Lecture 5.

Def: Let $F: \mathcal{C} \to \mathcal{D}$ be an additive functor between abelian categories $\mathcal{C}$ and $\mathcal{D}$.

F is <u>left-exact</u> if $\forall\ 0 \to A \to B \to C \to 0$ s.e.s in $\mathcal{C}$,

$$0 \to F(A) \to F(B) \to F(C) \quad \text{exact in } \mathcal{D}$$

F is <u>right-exact</u> if

$$F(A) \to F(B) \to F(C) \to 0 \quad \text{exact in } \mathcal{D}$$

F <u>exact</u> if both left & right exact:

$$0 \to F(A) \to F(B) \to F(C) \to 0 \text{ ses in } \mathcal{D}$$

Lem: Let $\mathcal{A}$ be an abelian category and let $M \in \mathrm{Obj}(\mathcal{A})$.
The functor $\mathrm{Hom}_{\mathcal{A}}(M, -): \mathcal{A} \to Ab$ is left exact:

$\forall$

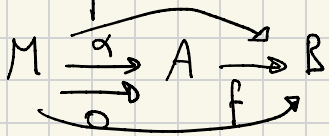
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{s.e.s}$$

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow[f_*]{} \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow[g_*]{} \mathrm{Hom}_{\mathcal{A}}(M, C)$$

exact.

Proof: If $\alpha \in \mathrm{Hom}_{\mathcal{A}}(M, A)$:

$f_*\alpha = f \circ \alpha = 0 \Rightarrow \alpha = 0$ since f is monic.



Exact: $g \circ f = 0$ so $g_* f_* \alpha = g f \alpha = 0$

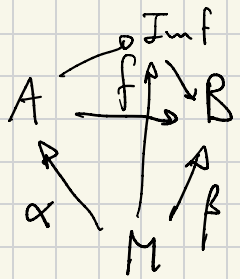
so $g_* f_* = 0$.

so $\mathrm{Im} f_* \subset \mathrm{Ker} g_*$

If $\beta \in \mathrm{Hom}_{\mathcal{A}}(M, B)$ and $g_* \beta = g\beta = 0$

then $\mathrm{Im}\beta \subset \mathrm{Ker} g = \mathrm{Im} f$ (exact)

so $\exists \alpha \in \mathrm{Hom}_{\mathcal{A}}(M, A)$ s.t. $\beta = f \circ \alpha = f_* \alpha$



□

Lemma: $\forall M \in \mathrm{obj}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(-, M) : \mathcal{A}^{op} \to Ab$ is left-exact.

Proof: $\mathcal{A}^{op}$ also abelian category. Apply previous lemma to $\mathcal{A}^{op}$. □

Ex: $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ in Ab

$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, -): 0 \to 0 \to 0 \to \mathbb{Z}/n\mathbb{Z}$

$\mathrm{Hom}(-, \mathbb{Z}/n\mathbb{Z}): 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z} \to 0$

Def: An object P is called projective if the functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to Ab \text{ is exact.}$$

that is: $\forall$ $B \to C \to 0$ exact

$\exists$ P

An object I is called injective if the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to Ab \text{ is exact.}$$

that is: $\forall$ $0 \to A \to B$ exact

I $\exists$

Ex: R: ring M: R-module (left) $\mathrm{Hom}_{R\text{-mod}}(R, M) = M$ as abelian groups

$\mathrm{Hom}(R, -) : R\text{-mod} \to Ab$ is the forget functor. Exact!

so R is projective in R-mod.

$\forall J$ set: $R^J = R^{\oplus J} = \bigoplus_J R$ free R-module on the basis J

$$\mathrm{Hom}(R^{\oplus J}, M) = \prod_J M$$

Exact, so free R-modules are projective in R.

Remark: If $P \in R\text{-mod}$ is projective.

Choosing generators of P, $\exists$ surjection (epi)

$$0 \to K \to R^J \to P \to 0.$$

$$\begin{array}{ccc} & & P \\ & \overset{\exists}{\nearrow} & \uparrow \mathrm{id} \\ & & P \end{array}$$

Split exact sequence: $R^J \simeq P \oplus K$

so P direct summand of a free R-module.

Exercise: Check the converse.

Thm: In R-mod, projectives are direct summands of free R-modules.

Def: Let $\mathcal{A}$ be an additive category. We call C compact if

$$\bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{A}}(C, b_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(C, \coprod_{i \in I} b_i)$$

Natural map of abelian group

$\forall \{b_i\}_{i \in I}$ objects in $\mathcal{A}$ s.t. $\coprod_{i \in I} b_i$ exist

$$\begin{array}{ccc} b_j & \to & \coprod_{i \in I} b_i \\ \uparrow & \nearrow & \\ C & & \end{array}$$

Rem: Automatic for finite I.

Ex: $\mathrm{Hom}_{R\text{-mod}}(R, \bigoplus_i M_i) = \bigoplus_i M_i = \bigoplus_i \mathrm{Hom}(R, M_i)$

so R compact in R-mod

R also generator: $\forall M, \exists I, \bigoplus_I R \to M \to 0$ epi

Thm: Let $\mathcal{A}$ be an abelian category with arbitrary coproducts. Assume that P is a compact projective generator. Then the functor

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \longrightarrow \underbrace{\mathrm{End}_{\mathcal{A}}(P)^{op}}_{\mathrm{Hom}_{\mathcal{A}}(P,P)}\text{-mod}$$

$\mathrm{Hom}_{\mathcal{A}}(P,P)$ Ring

is an equivalence of categories.

Rem: R & R' s.t. R-mod $\simeq$ R'-mod ? "Morita equivalence".

Sketch of proof: Construct an adjoint. Denote $R := \mathrm{End}_{\mathcal{A}}(P)^{op}$.

Want $R \to P$

$R^{\oplus I} \to P^{\oplus I}$

any $M \in R$-mod as a presentation: