] = \left\{ \sum_{(v,w) \in V \times W} a_{(v,w)} (v, w) \text{ with finite support} \right\}$$

Look at the subspace of $k[V \times W]$ spanned by:

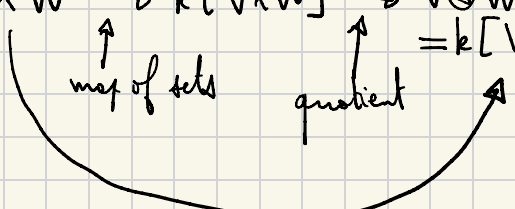
$$M = \left\{ \begin{array}{ll} (v + w', w) - (v, w) - (v', w) & (v, w + w') - (v, w) - (v, w') \\ (cv, w) - c(v, w) & (v, cw) - c(v, w) \\ \text{for all } c \in k & v, v' \in V \quad w, w' \in W \end{array} \right\}$$

By construction (first isomorphism theorem):

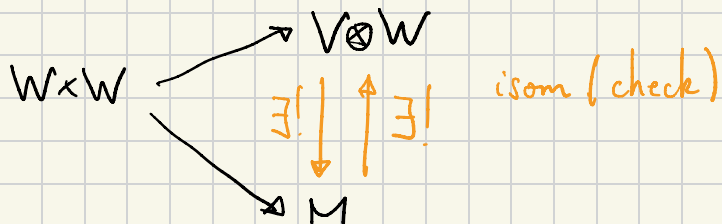
$$\mathrm{Hom}(k[V \times W]/M, L) = \mathrm{Bilin}(V, W, L)$$

so $V \otimes W := k[V \times W]/M$

Universal bilinear map: $V \times W \to k[V \times W] \to V \otimes W = k[V \times W]/M$



Note: $V \otimes W$ is given up to unique isomorphism. If M is another solution (universal bilinear map)



$\mathrm{Bilin}(V, W; L) \xrightarrow{\sim} \mathrm{Bil}(W, V; L)$ (flip order)

$\Rightarrow \quad V \otimes W \xrightarrow{\sim} W \otimes V$

Note: $k \otimes W \simeq W \otimes k \simeq W$

—

Trilinear maps: $\mathrm{Trilin}(V_1, V_2, V_3, L) \xrightarrow{\sim} \mathrm{Bilin}(V_1 \otimes V_2, V_3, L)$

$\xrightarrow{\sim} \mathrm{Bil}(V_1, V_2 \otimes V_3, L)$

$\Rightarrow (V_1 \otimes V_2) \otimes V_3 \xrightarrow{\sim} V_1 \otimes (V_2 \otimes V_3)$

Not to easy to prove using the quotient description

More generally, $\forall \sigma \in S_n$:

$$((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \cdots$$

$$\xrightarrow{\sim} (\cdots ((V_{\sigma(1)} \otimes V_{\sigma(2)}) \otimes V_{\sigma(3)}) \otimes \cdots$$

$(\mathrm{Vec}_k, \otimes)$ is a symmetric monoidal category.