Example: Show

$$a_n = \frac{n^2 + \sin(n^2)}{3n^2 + n + 1}$$

converges.

Thoughts: Recall we need to prove

$$\exists L \in \mathbb{R} \qquad \forall \varepsilon > 0 \qquad \exists N \in \mathbb{N} \qquad \forall n \geqslant N \qquad |a_n - L| < \varepsilon.$$

Firstly we need to decide what L is. The significant terms are the n^2s. Note

$$a_n = \frac{1 + n^{-2}(\sin n^2)}{3 + n^{-1} + n^{-2}} \approx \frac{1 + 0}{3 + 0 + 0} = \frac{1}{3},$$

so we should expect $L = 1/3$.

Solution: Note for $n \geqslant 1$ that

$$a_n \leqslant \frac{n^2 + 1}{3n^2 + n + 1} \leqslant \frac{n^2 + n}{3n^2 + n} = \frac{n + 1}{3n + 1} \leqslant \frac{n + 1}{3n} = \frac{1}{3} + \frac{1}{3n}$$

and that

$$a_n \geqslant \frac{n^2 - 1}{3n^2 + n + 1} \geqslant \frac{n^2 - n}{3n^2 + 2n} = \frac{n - 1}{3n + 2} = \frac{1}{3} - \frac{5}{3(3n + 2)}.$$

Let $\varepsilon > 0$. We wish to find N such that for $n \geqslant N$ we have

$$-\varepsilon < a_n - \frac{1}{3} < \varepsilon.$$

First we note the right inequality holds if

$$\frac{1}{3n} < \varepsilon \qquad \Longleftrightarrow \qquad n > \frac{1}{3\varepsilon}.$$

And the left inequality holds if

$$-\varepsilon < -\frac{5}{3(3n + 2)} \qquad \Longleftarrow \qquad n > \frac{5}{9\varepsilon}.$$

So take

$$N \geqslant \max\left\{\frac{1}{3\varepsilon}, \frac{5}{9\varepsilon}\right\} = \frac{5}{9\varepsilon},$$

and it follows that $|a_n - 1/3| < \varepsilon$.