Week 1: First Order IVPs and Maximal Intervals of Existence

When solving initial value problems (IVPs), in addition to finding a formula for the solution, it is important to determine the interval on which the solution is defined. The following examples illustrate why this final step is important.

Example 1: $y' = 1 + y^2$, with $y(0) = 1$.

Step 1: Separate variables and integrate:

$$\int \frac{dy}{1+y^2} = \int dx,$$

$$\Rightarrow \arctan(y) = x + C.$$

$$\Rightarrow y = \tan(x + C).$$

Step 2: Apply the initial condition

$$y(0) = 1 \quad \Rightarrow \quad 1 = \tan(C).$$

$$\Rightarrow C = \frac{\pi}{4} + n\pi, \qquad n \in \mathbb{Z}.$$

Since tangent has period π, these values generate the same solution:

$$y(x) = \tan\left(x + \frac{\pi}{4}\right).$$

Step 3: Determine interval of existence The tangent function is undefined whenever

$$x + \frac{\pi}{4} = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}.$$

Hence singularities occur at

$$x = \frac{\pi}{4} + k\pi.$$

The nearest singularities to $x = 0$ are

$$x = -\frac{3\pi}{4}, \qquad x = \frac{\pi}{4}.$$

Therefore the maximal interval containing the initial point $x = 0$ is $-\frac{3\pi}{4} < x < \frac{\pi}{4}$, and the solution of the IVP is

$$\boxed{y(x) = \tan\left(x + \frac{\pi}{4}\right), \qquad -\frac{3\pi}{4} < x < \frac{\pi}{4}.}$$

Remark The solution becomes unbounded as $x \to -3\pi/4^+$ and $x \to \pi/4^-$.

Example 2: $y' = y^2$, $y(0) = 1$

Step 1: Separate variables and integrate

$$\int \frac{dy}{y^2} = \int dx,$$

$$\Rightarrow -\frac{1}{y} = x + C \quad \text{or} \quad y = -\frac{1}{x + C}.$$

Step 2: Apply the initial condition, $y(0) = 1$

$$1 = -\frac{1}{C}, \quad \Rightarrow \quad y(x) = \frac{1}{1 - x}.$$

Step 3: Determine interval of existence The solution is undefined when $x = 1$. Since initial condition imposed at $x = 0$, maximal interval containing 0 is

$$(-\infty, 1).$$

Therefore

$$\boxed{y(x) = \frac{1}{1 - x}, \qquad -\infty < x < 1.}$$

Observe that

$$\lim_{x \to 1^-} \frac{1}{1 - x} = +\infty.$$

i.e. solution becomes infinite at finite time $x = 1$. This phenomenon is called *finite-time blow-up.*

Take-home Message:

When solving an IVP, find the formula for the solution and determine the maximal interval on which the solution exists.