Adjoint functors: $L: \mathcal{A} \to \mathcal{B}$ $\quad R: \mathcal{B} \to \mathcal{A}$ Lecture 4.

if $\forall A \in \mathcal{A}$, $B \in \mathcal{B}$ $\exists$ bijection $\tau_{AB}: \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, R(B))$

which is natural:

$\forall f: A' \to A$

$g: B \to B'$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) & \xrightarrow{\text{commutes}} & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{B}}(L(A), B') \\
\downarrow & & \downarrow \qquad\qquad \text{commutes} \qquad \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) & \xrightarrow{} & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \xrightarrow[R(g)\circ -]{} \mathrm{Hom}(A, R(B'))
\end{array}$$

i.e. τ is a natural isom between

$\mathrm{Hom}_{\mathcal{B}}(L-, -)$ and $\mathrm{Hom}_{\mathcal{A}}(-, R-)$

which are functors $\mathcal{A}^{op} \times \mathcal{B} \to \mathrm{Set}$

($\mathcal{A}^{op} \times \mathcal{B}$: pairs of objects & pairs of morphisms)

$\mathcal{C}$ and $\mathcal{D}$ are <u>equivalent</u> if $\exists$ $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ functors s.t. $\exists$ natural transfo $F \circ G \xrightarrow{\sim} \mathrm{id}$

$G \circ F \xrightarrow{\sim} \mathrm{id}$

Ex: Linear algebra $V \cong k^{\dim V}$ $\quad \mathrm{Vect}_k^{f.d.} \longleftrightarrow \{k^n\}$

equivalence of category (check!)

Abelian categories

Ab-enriched category $\mathcal{C}$: $\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X,Y \in \mathrm{Obj}(\mathcal{C})$

and composition bi-additive (extra structure)

Ex: Ab, Vec_k, ...

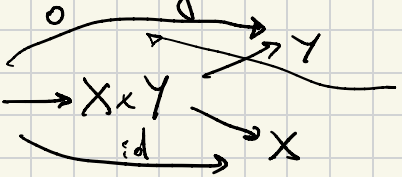
Def: A functor $F: \mathcal{C} \to \mathcal{D}$ between Ab-enriched categories is additive if $\forall X,Y$: $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$ is a group homomorphism.

Def: An Ab-enriched category is called additive if it admits finite coproducts (property).

Prop: In an Ab-enriched category,

a finite product is also a coproduct

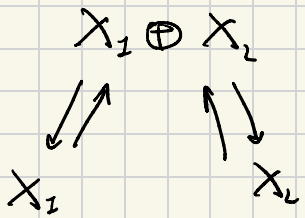
(so an initial object is also final, hence a zero object)

Idea: 

$\exists 0$-map because $\mathrm{Hom}(X,Y)$ group.

Notation: $X_1 \times X_2 = X_1 \sqcup X_2 = X_1 \oplus X_2$

In Vect_k:



Rem: Does not work for infinite products: $\coprod_{i \in I} X_i = \bigoplus_{i \in I} X_i \neq \prod_{i \in I} X_i$ in Ab.

Def: An additive category $\mathcal{C}$ is pre-abelian if any morphism has a kernel and a cokernel.

If $\mathcal{C}$ is pre-abelian:

$$\mathrm{Ker}(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker}(f)$$

$$\mathrm{Coker}(\mathrm{Ker}(f)) \longrightarrow \mathrm{Ker}(\mathrm{Coker}(f))$$

$\mathrm{Coker}(\mathrm{Ker}(f))$ = "$\mathrm{CoIm}(f)$"

$\exists!$ (universal properties)

$\mathrm{Ker}(\mathrm{Coker}(f))$ = "$\mathrm{Im}(f)$"

Def: f is called strict if $\mathrm{CoIm}(f) \longrightarrow \mathrm{Im}(f)$.

Def: A pre-abelian category is *abelian* if all morphisms are strict.

Rem: This is equiv to every mono is a kernel and every epi is a cokernel

Ex: . $Ab = \mathbb{Z}$-mod is an abelian category (first isom thm)

R-mod ——— ———

. $\mathcal{A}$ ab category, $\mathcal{C}$ small category then $Fun(\mathcal{C}, \mathcal{A})$ also abelian.

. $Ban_{\mathbb{R}}$: category of $\mathbb{R}$-Banach spaces, is *not* abelian.
(with bounded linear maps) is pre-abelian.

$V \xrightarrow[i]{} W$ $\quad$ Coker $i = W/\bar{V}$

V not closed $\quad$ $\ker(W \to W/\bar{V}) \neq V$.

in W $\quad$ $\mathrm{Im}\, i = V$ $\quad$ $\times$ i not strict

(eg. $\ell^1(\mathbb{R}) \hookrightarrow \ell^2(\mathbb{R})$)

Ex: Top. ab group, Filtered ab groups, ... are not abelian

In an abelian category, any morphism has a factorization:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{\text{epi}} \quad \nearrow_{\text{mono}} & \\ & \mathrm{Im}(f) & \end{array}$$

Def: A seq of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if

$$\operatorname{Ker} g = \operatorname{Im} f$$

(as sub-objects of B)

[$\forall X$ {subsets of X} $= \left\{ Y \to X \text{ monomorphism} \right\} / \sim$

"set"

$Y_1 \to X$, $Y_1 \xrightarrow{\cong} Y_2$, $Y_2 \to X$]

A short exact sequence (ses) is a seq. of the form $0 \to A \xrightarrow[f]{} B \xrightarrow[g]{} C \to 0$, exact at every places, that is:

- f mono, g epi, $\operatorname{Ker} g = \operatorname{Im} f$ (so $f = \operatorname{Ker} g$, $g = \operatorname{Coker} f$).

Ex:

$0 \to A \to A \oplus C \to C \to 0$: ses.

Any ses isom to this is called split.