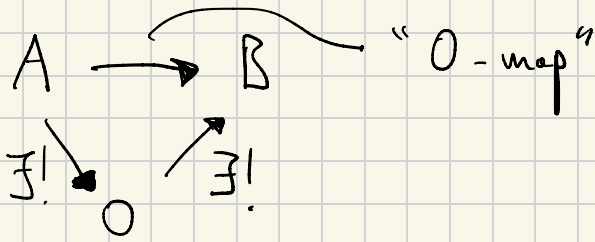
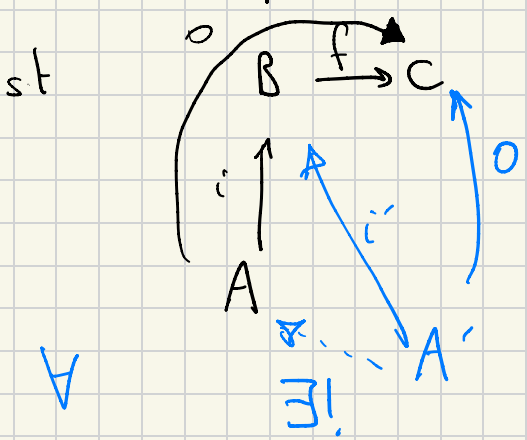
Lecture 3.

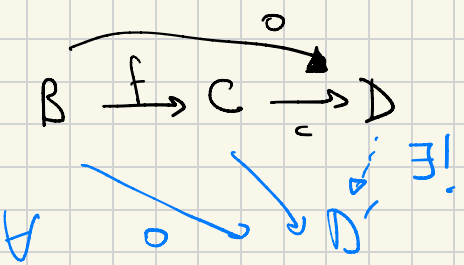


Kernel: Assume ∃ 0-object a kernel of $f: B \to C$

is a morphism $i: A \to B$ s.t. $f \circ i = 0$.



Cokernel



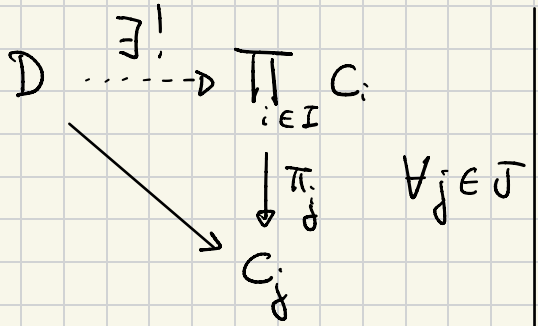
Opposite category: $\mathcal{C} \rightsquigarrow \mathcal{C}^{op}$ $ob(\mathcal{C}^{op}) = ob(\mathcal{C})$

$\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$ Ex: Opposite monoid.

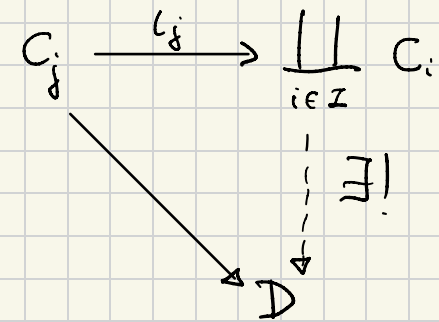
Ex: $f: B \to C$ monic in $\mathcal{C}$

$\iff$ $f: B \to C$ epi in $\mathcal{C}^{op}$

Products and coproducts: $\mathcal{C}$ category $\{C_i \mid i \in I\}$ objects



$\forall j \in J$



If they exist, unique up to unique isomorphism.

Note: $\mathrm{Hom}_{\mathcal{C}}(A, \prod_{i \in I} C_i) = \prod_{i \in I} \mathrm{Hom}(A, C_i)$

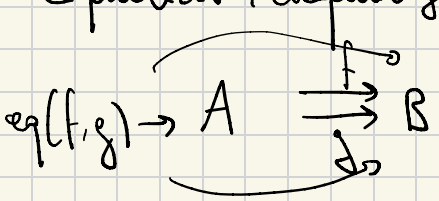
$\mathrm{Hom}_{\mathcal{C}}(\coprod_{i \in I} C_i, A) = \prod_{i \in I} \mathrm{Hom}(C_i, A)$

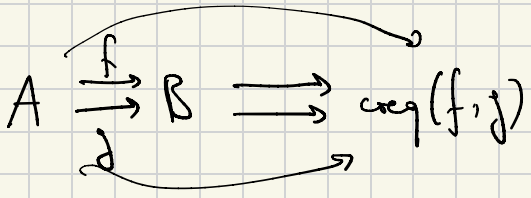
Ex: ① In Set $\prod$ = product of sets $\coprod$ = disjoint union

Top ...

② Ab abelian group $\prod = \prod$ $\coprod_{i \in I} = \bigoplus_{i \in I}$

Equalizer / coequalizer





Ex: If $\exists$ 0-object $eq(f,0) = \ker(f)$, $coeq(f,0) = \mathrm{Coker}(f)$

Functors: $F: \mathcal{E} \to \mathcal{D}$ $\quad F: ob(\mathcal{E}) \to ob(\mathcal{D})$

Def:

and $\forall f: C_1 \to C_2$, $F(f): F(C_1) \to F(C_2)$

$\forall C_1, C_2 \in ob(\mathcal{E})$ that is $Hom_{\mathcal{E}}(C_1, C_2) \to Hom_{\mathcal{D}}(F(C_1), F(C_2))$

s.t:

1) $F(gf) = F(g)F(f)$

2) $F(id_C) = id_{F(C)}$.

—

F faithful if $\forall C_1, C_2$, $Hom_{\mathcal{E}}(C_1, C_2) \to Hom_{\mathcal{D}}(F(C_1), F(C_2))$ one-to-one.

— fully faithful

bijection

Ex: Let $\mathcal{E}$ be a category and $\{C_i \mid i \in I\}$ objects in $\mathcal{E}$.

The full subcategory on $\{C_i\}_{i \in I}$ is the category $\{C_i\}$ and morphisms the same as in $\mathcal{E}$.

$\{C_i\}_i \hookrightarrow \mathcal{C}$ example of fully fait

Ex: Forget functors $Ab \longrightarrow Set$ $\quad Ring \longrightarrow Set$

Ex: Free construction $\quad Set \longrightarrow Vect_k \quad Set \longrightarrow Comm$

$X \longmapsto \underbrace{k[X]}_{\text{free vector space over } X} \quad X \longmapsto Free(X)$

$\mathbb{Z}[X]$ polynomial

—

Natural transformations:

$F, G: \mathcal{C} \to \mathcal{D}$ functors. Natural transfo $\alpha: F \Rightarrow G$:

collection of $\{\alpha_x : F(x) \to G(x)\}_{x \in obj(\mathcal{C})}$

s.t. $\forall f: X \to Y$

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \alpha_X & & \downarrow \alpha_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$ commutes

Ex: $* : Vect_k \to Vect_k^{op}$

$V \longmapsto V^*$ dual functor

[Natural isomorphism / Isomorphism of functors if α_x isomorphism $\forall x$

$** : Vect_k \to Vect_k$

$V \longmapsto V^{**}$

Tautological map $V \to V^{**}$

$$\begin{array}{ccc} V & \to & V^{**} \\ \downarrow & & \downarrow \\ W & \to & W^{**} \end{array}$$

$\exists$ natural transfo $Id \Rightarrow **$.

$Fun(\mathcal{C}, \mathcal{D})$: category, obj: functors, morphisms: natural transfo.