Lecture 2

15/10/26

$\mathrm{Bilin}(V,W;L) = \mathrm{Hom}(V, \underline{\mathrm{Hom}}(W,L))$

$\| \quad \nearrow$

$\mathrm{Hom}(V \otimes W, L)$

$\mathrm{Hom}(- \otimes W, -) \simeq \mathrm{Hom}(-, \underline{\mathrm{Hom}}(W,-))$

"natural"

The functor $- \otimes W : \mathrm{Vect} \to \mathrm{Vect}$] left adjoint

functor $\underline{\mathrm{Hom}}(W,-) : \mathrm{Vect} \to \mathrm{Vect}$] right adjoint

Pair of adjoint functors

Non-linear version: Set

$\mathrm{Hom}(X \times Y, Z) = \mathrm{Hom}(X, \mathrm{Hom}(Y,Z))$

$\mathrm{Hom}(- \times Y, -) = \mathrm{Hom}(-, \mathrm{Hom}(Y,-))$

left-adj right-adj

$\dim(V \oplus W) = \dim V + \dim W$

$\dim(V \otimes W) = (\dim V)(\dim W)$

$\dim : \mathrm{Vect}_{f.d} \longrightarrow \mathbb{N}$ "categorification"

$\oplus \quad \otimes \qquad (\mathbb{N}, +, \times)$

R any commutative ring Mod_R $M \underset{R}{\otimes} N$ $\underline{\mathrm{Hom}}(N,L)$

$\mathrm{Hom}(M \otimes N, L) \xrightarrow{\sim} \mathrm{Hom}(M, \underline{\mathrm{Hom}}(N,L))$

Category theory

"objects"

Def: A category C consists of a set/class $\mathrm{obj}(C)$, and $\forall A, B \in \mathrm{obj}(C)$ a set $\mathrm{Hom}_C(A,B)$. $\forall A \in \mathrm{obj}(C)$, $\mathrm{id}_A \in \mathrm{Hom}_C(A,A)$,

$\forall X, Y, Z \in \mathrm{obj}(C)$ $\quad \mathrm{Hom}_C(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$

"morphisms" $\qquad\qquad\qquad\quad f \qquad\quad g \qquad\quad \mapsto g \circ f$

s.t: 1) $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f$

$$\mathrm{id}_X \circlearrowright X \xrightarrow{f} Y \circlearrowleft \mathrm{id}_Y$$

2) $(f \circ g) \circ h = f \circ (g \circ h)$ (associativity)

$$\forall \quad X \xrightarrow{h} Y \xrightarrow{g} Z \xrightarrow{f} W$$

Ex: a category with one object X gives a monoid

$$\mathrm{End}_C(X) := \mathrm{Hom}_C(X,X)$$

is a monoid

[composition, associative, unital]

Slogan: "a category is a monoid with many objects."

Examples: Sets, Vect_k, Mod_R, Grp, Rings, Manifolds, Top spaces, ...

Def: X, Y isom ...

Def: $f: B \to C$ a morphism is monic

if $\forall e_1, e_2 : A \to B$, $f \circ e_1 = f \circ e_2 \Rightarrow e_1 = e_2$

Ex: In Sets

$A = \{*\} \longrightarrow B$ (two parallel maps e_1, e_2)

$\Leftrightarrow$ elements in B (x_1, x_2)

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

f monic $\Leftrightarrow$ f one-to-one

Exercise.

$$A \underset{e_2}{\overset{e_1}{\rightrightarrows}} B \xrightarrow{f} C$$

Def: $f: B \to C$ epi / epimorphism if $\forall g_1, g_2 : C \to D$:

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$$

$$B \xrightarrow{f} C \underset{g_2}{\overset{g_1}{\rightrightarrows}} D$$

Rem: "Reverse arrows" **Ex:** In Sets, $\Leftrightarrow$ f onto

Note: In Ring: $\mathbb{Z} \hookrightarrow \mathbb{Q}$ / monic

epi in Ring!

$\mathbb{Z} \to \mathbb{Q} \rightrightarrows R$ 2 maps of rings $= \text{on } \mathbb{Z}$

not in set, not onto!

determined $\rightsquigarrow \mathbb{Q} = \text{Frac}(\mathbb{Z})$

but not isom

(f isom if $\exists g$ s.t. $f \circ g = id$, $g \circ f = id$)

Initial object I s.t. $\forall A \in \text{obj}(C)$

$\exists!\ I \to A \quad |\text{Hom}(I, A)| = 1$

Final object: T s.t. $\forall A$

$\exists!\ A \to T \quad |\text{Hom}(A, T)| = 1$

↖ Unique up to unique isom if they exist.

Ex: In Sets, $\emptyset$ initial

$\{*\}$ final

Zero object: both initial & final.

If 0 is a zero object, then $\forall A, B$